Robustness of option prices and their deltas in markets modelled by jump-diffusions

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A joint work with
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- In this presentation, we consider an SDE driven by a Lévy diffusion then we approximate the small jump part by a Brownian motion appropriately scaled.
- We aim to compute the delta of options written in such an asset.
- We also aim to study the convergence of the Δ^{ε} to Δ when ε goes to 0.

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Plan

1. Malliavin derivative for Lévy diffusions
2. Approximation of a Lévy diffusion
3. Computation of the delta and robustness

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- Let L be a Lévy process in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the filtration $\mathcal{F}_t$.
- The process L admits a Lévy-Ito representation

$$L(t) = at + bW(t) + Z(t) + \lim_{\varepsilon \downarrow 0} \widetilde{Z}_\varepsilon(t),$$

where

$$Z(t) = \sum_{s \in [0,t]} \triangle L(s) \mathbf{1}_{\{|\triangle L(s)| \geq 1\}},$$

$$\widetilde{Z}(t) = \sum_{s \in [0,t]} \triangle L(s) \mathbf{1}_{\{\varepsilon \leq |\triangle L(s)| < 1\}} - t \int_{\varepsilon \leq |z| < 1} z Q(dz),$$

and Q is the Lévy measure of L.

- Let $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$ be the canonical Brownian space. In Nualart (1995), a Malliavin derivative D^W of functionals of Brownian motion is defined in a subspace of $L^2(\Omega_W)$. We will denote by $Dom\ D^W$ its domain.
- In a more general setting, Di Nunno (2005), defined in a subspace of $L^2(\Omega)$ a stochastic derivative $D_{t,0}$ of functionals of Lévy processes. The idea is to exploit chaos expansion representations in the same manner as done for the Malliavin derivative in the Wiener space. See also Solé et al. (2006).

- The derivative $D_{t,0}$ is essentially, a derivative with respect to the Brownian part of L. In particular, if there is no jump part, that is
$$L(t) = at + bW(t),$$
then we have
$$D_{t,0}L = D_t^W L = b.$$
- In many cases the usual rules of classical Malliavin calculus are applied.
- Let $(\Omega_J, \mathcal{F}_J, \mathbb{P}_J)$ be the canonical pure jump Lévy space, i.e. $\mathcal{F}_J = \sigma\{ \int_0^s \int_A \widetilde{N}(du, dz); \quad s \leq t, \quad A \in \mathcal{B}(R_0)\}$. Consider $\Omega = \Omega_W \times \Omega_J$. The following results are due to Solé et al. (2006).

Chain rule

Proposition

Let $F = f(Z, Z') \in L^2(\Omega_W \times \Omega_J)$ with $Z \in \text{Dom } D^W$ and $Z' \in L^2(\Omega_J)$, and $f(x, y)$ is a continuously differentiable function with bounded partial derivatives in the variable x. Then F is $D_{t,0}$ Malliavin differentiable and

$$D_{t,0}F = \frac{\partial f}{\partial x}(Z, Z') D_t^W Z.$$

Duality formula

Lemma

A process $f \in L^2([0,T] \times \Omega, dt \times \mathbb{P})$ belongs to Dom δ if and only if there is a constant C such that for all $F \in$ Dom D,

$$\left|\mathbb{E}[\int_0^T f(z) D_z F dt]\right| \leq C(\mathbb{E}[F^2])^{\frac{1}{2}}.$$

If $f \in$ Dom δ, then

$$\mathbb{E}[\delta(f)F] = \mathbb{E}\Big[\int_0^T f(t) D_z F dt\Big],$$

for any $D_{t,0}$ Malliavin differentiable F.

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- We consider the following stochastic differential equation

$$\begin{aligned} X(t) &= x + \int_0^t \alpha(X(s-))\,ds + \int_0^t \beta(X(s-))\,dW(s) \\ &+ \int_0^t \int_{\mathbb{R}_0} \gamma_1(X(s-))g(z)\,\widetilde{N}(ds,dz)\,. \end{aligned}$$

Here the factor $g(z)$ satisfies

$$G^2(\infty) = \int_{\mathbb{R}_0} g^2(z)\,\ell(dz) < \infty,$$

which will ensure that $X(t)$ has finite variance.

