Aspects of market impact modeling and optimal trade execution

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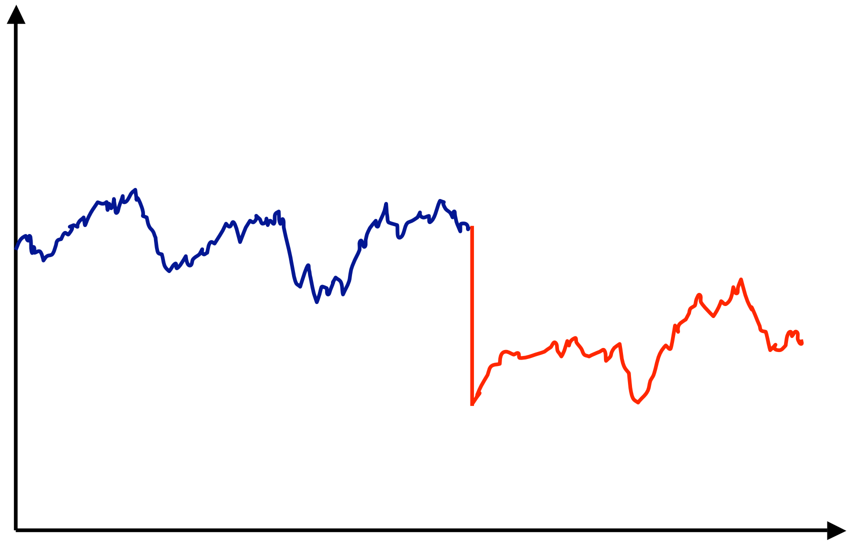
Joint work with Aurélien Alfonsi, Jim Gatheral, and Alla Slynko

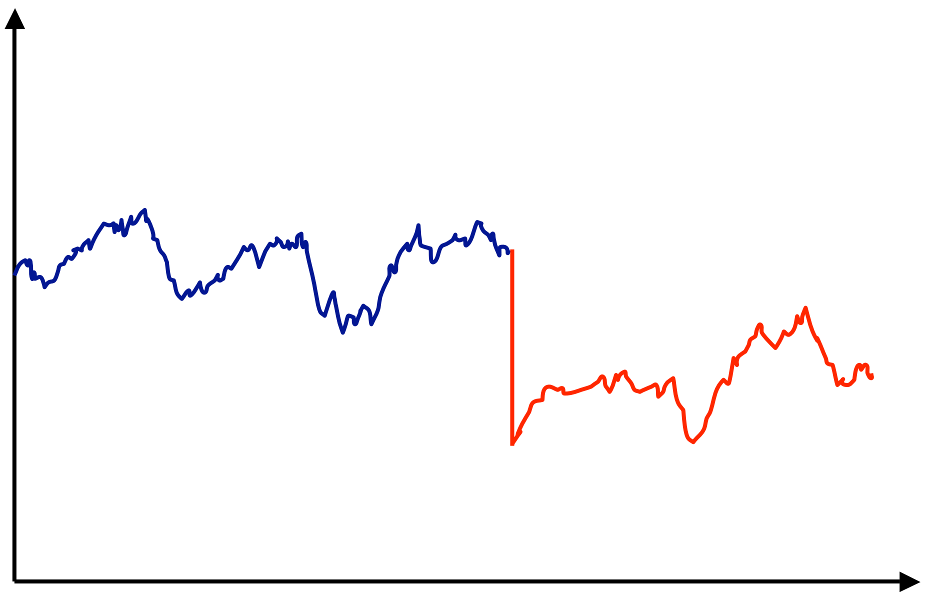
Market impact: adverse feedback effect on the quoted price of a stock caused by one's own trading

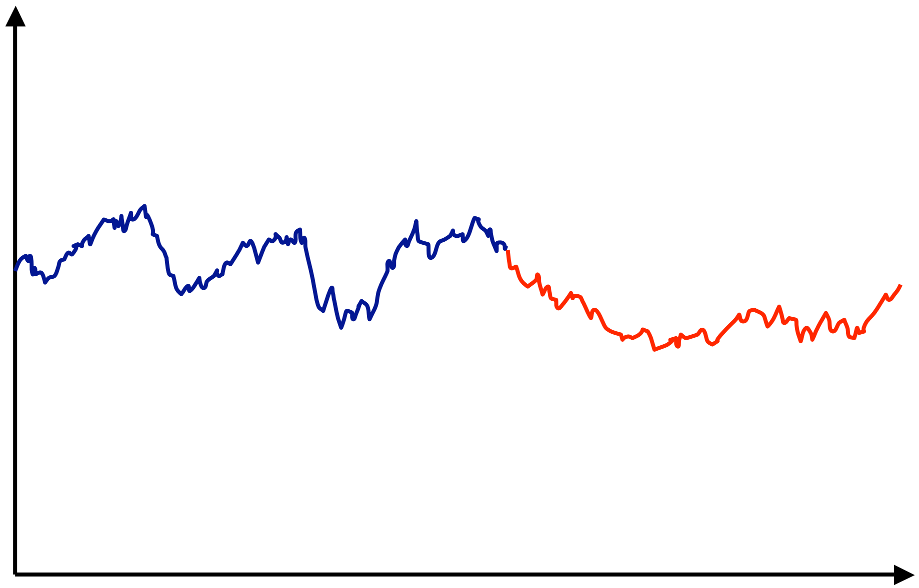


Empirical facts:

- Often nonlinear function of order size
- Mainly transient; resilience of prices

Important consequence of transience: liquidity costs of a large trade can be reduced signficantly by splitting the trade into a sequence of smaller trades (“child orders”), which are then spread out over a certain time interval.





Goal:

Realistic and tractable model for nonlinear transient price impact

Several models in the literature, e.g.:

- Alfonsi, Fruth, and A.S. (Quant. Finance, 2010),
  Alfonsi and A.S. (Preprint, 2009)
- Gatheral (Quant. Finance, forthcoming)

Both models are similar and coincide for linear price impact,
extend Obizhaeva and Wang (Preprint, 2005)

Our main concerns here:

- Can there be undesirable properties, e.g., "large-investor arbitrage"?
- Is model behavior stable and robust when parameters change?

1. Linear transient price impact

Concentrate on effects created by transience
of price impact

2. Nonlinear transient price impact

How to model nonlinearity?

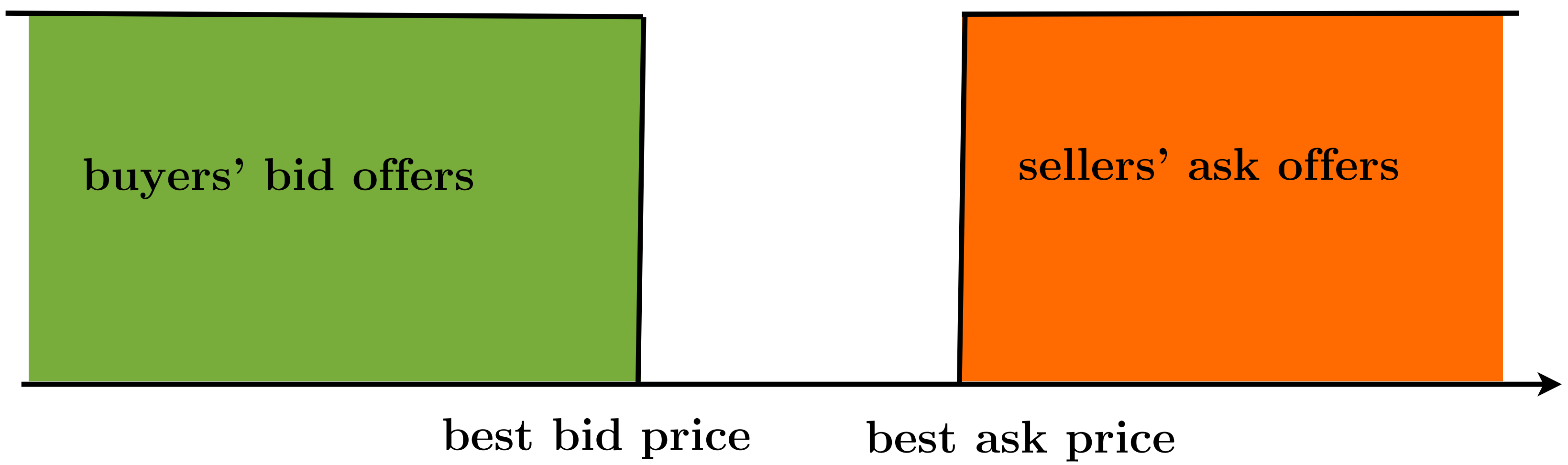
Possible effects of nonlinear impact?

Preliminary study....

