On refined volatility smile expansion in the Heston model

Stefan Gerhold
(joint work with P. Friz, A. Gulisashvili, and S. Sturm)

Vienna University of Technology, Austria

6th World Congress of the Bachelier Finance Society

Heston Model

- Dynamics

$$dS_t = S_t \sqrt{V_t} dW_t, \qquad S_0 = 1,$$
$$dV_t = (a + bV_t)\,dt + c\sqrt{V_t} dZ_t, \qquad V_0 = v_0 > 0,$$

- Correlated Brownian motions

$$d\langle W, Z \rangle_t = \rho dt, \qquad \rho \in [-1, 1]$$

- Parameters

$$a \geq 0, b \leq 0, c > 0$$

Density and smile asymptotics

- Consider a fixed maturity $T > 0$.
- $D_T :=$ density of S_T.
- How heavy are the tails?

$$D_T(x) \sim \,? \qquad (x \to 0, \infty)$$

- Implied Black-Scholes volatility ($k = \log K$ is the log-strike)

$$\sigma_{BS}^2(k, T) \sim \,? \qquad (k \to \pm\infty)$$

Known results

- Leading term of smile asymptotics: Lee's moment formula. Andersen, Piterbarg (2007); Benaim, Friz (2008)
- Drăgulescu, Yakovenko (2002): Stationary variance regime. Leading growth order of distribution function of S_T, by (non-rigorous) saddle-point argument
- Gulisashvili-Stein (2009): Precise density asymptotics for uncorrelated Heston model

Main results (right tail), SG et al. 2010

- Density asymptotics for $x \to \infty$

$$D_T(x) = A_1 x^{-A_3} e^{A_2\sqrt{\log x}} (\log x)^{-3/4+a/c^2} \left(1+O((\log x)^{-1/2})\right)$$

- Implied volatility for $k = \log K \to \infty$

$$\sigma_{BS}(k, T)\sqrt{T} = \beta_1 k^{1/2} + \beta_2 + \beta_3 \frac{\log k}{k^{1/2}} + O\left(\frac{\varphi(k)}{k^{1/2}}\right)$$

(φ arbitrary function tending to ∞)

Interpretation of smile expansion

- Implied volatility for $k = \log K \to \infty$

$$\sigma_{BS}(k,T)\sqrt{T} = \beta_1 k^{1/2} + \beta_2 + \beta_3 \frac{\log k}{k^{1/2}} + O\left(\frac{\varphi(k)}{k^{1/2}}\right)$$

- β_1 does not depend on $\sqrt{v_0}$
- β_2 depends linearly on $\sqrt{v_0}$
- Changes of $\sqrt{v_0}$ have second-order effects
- Increase $\sqrt{v_0}$: parallel shift, slope not affected
- Changes in mean-reversion level $\bar{v} = -a/b$ seen only in β_3

General remarks

- Constants depend on: critical moment, critical slope, critical curvature
- Critical moment etc. defined in a model-free manner
- Closed form of Fourier (Mellin) transform not needed
- Work only with affine principles (Riccati equations)

Lee's moment formula (2004)

- Model-free result
- Relates critical moment to implied volatility

$$s^* := \sup\{s : E[S_T^s] < \infty\}$$
$$s^* =: \frac{1}{2\beta_1^2} + \frac{\beta_1^2}{8} + \frac{1}{2}$$
$$\limsup_{k\to\infty} \frac{\sigma_{BS}(k,T)\sqrt{T}}{\sqrt{k}} = \beta_1$$

- Refinements by Benaim, Friz (2008), Gulisashvili (2009)

Heston Model: Mgf of log-spot X_t

- Moment generating function

$$E[e^{sX_t}] = \exp(\phi(s,t) + v_0\psi(s,t))$$

- Riccati equations

$$\partial_t\phi = F(s,\psi),\ \ \phi(0) = 0,$$
$$\partial_t\psi = R(s,\psi),\ \ \psi(0) = 0$$

$$F(s,v) = av,$$
$$R(s,v) = \frac{1}{2}(s^2 - s) + \frac{1}{2}c^2v^2 + bv + s\rho cv$$

- Explicit solution possible, but cumbersome expression

Moment explosion

- Critical moment for time T

$$s^* := \sup\{s \geq 1 : E[S_T^s] < \infty\}$$

- Explosion time for moment of order s

$$T^*(s) = \sup\{t \geq 0 : E[S_t^s] < \infty\}$$

- Critical slope, critical curvature:

$$\sigma := -\partial_s T^*|_{s^*} \geq 0 \qquad \text{and} \qquad \kappa := \partial_s^2 T^*|_{s^*}$$

Explicit Explosion time for the Heston model

- Explosion time for moment of order s

$$T^*(s) = \frac{2}{\sqrt{-\Delta(s)}} \left(\arctan \frac{\sqrt{-\Delta(s)}}{s\rho c + b} + \pi \right),$$

$$\Delta(s) := (s\rho c + b)^2 - c^2 \left(s^2 - s\right)$$

- Critical moment s^*: Find numerically from

$$T^*(s^*) = T.$$

Mellin (Fourier) inversion

- Mellin transform of spot: $M(u) = E[e^{(u-1)X_T}]$
- Analytic in a complex strip
- Density of S_T by Mellin inversion:

$$D_T(x) = \frac{1}{2i\pi} \int_{-i\infty}^{+i\infty} x^{-u} M(u) du.$$

- Valid for contour in analyticity strip of the Mellin transform
- Justification: exponential decay of $M(u)$ at $\pm i\infty$.

