When are path-dependent payoffs suboptimal?

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When are path-dependent payoffs suboptimal?

Path-dependent payoffs are known to be suboptimal e.g. in

- Black-Scholes models,
  as shown via stochastic control in

  Cox and Leland 1982
  On dynamic investment strategies.
  Published 2000 in the Journal of Economic Dynamics and Control 24.

  and more generally in all

- Exponential Lévy models with Esscher transform,
  where favorable path-independent payoffs for risk-averse investors are constructed in

  Vanduffel, Chernih, Maj and Schoutens 2009
  A note on the suboptimality of path-dependent payoffs in general markets.
  Applied Mathematical Finance 16(4).

When do risk-averse investors prefer path-independent payoffs?

Why?

Exponential Lévy model?

Esscher transform?

Setting the stage

- Discounted payoff X at investment horizon T
- Investor is strongly risk-averse
  - concave stochastic order, i.e.
  - Y preferable to X if $\mathbb{E}[U(Y)] \geq \mathbb{E}[U(X)]$ for all concave functions U
- Pricing kernel Z, i.e.
  - payoff X at time T
  - price $\mathbb{E}[Z_T X]$ today

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- Pricing kernel Z, i.e.
  - payoff X at time T
  - price $\mathbb{E}[Z_T X]$ today
- No specific assumptions on stochastic model and pricing kernel so far.
- No utility function specified, just strong risk aversion.

Risk-averse investors like conditioning

If X is $\mathcal{F}_T$-measurable and $\mathcal{G}$ is a sub-sigma-algebra

$$\mathbb{E}\Big[U(\mathbb{E}[X|\mathcal{G}])\Big] \geq \mathbb{E}\Big[U(X)\Big]$$

for all concave functions U (due to Jensen's inequality).

$\Longrightarrow$ Our investor prefers $\mathbb{E}[X|\mathcal{G}]$ over X.

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But can she actually afford $\mathbb{E}[X|\mathcal{G}]$?

That means, is $\mathbb{E}\left[Z_T\, \mathbb{E}[X|\mathcal{G}]\,\right] \leq \mathbb{E}\left[Z_T X\right]$ for some $\mathcal{G}$?

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Yes, she can afford $\mathbb{E}[X|Z_T]$!

Risk-averse investors like conditioning

Functions of kernel preferred

- Risk averse investors prefer $\mathbb{E}[X|Z_T]$ over X.
- $\mathbb{E}[X|Z_T]$ and X have the same price.
- Moreover, if we define $\mathbb{Q}$ via $d\mathbb{Q} = Z_T\, d\mathbb{P}$, $\mathbb{E}_{\mathbb{Q}}[X|Z_T] = \mathbb{E}[X|Z_T]$.

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Therefore

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In particular

If (S_t) is the discounted underlying
and $Z_T = g(S_T)$ is an injective function of S_T,
$\Longrightarrow$ the path-independent payoff $\mathbb{E}[X|S_T]$ is preferable.

Cheaper payoff with same distribution

Basic example

- Suppose that there are only two states ω_1 and ω_2, each occuring with probability one half.
- Let $S_T(\omega_1) = 1$ and $S_T(\omega_2) = 2$.
- Assume $g(1) > g(2)$.

Price of payoff $h(S_T)$ is $\frac{1}{2}g(1)\,h(1) + \frac{1}{2}g(2)\,h(2)$

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The following payoffs have the same distribution

$h(1) = 2$	|	$h(1) = 1$
$h(2) = 1$	|	$h(2) = 2$

but the increasing payoff is cheaper

$$g(1) + \frac{1}{2}g(2) \quad > \quad \frac{1}{2}g(1) + g(2)$$

Cheaper payoff with same distribution

Increasing payoffs preferable in case of decreasing kernel

If $Z_T = g(S_T)$ is a decreasing function of S_T and

F is the distribution function of S_T

$H^{\leftarrow}$ is the inverse distribution function of $h(S_T)$

$\hat{h}(x) := H^{\leftarrow}(F(x))$,

$\Longrightarrow$ $\hat{h}$ is increasing and

$\hat{h}(S_T) = H^{\leftarrow}(F(S_T))$ is distributed as $h(S_T)$

but $\hat{h}(S_T)$ is cheaper.

I.e. $\hat{h}(S_T)$ is preferable.