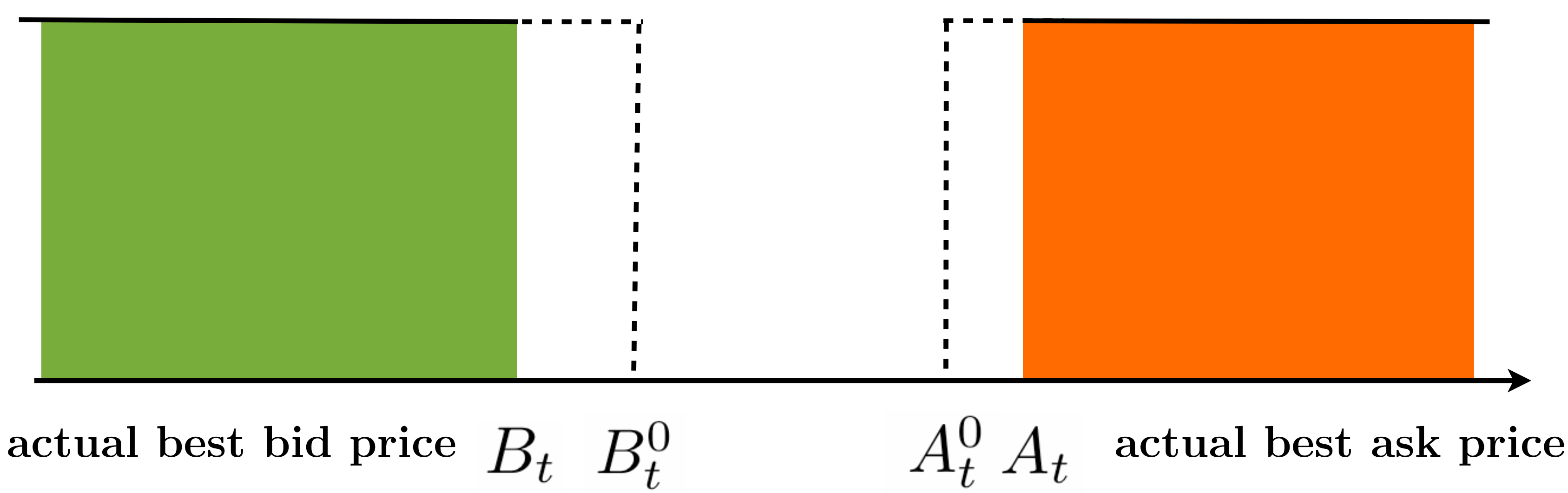
1. Linear transient price impact

Model based on intuition of electronic limit order book....

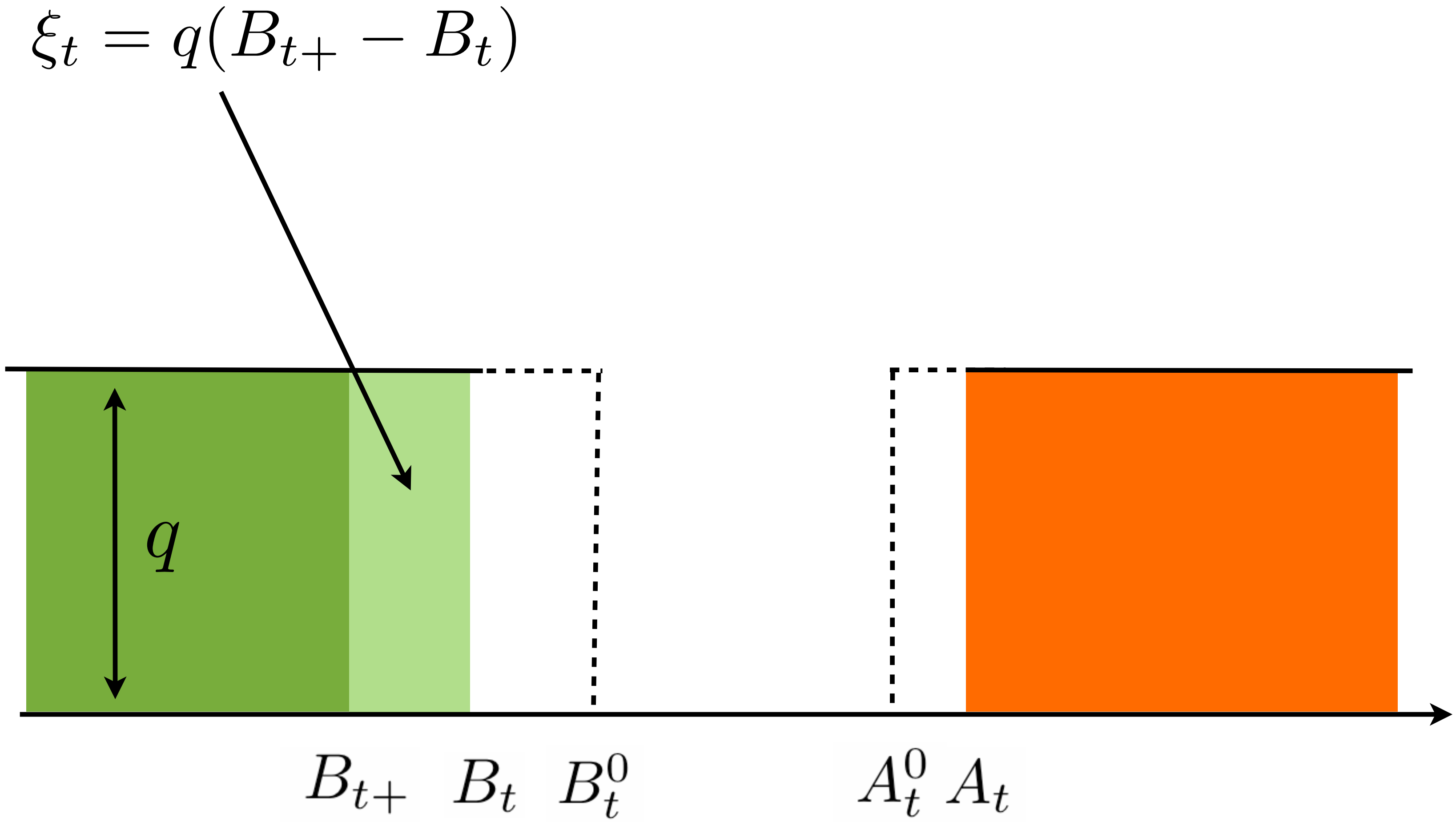
Obizhaeva and Wang (2005)
Alfonsi, Fruth, and A.S. (2008)
Alfonsi, A.S., and Slynko (2009)
Gatheral, A.S., and Slynko (2010)


buyers' bid offers
sellers' ask offers
best bid price
best ask price

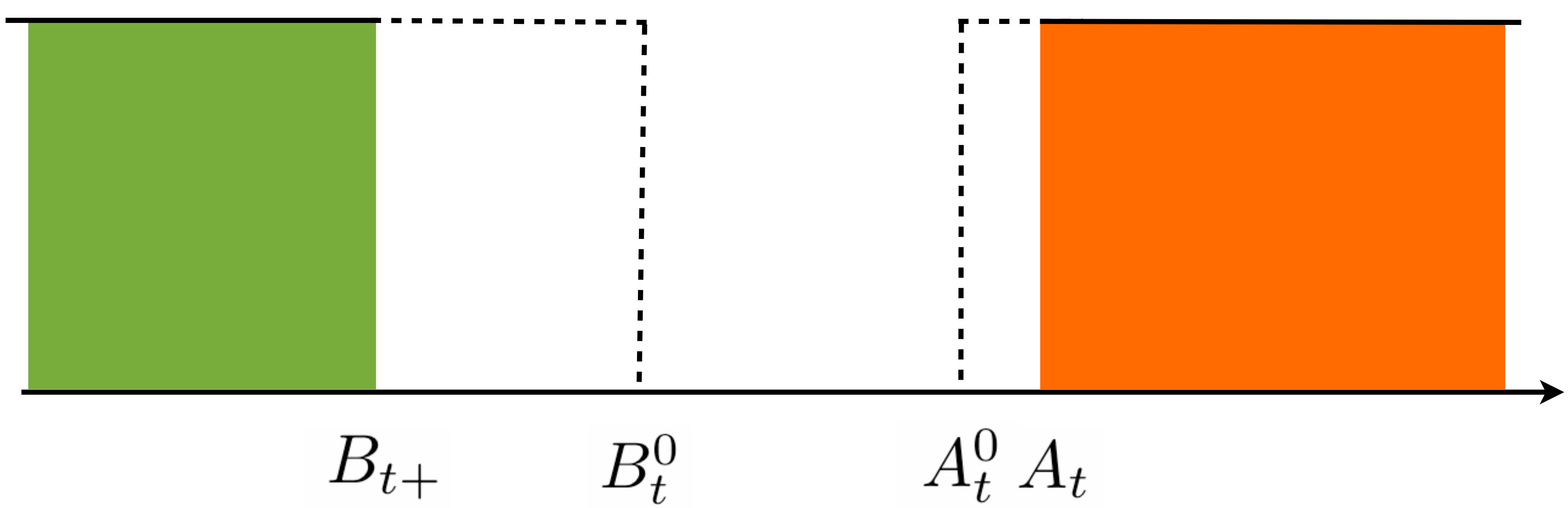
Limit order book model after large trades



