Numéraire-invariant choices in financial modeling

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Outline

Numéraire-invariant choices: the static case

Numéraire-invariant choices in a dynamic environment

Agent's optimal investment and consumption problem

The numéraire under random sampling

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Hors d'oeuvres in the static environment

Framework. For the "bulk consumption" case we work on ...

- a space of random outcomes $(\Omega, \mathcal{F})$, equipped with ...
- a class Π of equivalent probabilities on $(\Omega, \mathcal{F})$.
- Denote $\mathbb{L}^0 \equiv \mathbb{L}^0(\Pi)$, with usual topology.
- $\mathbb{L}^0_+ := \{ f \in \mathbb{L}^0 \mid f \geq 0 \}$; $\mathbb{L}^0_{++} := \{ f \in \mathbb{L}^0 \mid f > 0 \}$.

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Definitions. A set $\mathcal{C} \subseteq \mathbb{L}^0_+$ will be called:

- convex if $f \in \mathcal{C}$, $g \in \mathcal{C}$, $\alpha \in [0,1]$ imply $((1-\alpha)f + \alpha g) \in \mathcal{C}$.
- closed if it is closed in $\mathbb{L}^0$.
- bounded if $\lim_{\ell \to \infty} \sup_{f \in \mathcal{C}} \mathbb{P}[f > \ell] = 0$ (for any $\mathbb{P} \in \Pi$).

Preferences on $\mathbb{L}^0_+$ via expected relative rate of return

Preference on $\mathbb{L}^0_+$ via e.r.r.o.r: Fix $\mathbb{P} \in \Pi$. Set

$$f \preccurlyeq_{\mathbb{P}} g \iff \mathrm{rel}_{\mathbb{P}}(f \mid g) := \mathbb{E}_{\mathbb{P}}\left[(f-g)/g\right] \leq 0$$

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Observations:

- $\preccurlyeq_{\mathbb{P}}$ is a *numéraire-invariant* relation: for $h \in \mathbb{L}^0_{++}$,

$$f \preccurlyeq_{\mathbb{P}} g \iff \frac{f}{h} \preccurlyeq_{\mathbb{P}} \frac{g}{h}.$$

- $\preccurlyeq_{\mathbb{P}}$ is *neither* complete *nor* transitive (*nor* additive)!

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Connection with log-utility maximization. Let $\mathcal{C} \subseteq \mathbb{L}^0_+$ be convex and closed. Suppose that there exists $\widehat{f} \in \mathcal{C}$ such that

$$\mathbb{E}_{\mathbb{P}}\left[\log f\right] \le \mathbb{E}_{\mathbb{P}}\left[\log \widehat{f}\,\right] < \infty, \quad \forall f \in \mathcal{C}.$$

Formal first-order conditions give, for all $f \in \mathcal{C}$:

$$\mathbb{E}_{\mathbb{P}}\left[\left(f - \widehat{f}\right)/\,\widehat{f}\,\right] \le 0 \implies f \preccurlyeq_{\mathbb{P}} \widehat{f}.$$

Axiomatic foundation of numéraire-invariant preferences

Theorem. Suppose that $\preccurlyeq$ is a binary relation on $\mathbb{L}^0_+$ that satisfies the following *axioms*:

A1. $f \preccurlyeq g \iff (f/g) \preccurlyeq 1$.

A2. If $f \leq 1$, then $f \preccurlyeq 1$. If, furthermore, $f \neq 1$, then $f \prec 1$.

A3. The lower-contour set $\{f \in \mathbb{L}^0_+ \mid f \preccurlyeq 1\}$ is convex.

A4. If $\mathcal{C} \subseteq \mathbb{L}^0_+$ is convex, closed and bounded, $\exists\ \widehat{f} \in \mathcal{C}$ such that

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Remark. The above $\mathbb{P} \in \Pi$ is the subjective probability of a risk-averse individual with numéraire-invariant preferences:

- if $f \preccurlyeq_{\mathbb{P}} \mathbb{E}_{\mathbb{Q}}[f]$ for all $f \in \mathbb{L}^\infty_+$, then $\mathbb{Q} = \mathbb{P}$.

Extending the preference structure: expected log utility

Theorem. Let $\preccurlyeq$ satisfy the previous axioms A1, A2, A3, A4. Then, there exists a binary relation $\trianglelefteq$ on $\mathbb{L}^0_{++}$ such that:

1. $f \trianglelefteq g \iff (f/g) \trianglelefteq 1$.
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3. $\trianglelefteq$ is transitive.
4. $\trianglelefteq$ has weak continuity properties.