- Define
$$G^2(\varepsilon) = \int_{|z|<\varepsilon} g^2(z)\,\ell(dz).$$
- Introduce the following approximative jump-diffusion dynamics
$$\begin{aligned} X_\varepsilon(t) = x &+ \int_0^t \alpha(X_\varepsilon(s-))\,ds + \int_0^t \beta(X_\varepsilon(s-))\,dW(s) \\ &+ \int_0^t G(\varepsilon)\gamma_1(X_\varepsilon(s-))dB(s) \\ &+ \int_0^t \int_{|z|\geq\varepsilon} \gamma_1(X_\varepsilon(s-))g(z)\,\widetilde{N}(ds,dz)\,. \end{aligned}$$

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Proposition

For every $0 \leq t \leq T < \infty$, we have

$$\|X(t) - X_\varepsilon(t)\|_2^2 \leq CG^2(\varepsilon),$$

where C is a positive constant depending on T, but independent of ε.

Corollary

Suppose f is a Lipschitz continuous function. Then, for every $0 \leq t \leq T < \infty$, there exists a positive constant C depending on T but independent of ε such that

$$|\mathbb{E}[f(X_\varepsilon(t))] - \mathbb{E}[f(X(t))]| \leq CG^2(\varepsilon).$$

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Separability assymption

- Assume that X is of the form

$$X(t) = h(X^c(t), X^d(t)), \quad X(0) = x\,,$$

where X^c satisfies a stochastic differential equation

$$\begin{aligned} dX^c(t) &= \alpha_c(X^c(t))dt + \beta_c(X^c(t))dW(t), \\ X^c(0) &= x = h(X^c(0), X^d(0)) \end{aligned}$$

and X^d is adapted to the natural filtration $\mathcal{F}_J$.

- The jump-diffusion process X in that case is called *separable*.

- We associate to the process X^c, a process V given by

$$V(t) = 1 + \int_0^t \alpha_c'(X^c(s))V(s)ds + \int_0^t \beta_c'(X^c(s))V(s)dW(s),$$

- The process V is called *the first variation process* for X^c and we have

$$V(t) = \frac{\partial X^c(t)}{\partial x}.$$

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Example

Consider a jump-diffusion of the form

$$\begin{aligned} dX(t) &= \alpha X(t-)dt + \beta X(t-)dW(t) \\ &+ \int_{\mathbb{R}_0} (e^z - 1)X(t-)\widetilde{N}(dt,dz), \end{aligned}$$

We introduce the process $X^c(t)$ defined by

$$\begin{aligned} dX^c(t) &= \Big\{\alpha + \int_{\mathbb{R}_0} (1 + z - e^z)\ell(dz)\Big\} X^c(t)dt + \beta X^c(t)dW(t), \\ X^c(0) &= X(0) = x. \end{aligned}$$

Then by applying the Itô formula to $\widehat{X}(t) = e^{\widetilde{Z}(t)} X^c(t)$, where $d\widetilde{Z}(t) = \int_{\mathbb{R}_0} z\widetilde{N}(dt,dz)$ we get $\widehat{X}(t) = X(t)$, a.e.

Computation of the delta

Theorem

Assume that X_ε is separable. Define

$$\Gamma = \left\{ a \in L^2[0,T] | \int_0^T a(t)dt = 1 \right\}.$$

Let $a \in \Gamma$, V_ε the first variation process of X_ε^c and $f(X_\varepsilon(T)) \in L^2(\Omega)$. Then

$$\Delta_\varepsilon = \mathbb{E}\left[f(X_\varepsilon(T)) \int_0^T a(t)\beta_c^{-1}(X_\varepsilon^c(t))V_\varepsilon(t)dW(t) \right], \quad (1)$$

$$\Delta_\varepsilon = \mathbb{E}\left[f(X_\varepsilon(T)) \int_0^T a(t)\gamma_{1,c}^{-1}(X_\varepsilon^c(t))\frac{V_\varepsilon(t)}{G(\varepsilon)}dB(t) \right]. \quad (2)$$

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Proof

Assume that $f \in C_K^\infty(\mathbb{R})$. Then

$$\frac{\partial}{\partial x}\mathbb{E}\Big[f(X_\varepsilon(T))\Big] = \mathbb{E}\Big[f'(X_\varepsilon(T))\frac{\partial X_\varepsilon(T)}{\partial x}\Big] = \mathbb{E}\Big[f'(X_\varepsilon(T))\frac{\partial X_\varepsilon(T)}{\partial X_\varepsilon^c(T)}V_\varepsilon(T)\Big], \quad (3)$$

where V_ε is the first variation process for X_ε^c.