Limit order book model at large trade



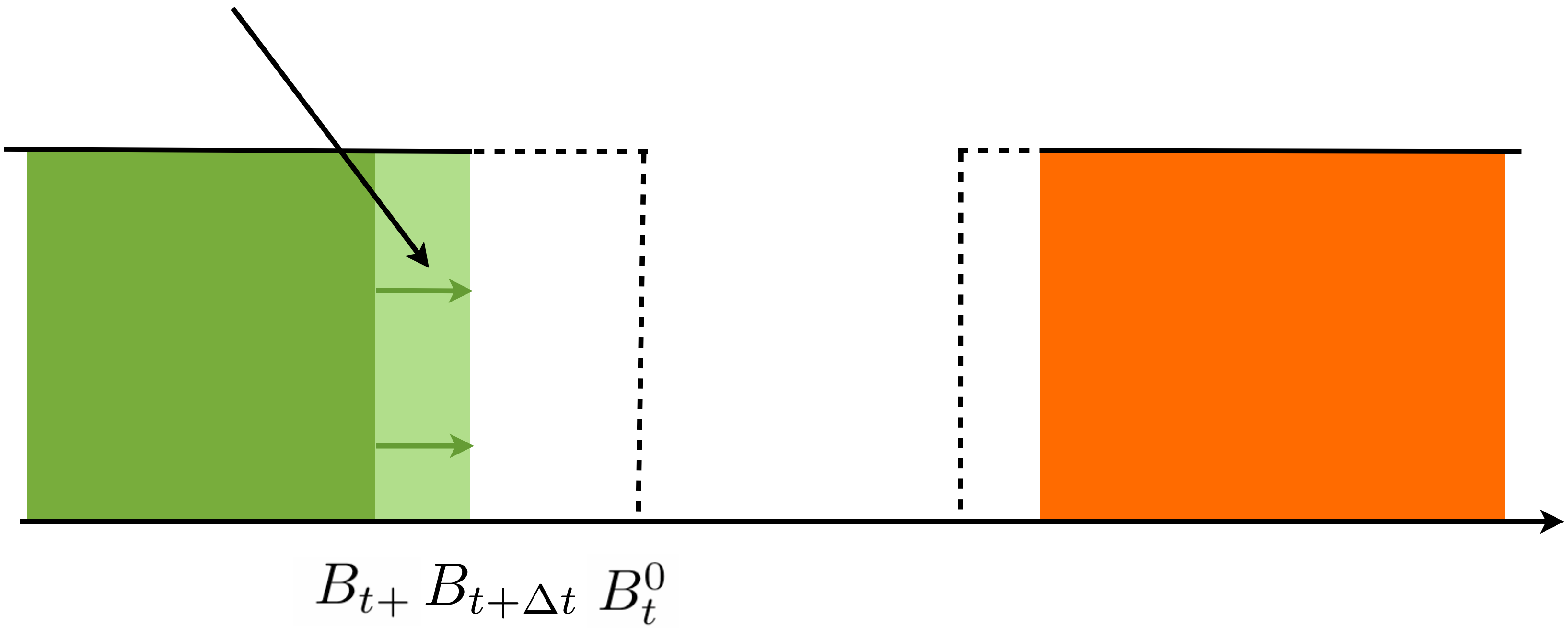
Limit order book model immediately after large trade



Resilience of the limit order book

$$G : [0, \infty[\to [0, \infty[, \quad G(0) = \frac{1}{q}$$

$$\xi_t \cdot G(\Delta t) + \text{decay of previous trades}$$



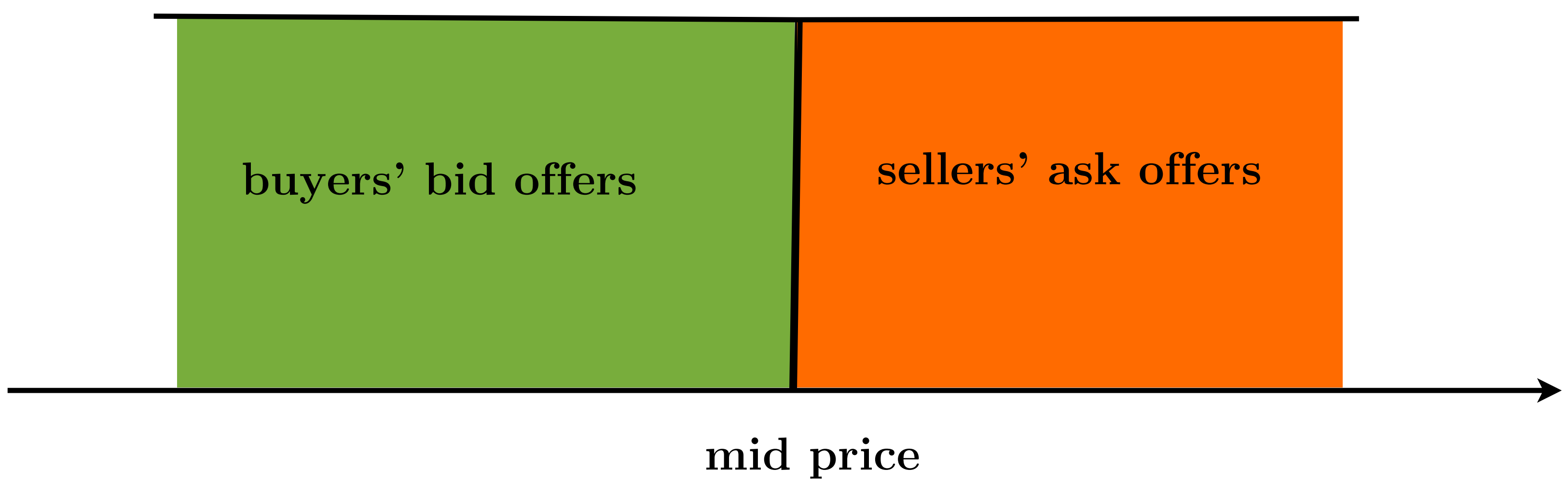
Observation: Expected costs of a sequence of trades can be compared to cost in a simplified **zero-spread model**.

Unaffected price process: martingale S^0

Admissible trategy: adapted process $X = (X_t)$ that describes the number of shares held by the trader

- $t \longrightarrow X_t$ is leftcontinuous with finite total variation
- the signed measure dX_t has compact support
- $X_t = 0$ for $t \geq T$

Block-shaped zero-spread order book model



Impacted price process:

$$S_t = S_t^0 + \int_{\{s<t\}} G(t-s)\,dX_s,$$

where

$$G : (0,\infty) \to [0,\infty)$$

is the decay kernel. It describes the resilience of price impact between trades; see Bouchaud et al. (2004), Obizhaeva and Wang (2005), Alfonsi et al. (2008, 2007), Gatheral (2008).

We first assume

(1) $$G \text{ is bounded and } G(0) := \lim_{t\downarrow 0} G(t) \text{ exists.}$$

Costs of a strategy X:

When X is continuous at t, then the infinitesimal order dX_t is executed at price S_t, so $S_t\,dX_t$ is the cost increment.
Thus, the total costs of a continuous strategy are

$$\int S_t\,dX_t = \int S_t^0\,dX_t + \int\int_{\{s<t\}} G(t-s)\,dX_s\,dX_t.$$

When X has a jump ΔX_t, then the price is moved from S_t to

$$S_{t+} = S_t + \Delta X_t G(0)$$

This linear price impact corresponds to a constant supply curve for which $G(0)^{-1}\,dy$ buy or sell orders are available at each price y. The trade ΔX_t is thus carried out at the following cost,

$$\int_{S_t}^{S_{t+}} yG(0)^{-1}\,dy = \frac{1}{2G(0)}\big(S_{t+}^2 - S_t^2\big) = \frac{G(0)}{2}(\Delta X_t)^2 + \Delta X_t S_t.$$

Hence, the total costs of an arbitrary admissible strategy X are given by

$$\begin{aligned}
&\int S_t\,dX_t + \frac{G(0)}{2}\sum(\Delta X_t)^2 \\
&\quad = \int S_t^0\,dX_t + \int\int_{\{s<t\}} G(t-s)\,dX_s\,dX_t + \frac{G(0)}{2}\sum(\Delta X_t)^2 \\
&\quad = \int S_t^0\,dX_t + \frac{1}{2}\int\int G(|t-s|)\,dX_s\,dX_t.
\end{aligned}$$

It therefore follows from the martingale property of S^0 that the **expected costs** of an admissible strategy with X_0 deterministic are

$$\mathbb{E}\Big[\int S_t^0\,dX_t\Big] + \frac{1}{2}\mathbb{E}[\,\mathcal{C}(X)\,] = X_0 S_0 + \frac{1}{2}\mathbb{E}[\,\mathcal{C}(X)\,],$$

where

$$\mathcal{C}(X) := \int\int G(|t-s|)\,dX_s\,dX_t.$$

Questions:

- Can there be model irregularities?
- Existence, uniqueness, and structure of strategies minimizing the expected costs?
- Stability of these strategis?

Definition 1 (Huberman and Stanzl (2004)). A round trip is an admissible strategy with $X_0 = 0$. A price manipulation strategy is a round trip with strictly negative expected costs.