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Let $\trianglelefteq$ be *any* such binary relation on $\mathbb{L}^0_{++}$. With $\mathbb{P}$ generating $\preccurlyeq$,

$$f \trianglelefteq g \iff \mathbb{E}_{\mathbb{P}} [\log (f/g)] \leq 0$$

holds whenever $\mathbb{E}_{\mathbb{P}} \left[\log_+ (f/g)\right] < \infty$.

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Dynamic framework

Filtered probability space: $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$.

- $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$: flow of information.
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Quantities of interest: cumulative consumption streams...

- ...i.e., nondecreasing, right-continuous, adapted processes ...
- ...whose densities with respect to some "consumption clock" live on $(\Omega \times \mathbb{R}_+, \mathcal{O})$, where $\mathcal{O}$ is the *optional* sigma-algebra.
- Π: collection of equivalent measures with unit mass ("probabilities") on $(\Omega \times \mathbb{R}_+, \mathcal{O})$, generically denoted by p.

A canonical representation of unit-mass optional measures

Theorem. On $(\Omega \times \mathbb{R}_+, \mathcal{O})$, let p with $p[\Omega \times \mathbb{R}_+] = 1$ and $p[A] = 0$ for evanescent $A \in \mathcal{O}$. There exists (L, K) such that:

1. L is a nonnegative local martingale with $L_0 = 1$.
2. K is adapted, right-continuous, nondecreasing, $0 \leq K \leq 1$.
3. $\int_{\Omega \times \mathbb{R}_+} V \mathrm{d}p = \mathbb{E}\big[\int_{\mathbb{R}_+} V_t \, L_t \mathrm{d}K_t \big]$, for all V nonnegative.

Under an additional "minimality" condition, the pair (L, K) that satisfies the above requirements is essentially unique.

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Remark. It *may* happen that L is *not* a true martingale. (It *may* also happen that $\mathbb{P}[K_\infty = 1] < 1$.) Seemingly a technicality, this has deep economic consequences, related to market *bubbles*.

Numéraire-invariant choice on consumption streams

Preferences. For $p \in \Pi$ with canonical pair (L, K), define

$$\mathsf{rel}_p(F \,|\, G) \;:=\; \int_{\Omega \times \mathbb{R}_+} \left(\frac{\mathrm{d}F - \mathrm{d}G}{\mathrm{d}G} \right) \mathrm{d}p = \mathbb{E}\left[\int_{\mathbb{R}_+} \left(\frac{\mathrm{d}F_t - \mathrm{d}G_t}{\mathrm{d}G_t} \right) L_t \mathrm{d}K_t \right]$$

for all consumption streams F and G. Then, define

$$F \preccurlyeq_p G \iff \mathsf{rel}_p(F \,|\, G) \leq 0.$$

Such preferences stem from axiomatic foundations, . . .

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$$\mathsf{rel}_p(F \mid G) := \int_{\Omega \times \mathbb{R}_+} \left(\frac{\mathrm{d}F - \mathrm{d}G}{\mathrm{d}G} \right) \mathrm{d}p = \mathbb{E}\left[\int_{\mathbb{R}_+} \left(\frac{\mathrm{d}F_t - \mathrm{d}G_t}{\mathrm{d}G_t} \right) L_t \mathrm{d}K_t \right]$$

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Special case: If $L \equiv (\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P})|_{\mathcal{F}_\cdot}$, then

$$\mathsf{rel}_p(F \mid G) = \mathbb{E}_{\mathbb{Q}}\left[\int_{\mathbb{R}_+} \left(\frac{\mathrm{d}F_t - \mathrm{d}G_t}{\mathrm{d}G_t} \right) \mathrm{d}K_t \right].$$

- $\mathbb{Q}$: *subjective views* of the agent.
- K: agent's *consumption clock*.

Outline

The market

Discounted asset-prices. There are d liquid assets with dynamics:

$$\frac{\mathrm{d}S_t^i}{S_t^i} = \alpha_t^i \mathrm{d}t + \sum_{j=1}^{m} \sigma_t^{ij} \mathrm{d}W_t^j, \quad i = 1, \ldots, d.$$

Notation:

- $W = (W^j)_{j=1,\ldots,m}$ is standard BM, and $d \leq m$.
- $c := \sigma\sigma^\top$: local covariation $(d \times d)$-matrix-valued process.
- $\alpha = (\alpha^i)_{i=1,\ldots,d}$: local excess rates of return.

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Market viability. Assume that there exists a process ρ that solves $c\rho = \alpha$, such that the integrated-squared-Sharpe-ratio process

$$\int_0^\cdot \left(\rho_t^\top c_t \rho_t\right) \mathrm{d}t = \int_0^\cdot \left| c_t^{-1/2} \alpha_t \right|^2 \mathrm{d}t$$

$\mathbb{P}$-a.s. does *not* explode in finite time.