Attractive payoffs

- Risk-averse investors prefer payoffs of type $\tilde{h}(Z_T)$.
  Distribution $\rightsquigarrow \tilde{h}(Z_T) = \mathbb{E}[X|Z_T]$.
- If $Z_T = g(S_T)$ is an injective function of S_T, path-independent payoffs $h(S_T)$ are preferred.
  Distribution $\rightsquigarrow h(S_T) = \mathbb{E}[X|S_T]$.
- If g is decreasing, $\hat{h}(S_T)$ with increasing $\hat{h}$ are preferred.
  Distribution $\rightsquigarrow \hat{h}(S_T) := H^{\leftarrow}(F(S_T))$.

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Esscher transform: $Z_T = g(S_T) = \dfrac{S_T^{\gamma}}{\mathbb{E}\left[S_T^{\gamma}\right]}$

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If (increasing and convex) utility function U is known, $\tilde{h}(Z_T)$ is optimal if $Z_T \propto U'(\tilde{h}(Z_T))$.
If $Z_T = g(S_T)$ and $U' \propto g \circ h^{\leftarrow}$, $h(S_T)$ is optimal.

Conditional expectation

for Geometric Brownian motion

$$S_t = e^{\sigma B_t + \mu t}$$

with a $\mathbb{P}$-standard Brownian motion (B_t), and let g be a deterministic function. Then,

$$\mathbb{E}\left[e^{\int_0^T g(u)\, d(\log S_u)} \middle| S_T \right]$$

$$= S_T^{\frac{1}{T}\int_0^T g(u)\, du}\, e^{\frac{\sigma^2 T}{2}\left(\frac{1}{T}\int_0^T g(u)^2\, du - \left(\frac{1}{T}\int_0^T g(u)\, du\right)^2\right)}.$$

Early payment and geometric average

for Geometric Brownian motion

$$S_t = e^{\sigma B_t + \mu t}$$

Early payment

$$\mathbb{E}\left[S_u^{\lambda} \middle| S_T \right] = S_T^{\lambda \frac{u}{T}} e^{\lambda^2 \frac{\sigma^2 u}{2} \left(1 - \frac{u}{T}\right)}.$$

Continuous geometric average

$$\mathbb{E}\left[e^{\frac{1}{T} \int_0^T (\log S_u)\, du} \middle| X_T \right]$$
$$= \mathbb{E}\left[e^{\int_0^T \left(1 - \frac{u}{T}\right) d(\log S_u)} \middle| X_T \right] = \sqrt{S_T}\, e^{\frac{\sigma^2 T}{24}}.$$

Supremum

for Geometric Brownian motion

$$S_t = e^{\sigma B_t + \mu t}$$

The supremum of (S_u) on $[0, T]$ conditional on the terminal value S_T is

$$\mathbb{E}\left[\sup_{u \le T} S_u \middle| S_T\right] = (S_T \vee 1)\left\{1 + \frac{\sigma\sqrt{T}\,\Phi\left(-\frac{|\log S_T|}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}\right)}{2\,\varphi\left(-\frac{|\log S_T|}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}\right)}\right\}.$$

Here, φ is the density function and Φ is the cumulative distribution function of a standard normal distribution.

Conditional expectation

for independent increments

Suppose that L is a process such that the increments $L_s - L_0$ and $L_t - L_s$ are independent.

Moreover, suppose that the distribution of these increments admits densities $f_{0,s}$ and $f_{s,t}$. Then,

$$\mathbb{E}[\,c(L_s)|\,L_T] = h_c(L_T), \quad \text{where}$$

$$h_c(x) = \frac{\int_{-\infty}^{\infty} c(y)\, f_{0,s}(y)\, f_{s,T}(x-y)\, dy}{\int_{-\infty}^{\infty} f_{0,s}(y)\, f_{s,T}(x-y)\, dy}$$

$$= \frac{\mathbb{E}[c(L_s)\, f_{s,T}(x-L_s)]}{\mathbb{E}_{\mathbb{P}}[f_{s,T}(x-L_s)]}.$$

Summary

- If the pricing kernel is path-independent, risk-averse investors prefer path-independent payoffs.
  - E.g. exponential Lévy with Esscher transform.
  - However, a path-independent net position can consist of several path-dependent payoffs...
- On the other hand, if the pricing kernel is path-depenent, path-dependent payoffs are attractive.
  - E.g. exponential Lévy with minimal entropy martingale measure, q-optimal martingale measure etc.
  - However, complex path-dependent products may not be available at a competitive price...