Clearly, there is no price manipulation when

$$\mathcal{C}(X) \geq 0 \qquad \text{for all strategies } X.$$

Proposition 1 (Straightforward extension of Bochner's thm). $\mathcal{C}(X) \geq 0$ *for all strategies* $X \iff G(|\cdot|)$ *can be represented as the Fourier transform of a positive finite Borel measure* μ *on* $\mathbb{R}$, *i.e.,*

$$G(|x|) = \int e^{ixz}\, \mu(dz);$$

(G is **positive definite***).*

If, in addition, the support of μ *is not discrete, then* $\mathcal{C}(X) > 0$ *for every nonzero admissible strategy* X *(G is* **strictly positive definite***).*

Optimal trade execution problem: Minimizing expected costs,

$$S_0^0 y + \frac{1}{2}\mathbb{E}[\,\mathcal{C}(X)\,]$$

for strategies that liquidate a given long or short position of y shares within a given time frame.

Time constraint: compact set $\mathbb{T} \subset [0, \infty)$.

Boils down to minimizing $\mathcal{C}(\cdot)$ over

$$\mathcal{X}(y, \mathbb{T}) := \Big\{ X \,\Big|\, \text{deterministic strategy with } X_0 = y \text{ and support in } \mathbb{T} \Big\}.$$

Simple when $\mathbb{T}$ is discrete. Existence of minimizers not clear when $\mathbb{T}$ is not discrete.

Minimization of expected costs used here to analyze model behavior.

Proposition 2. *Suppose that G is strictly positive definite. Then:*

- *The optimal strategy $X^* \in \mathcal{X}(y, \mathbb{T})$ is unique, if it exists*
- *X^* optimal if and only if it is a measure-valued solution of the Fredholm integral equation*

$$\int G(|t-s|)\, dX_s^* = \lambda \qquad \text{for all } t \in \mathbb{T}, \tag{2}$$

for some constant λ.

Examples

Example 1 (Exponential decay). For the exponential decay kernel

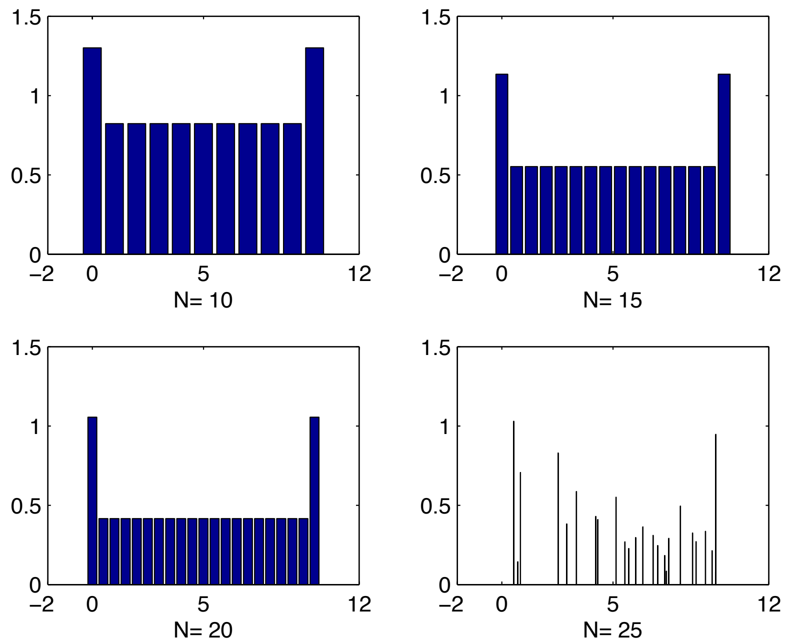
$$G(t) = e^{-\rho t},$$

$G(|\cdot|)$ is the Fourier transform of the positive measure

$$\mu(dt) = \frac{1}{\pi} \frac{\rho}{\rho^2 + t^2} \, dt$$

Hence, G is strictly positive definite.

Optimal strategies for $G(t) = e^{-\rho t}$ and discrete $\mathbb{T}$:



For $\mathbb{T} = [0, T]$:

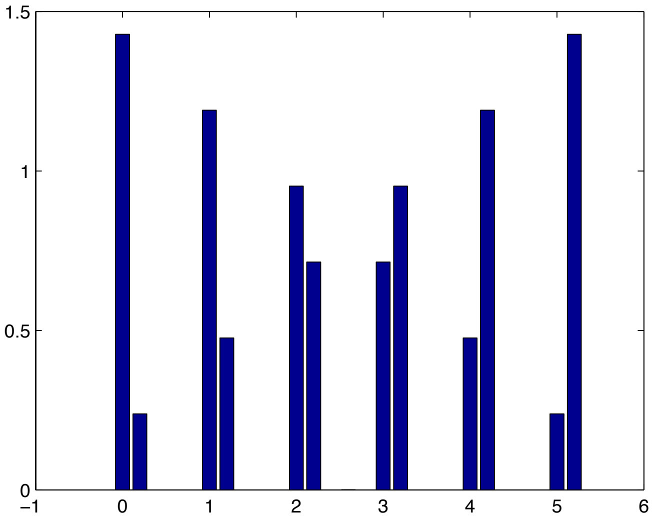
$$dX_s^* = \frac{x}{\rho T + 2}\Big(\delta_0(ds) + \rho\, ds + \delta_T(ds)\Big).$$

Example 2 (Capped linear decay). $G(t) = (1 - \rho t)^+$

The unique optimal strategy X^* for $\mathbb{T} = [0, T]$ is

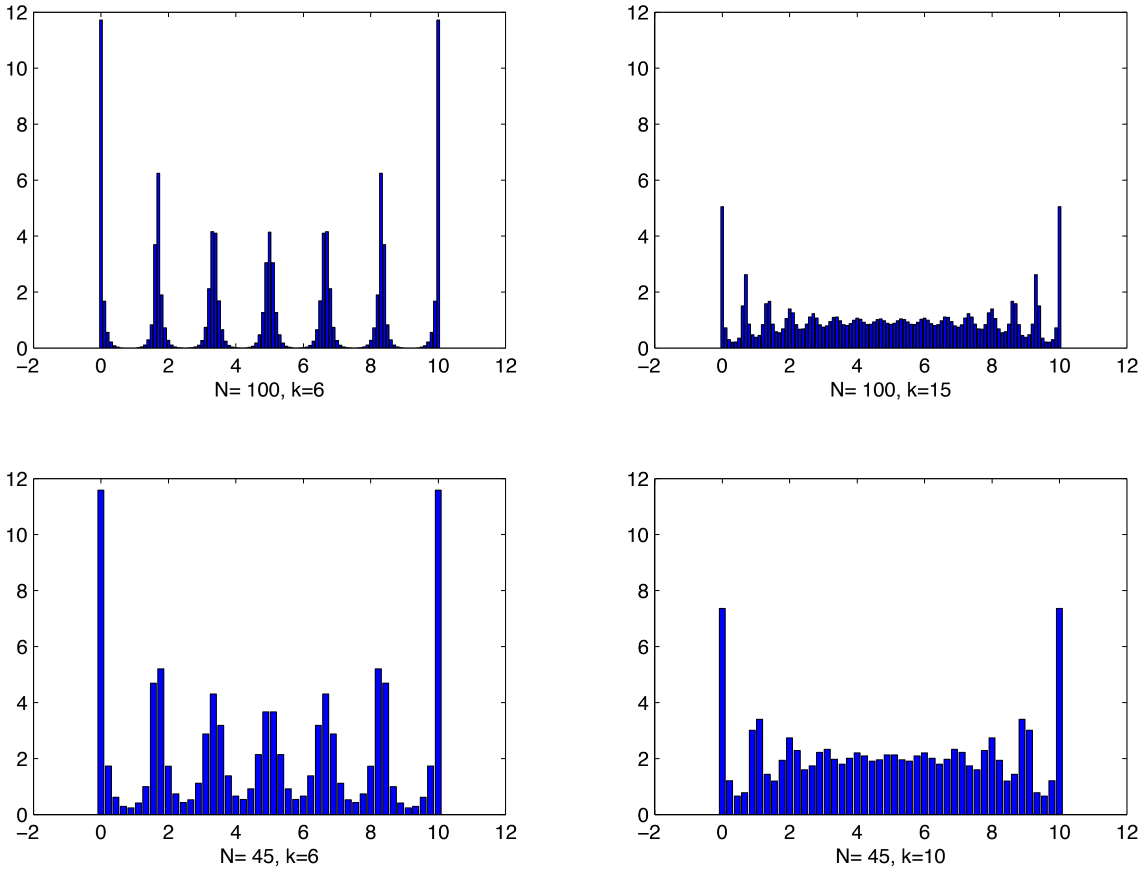
$$dX^* = \frac{X_0}{2+N} \sum_{i=0}^{N} \Big(1 - \frac{i}{N+1}\Big)\big(\delta_{\frac{i}{\rho}} + \delta_{T-\frac{i}{\rho}}\big),$$

where $N := \lfloor \rho T \rfloor$.