By the chain rule, we have

$$D_{t,0}X_\varepsilon(T) = \frac{\partial X_\varepsilon(T)}{\partial X_\varepsilon^c(T)} D_t^W X_\varepsilon^c(T) = \frac{\partial X_\varepsilon(T)}{\partial X_\varepsilon^c(T)} V_\varepsilon(T)(V_\varepsilon(t))^{-1}\beta_c(X_\varepsilon^c(t)).$$

Therefore,

$$\frac{\partial X_\varepsilon(T)}{\partial X_\varepsilon^c(T)} V_\varepsilon(T) = D_{t,0}X_\varepsilon(T) V_\varepsilon(t)\beta_c^{-1}(X_\varepsilon^c(t)).$$

Multiply by $a(t)$ and integrate,

$$\frac{\partial X_\varepsilon(T)}{\partial X_\varepsilon^c(T)} V_\varepsilon(T) = \int_0^T D_{t,0}X_\varepsilon(T) a(t)\beta_c^{-1}(X_\varepsilon^c(t)) V_\varepsilon(t) dt. \qquad (4)$$

Inserting (4) in (3), the chain rule and the Duality formula yield

$$
\begin{aligned}
&\frac{\partial}{\partial x}\mathbb{E}\Big[f(X_\varepsilon(T))\Big] \\
&= \mathbb{E}\Big[\int_0^T f'(X_\varepsilon(T))D_{t,0}X_\varepsilon(T)a(t)\beta_c^{-1}(X_\varepsilon^c(t))V_\varepsilon(t)dt\Big] \\
&= \mathbb{E}\Big[\int_0^T D_{t,0}f(X_\varepsilon(T))a(t)\beta_c^{-1}(X_\varepsilon^c(t))V_\varepsilon(t)dt\Big] \\
&= \mathbb{E}\Big[f(X_\varepsilon(T))\int_0^T a(t)\beta_c^{-1}(X_\varepsilon^c(t))V_\varepsilon(t)dW(t)\Big].
\end{aligned}
$$

Then we can extend this formula to $f(X_\varepsilon(T)) \in L^2(\Omega)$.

Remark

- *We assume* $\beta(x) = \gamma_1(x)$. *Then*

$$\Delta_\varepsilon = \mathbb{E}\Big[f(X_\varepsilon(T)) \int_0^T a(t)\gamma_{1,c}^{-1}(X_\varepsilon^c(t)) \frac{V_\varepsilon(t)}{\sqrt{G^2(\varepsilon)+1}} d\widetilde{W}_\varepsilon(t)\Big],$$

where $\widetilde{W}_\varepsilon(t) = \frac{1}{\sqrt{G^2(\varepsilon)+1}} W(t) + \frac{G(\varepsilon)}{\sqrt{G^2(\varepsilon)+1}} B(t)$.

- *If we approximate the small jumps of* $X(t)$ *by* $X_\varepsilon(t)$, *where* $B(t) = W(t)$, *then*

$$\begin{aligned}\Delta_\varepsilon &= \mathbb{E}\Big[f(X_\varepsilon(T)) \int_0^T a(t)\{G(\varepsilon)\gamma_{1,c}(X_\varepsilon^c(t)) \\ &+ \beta_c(X_\varepsilon^c(t))\}^{-1} V_\varepsilon(t) dW(t)\Big].\end{aligned}$$

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Robustness of the delta

Proposition

Denote by $\widehat{f}(u)$ *the Fourier transform of* f*. Let* $u\widehat{f}(u) \in L^1(\mathbb{R})$*. For* $0 \le t \le T$*, it holds that*

$$\lim_{\varepsilon \downarrow 0} \frac{\partial}{\partial x} \mathbb{E}[f(X_\varepsilon(t))] = \frac{\partial}{\partial x} \mathbb{E}\left[f(X(t))\right] .$$

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- These results are an extension of a paper by Benth, Di Nunno, and Khedher (2009), in which we considered Lévy processes and we studied the delta using the conditional density method.

Benth, F. E., Di Nunno, G., and Khedher, A.
Lévy-models robustness and sensitivity
To appear in Volume XXV of the series QP-PQ, Quantum Probability and White Noise Analysis, H. Ouerdiane and A. Barhoumi (eds.), World Scientific.

Di Nunno, G
On Orthogonal Polynomials and the Malliavin derivative for Lévy stochastic measures.
Seminaires et Congrès SMF (2008) 16,55-70.
Colloque Analyse et Probabilités, Hammamet 2003.

Sol, J. L., Utzet, F., and Vives, J.
Canonical Lévy process and Malliavin calculus.
Stochastic processes and their Applications, 117(2006), 165-187.