Investment and consumption

Investment-consumption: with initial capital $x \in \mathbb{R}_+$, the pair (π, κ) generates wealth $X^{(x;\pi,\kappa)}$ satisfying $X_0^{(x;\pi,\kappa)} = x$ and

$$\frac{\mathrm{d}X_t^{(x;\pi,\kappa)}}{X_t^{(x;\pi,\kappa)}} = \sum_{i=1}^{d} \pi_t^i \left(\frac{\mathrm{d}S_t^i}{S_t^i} \right) - \kappa_t \mathrm{d}t$$

- Solving the last linear SDE, $X^{(x;\pi,\kappa)}$ is given by:

$$x \exp \left(\int_0^{\cdot} \left(\pi_t^\top \alpha_t - \frac{1}{2} \pi_t^\top c_t \pi_t - \kappa_t \right) \mathrm{d}t + \int_0^{\cdot} (\pi_t^\top \sigma_t) \mathrm{d}W_t \right)$$

- Consumption rate at $t \in \mathbb{R}_+$ is $X_t^{(x;\pi,\kappa)} \kappa_t$. Therefore, consumption streams financeable by $x \in \mathbb{R}_+$ are of the form:

$$F^{(x;\pi,\kappa)} := \int_0^{\cdot} X_t^{(x;\pi,\kappa)} \kappa_t \mathrm{d}t.$$

Agent's optimal investment and consumption

Agent has preferences with canonical representation (L, K), where

$$\frac{\mathrm{d}L_t}{L_t} = \lambda_t^\top \mathrm{d}W_t + \mathrm{d}\big(\text{local mart} \perp \text{to } W\big)_t,$$
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Separation of the problems of investment and consumption:

- The optimal portfolio π^* solves

$$c\pi^* = \alpha + \sigma\lambda.$$

- The optimal relative-to-wealth consumption rate κ^* satisfies:

$$\kappa_t^* \mathrm{d}t = \frac{\mathrm{d}K_t}{1 - K_t}, \ \ t \in \mathbb{R}_+ \iff \kappa^* = \frac{\dot{K}}{1 - K}$$

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The problem

Definition. Let $\mathcal{X}$ be a wealth-process set and T be a *random time*. Then, $\widehat{X}$ is the numéraire in $\mathcal{X}$ sampled at T if $\widehat{X}_0 = 1$ and

$$\mathbb{E}\left[\frac{X_T}{\widehat{X}_T}\right] \leq \frac{X_0}{\widehat{X}_0} = X_0, \text{ for all } X \in \mathcal{X}. \qquad \text{(NUM)}$$

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Observations and question:

- The numéraire is essentially the log-optimal portfolio.
- When T ranges in the class of *stopping* times, $\widehat{X}$ that satisfies (NUM) is always the same, simply called the numéraire in $\mathcal{X}$;
- ... but what if T is *not* a stopping time? How can we characterize $\widehat{X}$ that satisfies (NUM)?

The (abstract) solution

The numéraire sampled at a random time T: Define p via

$$\mathbb{E}[V_T] = \int_{\Omega \times \mathbb{R}_+} V \mathrm{d}p = \mathbb{E}_{\mathbb{Q}}\left[\int_{\mathbb{R}_+} V_t \mathrm{d}K_t\right], \quad \text{for } V \geq 0,$$

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- With $\widehat{X}^{\mathbb{Q}}$ being the numéraire under $\mathbb{Q}$,

$$\mathbb{E}\left[\frac{X_T}{\widehat{X}^{\mathbb{Q}}_T}\right] = \mathbb{E}_{\mathbb{Q}}\left[\int_{\mathbb{R}_+} \frac{X_t}{\widehat{X}^{\mathbb{Q}}_t}\mathrm{d}K_t\right] = \ldots \leq \frac{X_0}{\widehat{X}^{\mathbb{Q}}_0} = X_0,$$

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<u>Solution</u>: $\widehat{X}^{\mathbb{Q}}$ is the numéraire in $\mathcal{X}$ sampled at T.

The numéraire as an index of market status

Theorem. Consider a viable market with continuous asset prices. Suppose that the numéraire $\widehat{X}$ (under $\mathbb{P}$) is such that $\lim_{t \to \infty} \widehat{X}_t = \infty$. Let T be any random time such that

$$\widehat{X}_T = \min_{t \in \mathbb{R}_+} \widehat{X}_t.$$

Then,

$$\mathbb{E}_{\mathbb{P}} \left[X_T \mid \widehat{X}_T \right] \leq X_0$$

holds for all $X \in \mathcal{X}$.

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Thank You!

- *Numéraire-invariant preferences in financial modeling*, to appear in the *Annals of Applied Probability*. Preprint at *www.arxiv.org*.