Here $\rho = 1$, $T = 5.15$, and hence $N = 5$

Otherwise, for discrete equistant grid $\mathbb{T}$,

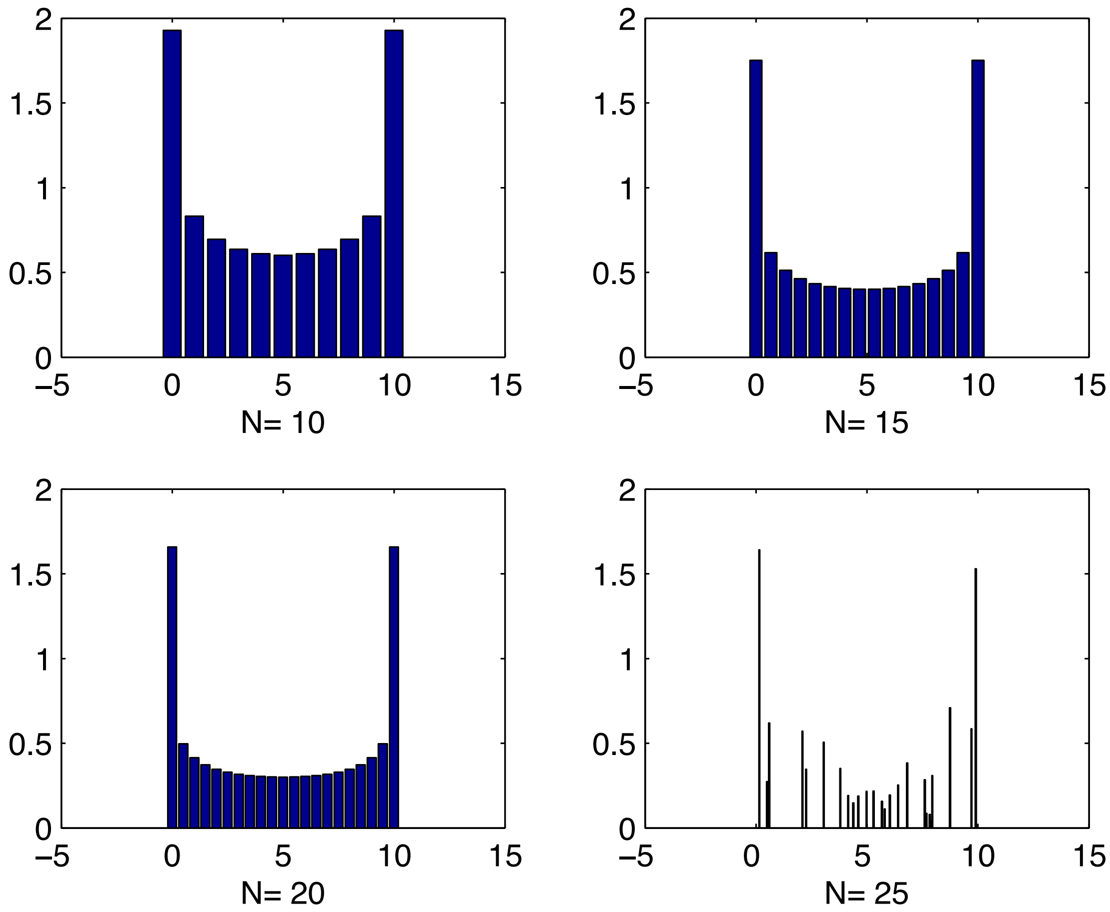


More generally: Convex decay

G is convex, decreasing, nonnegative, and nonconstant $\Longrightarrow$
$G(|\cdot|)$ is strictly positive definite.

[Carathéodory (1907), Toeplitz (1911), Young (1912)]

Example 3 (Power law decay). $G(t) = (1 + t)^{-\alpha}$ and equidistant grid $\mathbb{T}$,



So everything looks nice for

$$G(t) = \frac{1}{(1+t)^2}$$

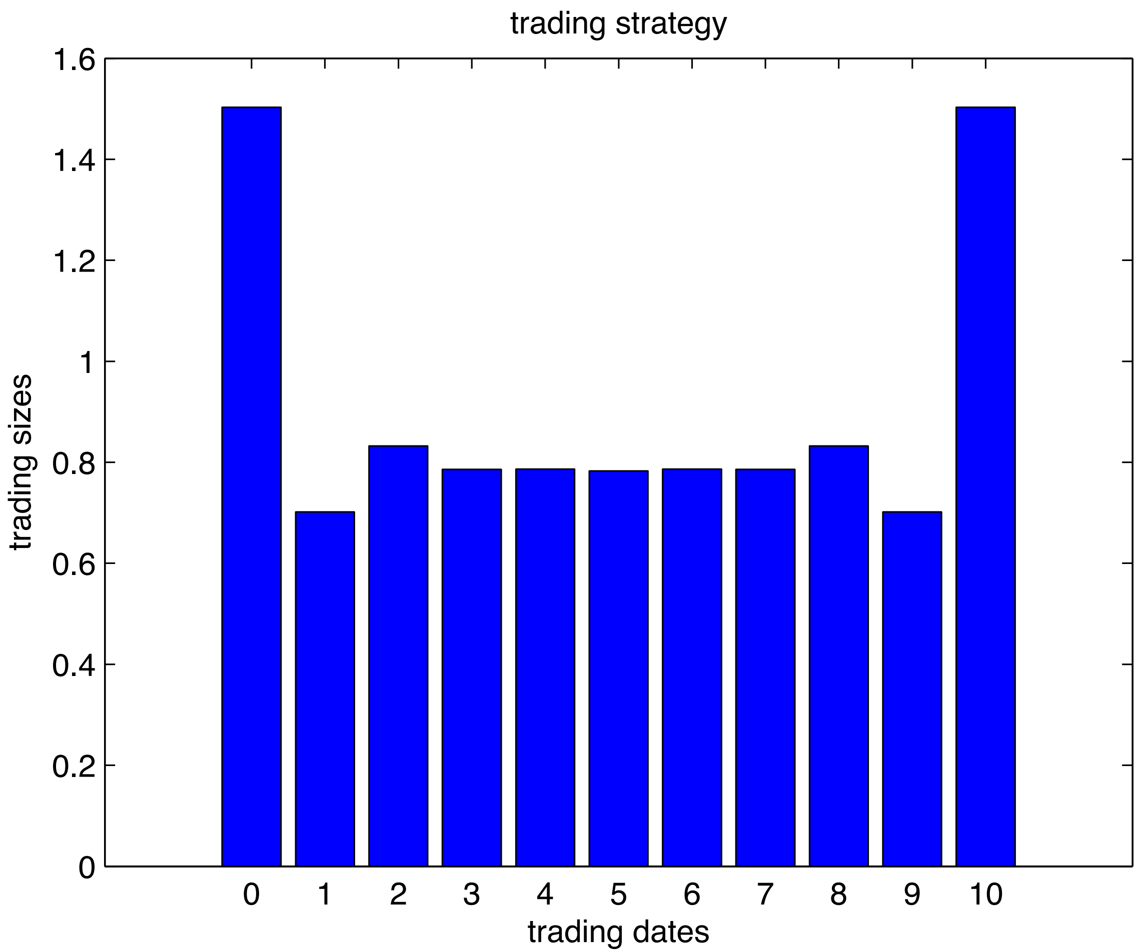
Let's look at:

Example 4 (Modified power-law decay). The decay kernel

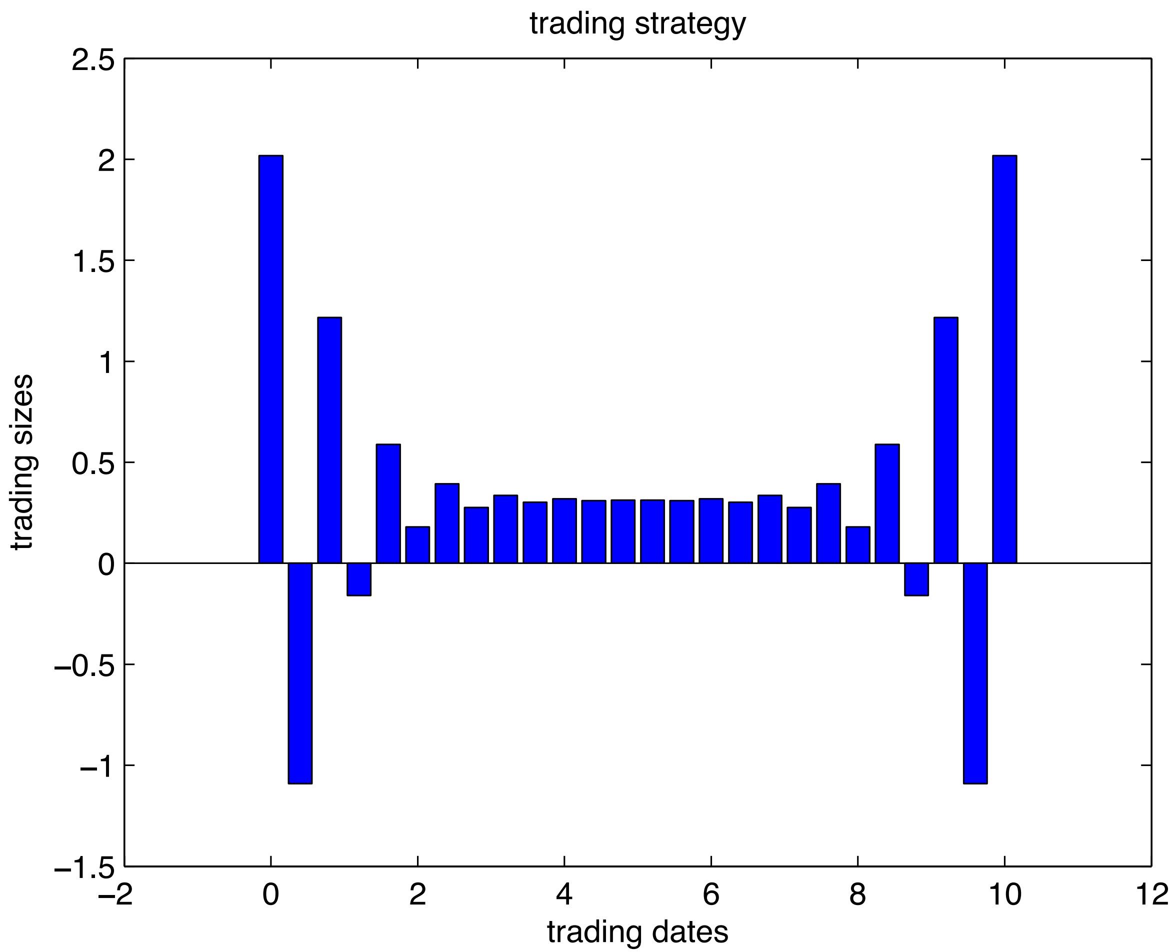
$$G(t) = \frac{1}{1+t^2}$$

is the Fourier transform of the function $\frac{1}{2}e^{-|x|}$. So it is strictly positive definite.

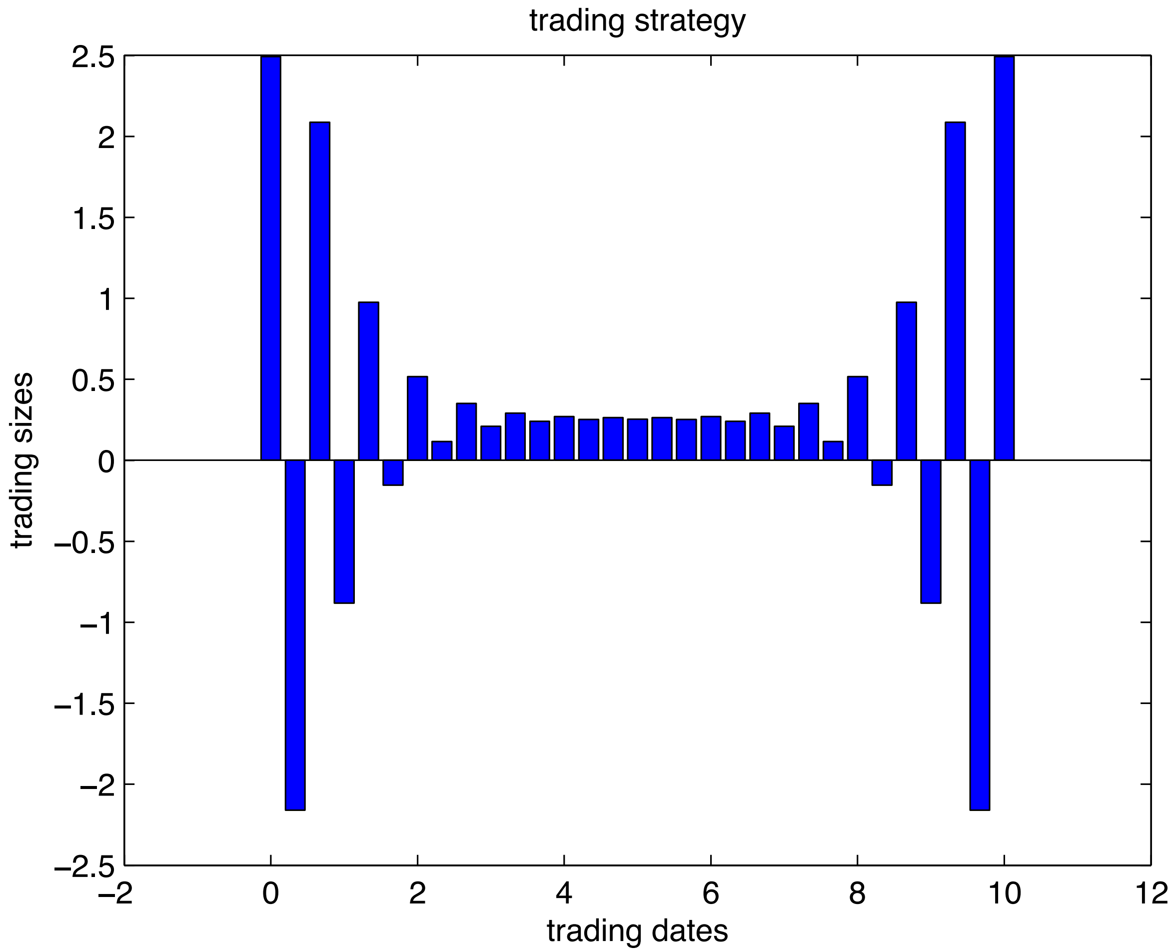
Modified power-law decay $G(t) = 1/(1+t^2)$, $N = 10$



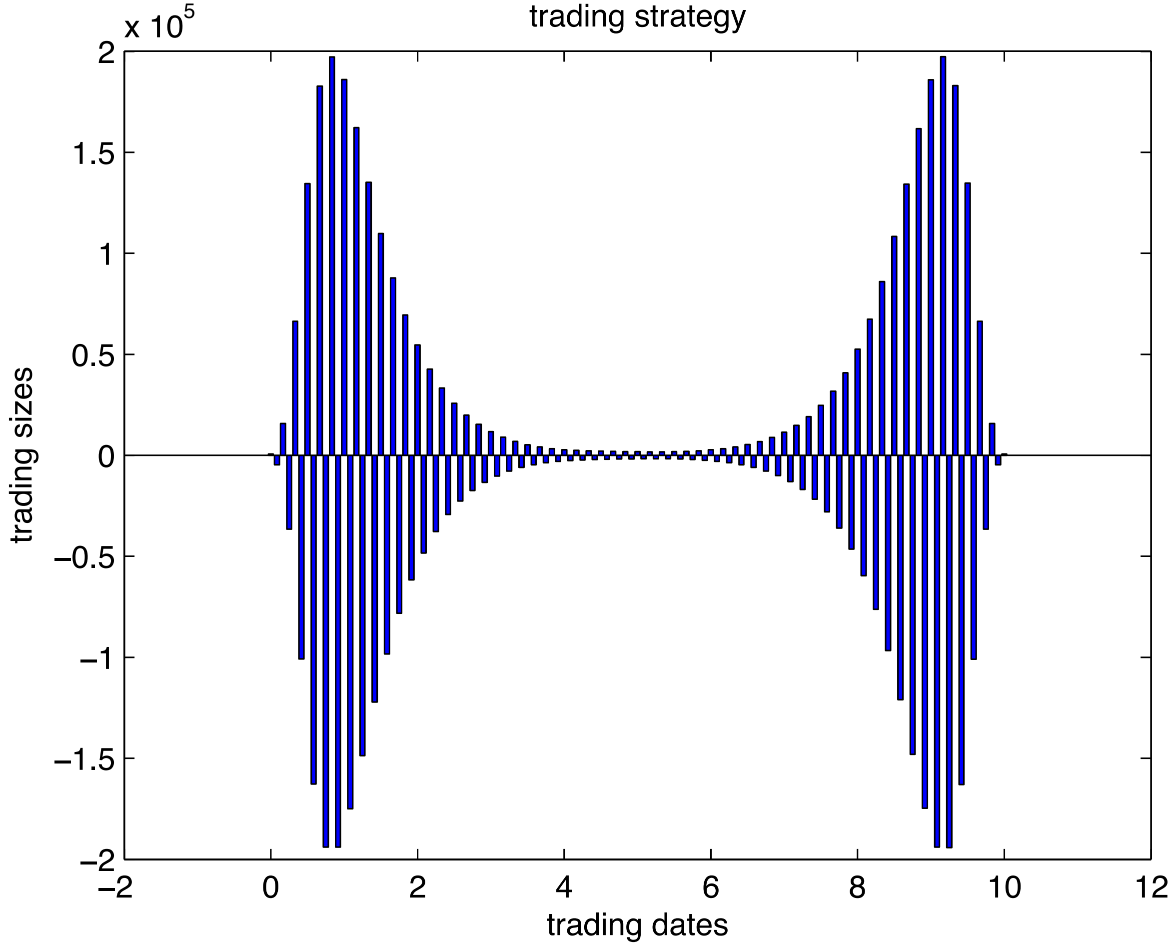
Modified power-law decay $G(t) = 1/(1+t^2)$, $N = 25$



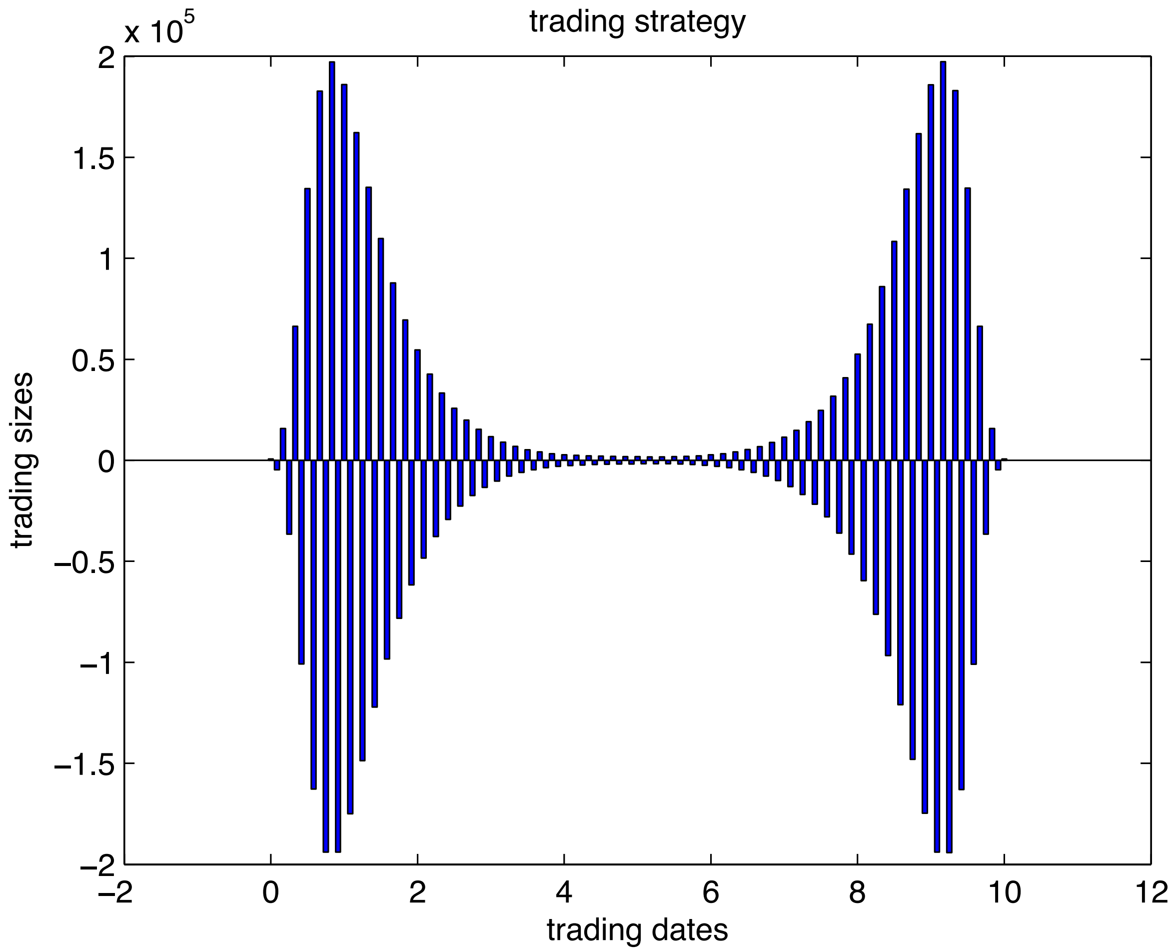
Modified power-law decay $G(t) = 1/(1 + t^2)$, $N = 30$



Modified power-law decay $G(t) = 1/(1+t^2)$, $N = 120$



Modified power-law decay $G(t) = 1/(1+t^2)$, $N = 120$



$\Rightarrow$ absence of price manipulation strategies is not enough

Definition [Hubermann & Stanzl (2004)]
A market impact model admits

price manipulation

if there is a round trip with negative expected liquidation costs.

Definition [Alfonsi, A.S., & Slynko (2009)]
A market impact model admits

transaction-triggered price manipulation

if the expected liquidation costs of a sell (buy) program can be decreased by intermediate buy (sell) trades.

Situation for non-discrete $\mathbb{T}$:

Theorem 1. *Suppose that $G(|\cdot|)$ is the Fourier transform of a finite Borel measure μ for which*

$$\int e^{\varepsilon x}\,\mu(dx) < \infty \qquad \textit{for some } \varepsilon > 0. \tag{3}$$

Suppose furthermore that the support of μ is not discrete. Then there are no optimal strategies in $\mathcal{X}(y,\mathbb{T})$ when $y \neq 0$ and $\mathbb{T}$ is not discrete.

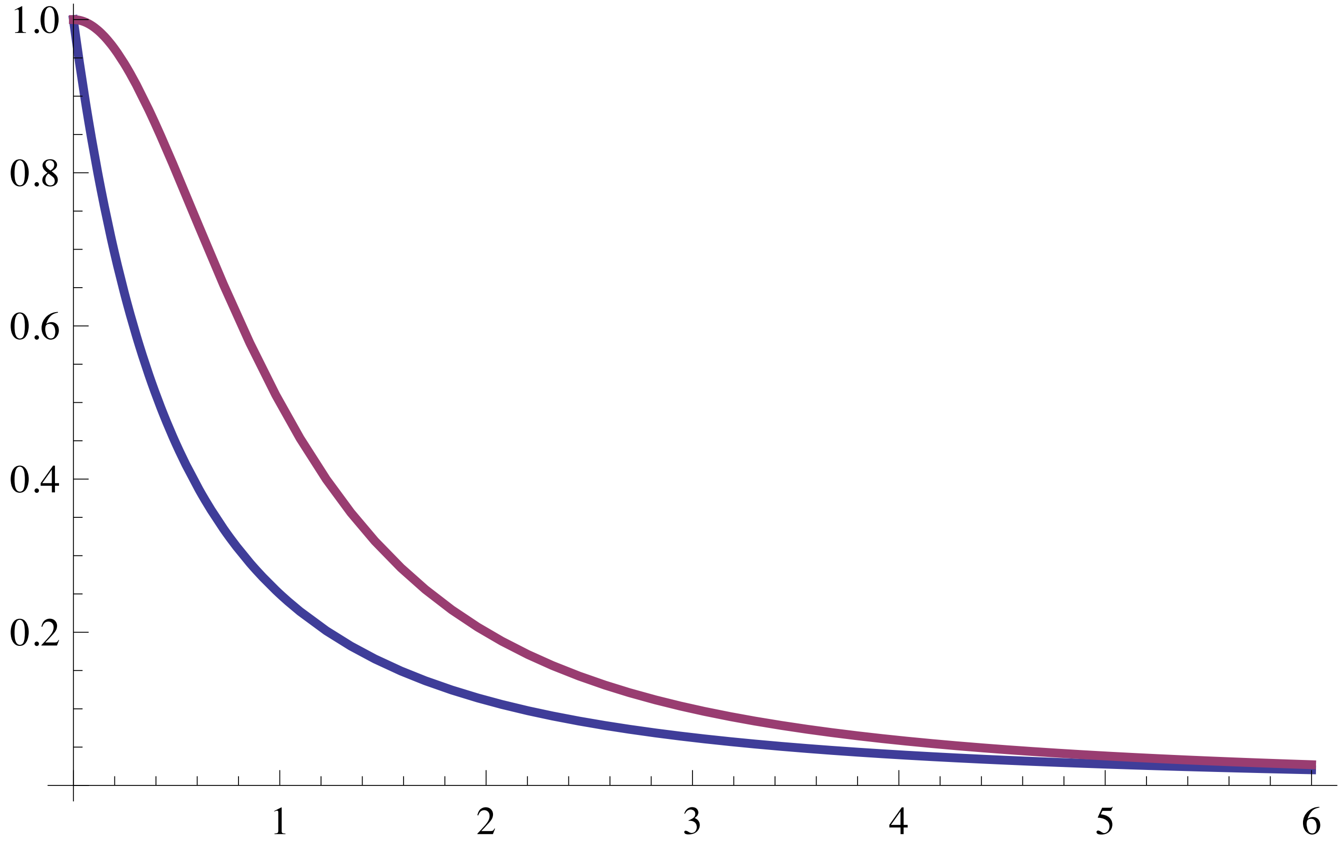
Examples:

$$G(t) = e^{-t^2} \qquad \text{or} \qquad G(t) := \frac{1}{1+t^2},$$

$$\text{or} \qquad G(t) = 2\frac{1-\cos t}{t^2} \qquad \text{or} \qquad G(t) = 1 + \frac{\sin t}{t},$$

Small alterations of G lead to dramatic change of model behavior

$\Longrightarrow$ **non-stability, non-robustness?**



The decay kernels $G(t) = \frac{1}{1+t^2}$ and $G(t) = \frac{1}{(1+t)^2}$

Can we characterize when optimal strategies exist and are monotone functions of time?

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For discrete $\mathbb{T} = \{t_0, \dots, t_N\}$: When does the minimizer $\boldsymbol{x}^*$ of

$$\sum_{i,j} x_i x_j G(|t_i - t_j|) \qquad \text{with} \quad \sum_i x_i = X_0$$

have only nonnegative components?

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Related to the **positive portfolio problem** in finance:

When are there no short sales in a Markowitz portfolio?

I.e. when is the solution of the following problem nonnegative

$$\boldsymbol{x}^\top M \boldsymbol{x} - \boldsymbol{m}^\top \boldsymbol{x} \to \min \qquad \text{for } \boldsymbol{x}^\top \mathbf{1} = X_0,$$

where M is a covariance matrix of assets and $\boldsymbol{m}$ is the returns vector?

Partial results, e.g., by Green (1986), Nielsen (1987)

Theorem 2. [Alfonsi, A.S., Slynko (2009)]

- *If G is* **convex** *then all components of $\boldsymbol{x}^*$ are nonnegative.*
- *If G is strictly convex, then all components are strictly positive.*

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- *If G is strictly convex, then all components are strictly positive.*

Proposition 3. *For positive definite G there is tranaction-triggered price manipulation if, e.g.,*

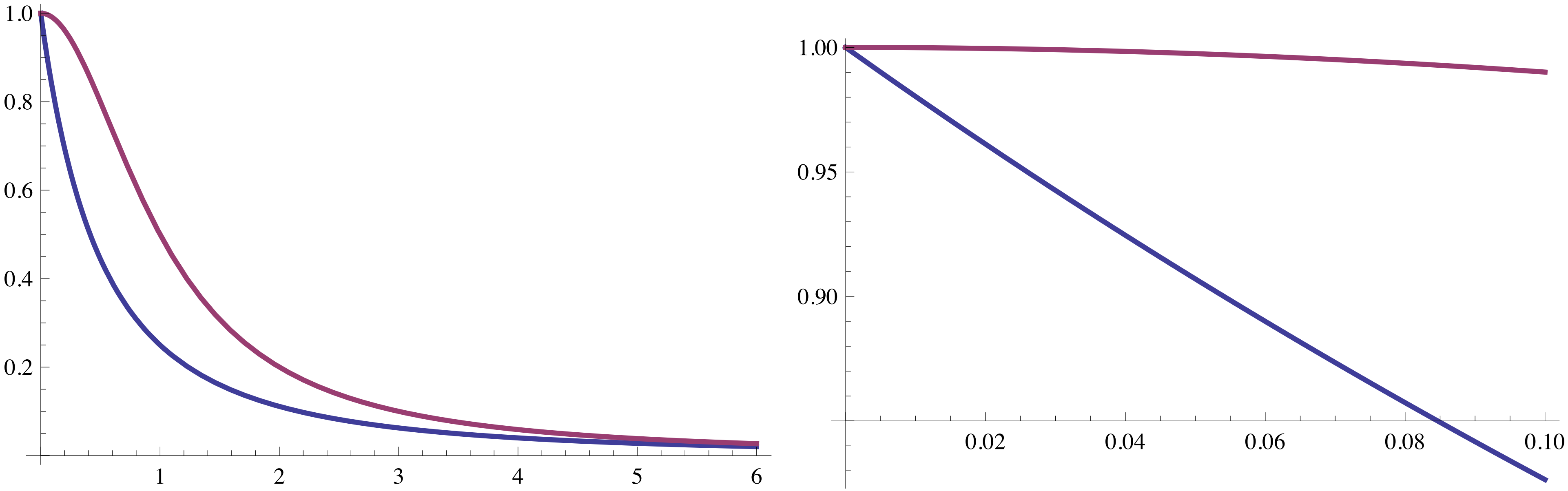
(a) *There are s, $t > 0$, $s \neq t$, such that*

$$G(0) - G(s) < G(t) - G(t + s).$$

(b) $G'(0+) = 0$

(c) $\int |z| \, \mu(dz) < \infty$

Economic intuition of convex decreasing decay: Markets react instantaneously, and hence efficiently (Fama), to price shocks



The decay kernels $G(t) = \frac{1}{1+t^2}$ and $G(t) = \frac{1}{(1+t)^2}$

We can thus completely solve the two problems of existence and monotonicity of strategies:

Theorem 3. *If G is nonconstant, nonincreasing, and convex, then there exists a unique optimal strategy X^* within each class $\mathcal{X}(y, \mathbb{T})$. Moreover, X_t^* is a monotone function of t.*

Works also if we relax the boundedness of G and assume

$$G \text{ is nonconstant, nonincreasing, convex, and } \int_0^1 G(t)\,dt < \infty.$$

E.g.,

$$G(t) = t^{-\gamma} \qquad \text{for } 0 < \gamma < 1, \text{ or}$$
$$G(t) = \log^-(t).$$

Let

$$\mathcal{X}_G(y, \mathbb{T}) := \Big\{ X \in \mathcal{X}(y, \mathbb{T}) \,\Big|\, \int\!\!\int G(|t-s|)\, d|X|_s\, d|X|_t < \infty \Big\}$$

Note: $\mathcal{X}_G(y, \mathbb{T})$ can be empty, e.g., for discrete $\mathbb{T}$.

Theorem 3. *When $\mathcal{X}_G(y, \mathbb{T}) \neq \emptyset$, there exists a unique optimal strategy X^* in $\mathcal{X}_G(y, \mathbb{T})$. Moreover, X^*_t is a monotone function of t.*

Application to Potential Theory:

Yields existence of capacitary potential for convex decreasing G without application of Cartan's theorem.
Minimizer μ^* of

$$\int\int G(|t-s|)\,\mu(ds)\,\mu(dt)$$

positive even when minimum taken over all signed measures of finite energy.

Approach seems to be limited to $d = 1$

Example: (Power-law decay kernel) $G(t) = t^{-\gamma}$ with $0 < \gamma < 1$

$$\int_0^1 \frac{u(s)}{|t-s|^\gamma}\,ds = 1 \qquad \text{for } 0 \le t \le 1,$$

is solved by

$$u^*(s) = \frac{c}{(s(1-s))^{\frac{1-\gamma}{2}}},$$

where c is a suitable constant. Thus, the unique optimal strategy in $\mathcal{X}_G(y,[0,1])$ is

$$X_t^* = y\left(1 - \frac{\Gamma(1+\gamma)}{\Gamma\left(\frac{1+\gamma}{2}\right)^2}\int_0^t \frac{1}{(s(1-s))^{\frac{1-\gamma}{2}}}\,ds\right).$$

Example: (Logarithmic decay kernel) $G(t) = \log^{-}(t)$

$$\int_0^1 u(s)G(|t-s|)\,ds = -\int_0^1 u(s)\log|t-s|\,ds = 1 \qquad \text{for } 0 \le t \le 1$$

solved by

$$u^*(s) = \frac{ds}{2\pi \log 2\sqrt{s(1-s)}}.$$

This fact was discovered by Carleman (1922). The unique optimal strategy in $\mathcal{X}_G(y,[0,1])$ is thus given by

$$X_t^* = y\Big(1 - \frac{1}{\pi}\int_0^t \frac{1}{\sqrt{s(1-s)}}\,ds\Big) = \frac{2x}{\pi}\arccos\sqrt{t}.$$

2. Nonlinear transient price impact

First possibility: Alfonsi, Fruth, A.S. (2010), Alfonsi, A.S. (2009)

Use nonlinear shape of limit order book

Simplified zero-spread model

Unaffected price process S^0, is martingale

Define the volume impact process with exponential resilience via

$$dE_t = dX_t - \rho E_t \, dt$$

Next, the number of shares offered at price $S_t^0 + x$ is given by $f(x)\,dx$
Thus, volume impact of E_t shares corresponds to a price impact of D_t, which is given implicitly via

$$F(D_t) := \int_0^{D_t} f(x)\,dx = E_t$$

Hence,

$$S_t = S_t^0 + D_t$$

Coincides with linear model if $f \equiv q$

Theorem 4 (Alfonsi and A.S. (2009)).
There is neither price manipulation nor transaction-triggered price manipulation for a large class of shape functions f.

Extension to more general Markovian resilience and larger class of continuous F possible via arguments by
Predoiu, Shaikhet, and Shreve (2010)

Second possibility: Gatheral (Quant. Finance, forthcoming)

X must be absolutely continuous. Then

$$S_t = S_t^0 + \int_0^t h(\dot{X}_s) G(t-s)\, ds$$

Theorem 5 (Gatheral).
There is price manipulation as soon as h is nonlinear and

$$G(t) = e^{-\rho t}$$

What is going on?

Generalization of Gatheral's theorem:

Theorem 6 (A.S. and Slynko (2010)). *There is price manipulation as soon as h is nonlinear and $G(t)$ is bounded and continuous on $(0, \infty)$.*

Thus, G must have a singularity such as

$$G(t) = t^{-\gamma} \qquad \text{for some } \gamma \in (0, 1)$$

Conjecture (Gatheral). If $G(t) = t^{-\gamma}$ and $h(x) = x^{\delta}$ there is no price manipulation if and only if $\gamma + \delta \geq 1$.

Comparison of the two models

In our model, $dE_t = dX_t - \rho E_t \, dt$ which is solved by

$$E_t = \int_{[0,t)} e^{-\rho(t-s)} \, dX_s$$

Generalization to arbitrary resilience:

$$E_t = \int_{[0,t)} \psi(t-s) \, dX_s$$

Hence,

$$S_t = S_t^0 + F^{-1}\Big(\int_{[0,t)} \psi(t-s) \, dX_s \Big)$$

In Gatheral's model,

$$S_t = S_t^0 + \int_0^t h(\dot{X}_s) G(t-s) \, ds$$

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Conclusion:

- Transient market impact can create new types of irregularities: price manipulation, transaction-triggered price manipulation
- This does not occur for convex decay of linear price impact
- Not clear how to best model nonlinear price impact.

Thank you