Analyticity and growth

- Mellin transform analytic in a strip

$$u_- < \Re(u) < u^* = s^* + 1$$

- Leading order of density for $x \to \infty$

$$x^{-u^*-\varepsilon} \ll D_T(x) \ll x^{-u^*+\varepsilon},$$

  depends on *location* of singularity
- Refinement: lower order factors depend on *type* of singularity

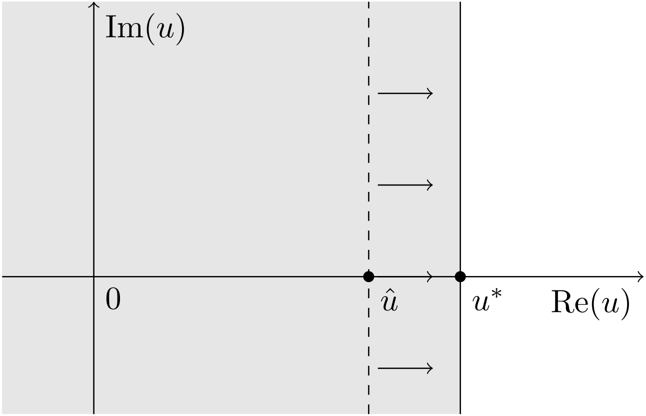
Saddle point method

- Recall:

$$D_T(x) = \frac{1}{2i\pi} \int_{-i\infty}^{+i\infty} x^{-u} M(u) du$$

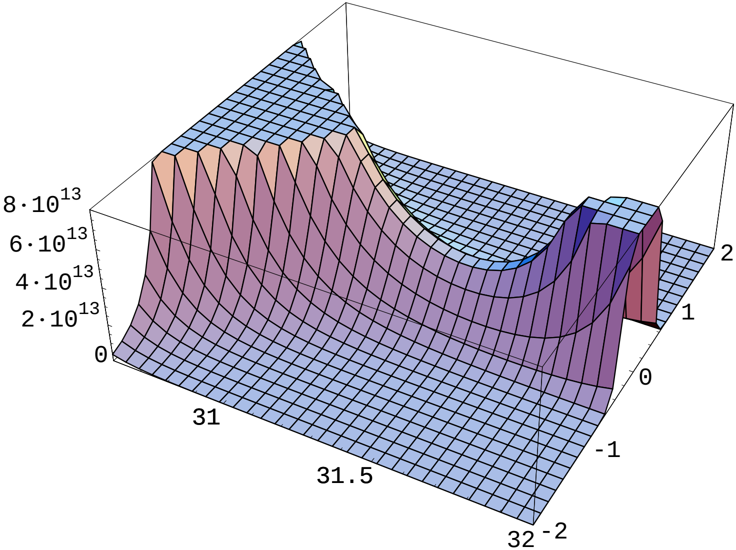
- Shift contour to the right, close to the singularity.
- Let it pass through a saddle point of the integrand.
- For large x, the integral is concentrated around the saddle.
- Local expansion of integrand yields expansion of whole integral.
- (Laplace, Riemann, Debye...)

New integration contour



- Contour runs through saddle point $\hat{u} = \hat{u}(x)$
- Moves to the right as $x \to \infty$

The surface $|x^{-u}M(u)|$



Asymptotics of ψ and ϕ near critical moment

- Recall $M(u) = \exp(\phi(u-1,t) + v_0\psi(u-1,t))$
- For $u \to u^*$ we have (with $\beta := \sqrt{2v_0}/c\sqrt{\sigma}$)

$$\psi(u-1,T) = \frac{\beta^2}{u^*-u} + const + O(u^*-u),$$
$$\phi(u-1,T) = \frac{2a}{c^2}\log\frac{1}{u^*-u} + const + O(u^*-u)$$

- Found from Riccati equations

Saddle point method

- Finding the saddle point: $0 =$ derivative of integrand
- Use only first order expansion:

$$0 = \frac{\partial}{\partial u} x^{-u} \exp\left(\frac{\beta^2}{u^* - u}\right)$$

- Approximate saddle point at

$$\hat{u}(x) = u^* - \beta / \sqrt{\log x}$$

New integration contour

- Contour depends on x:

$$u = \hat{u}(x) + iy, \qquad -\infty < y < \infty$$

- Divide contour into three parts:

$$|y| < (\log x)^{-\alpha} \quad \text{(central part)},$$

upper tail, lower tail (symmetric)

- Uniform local expansion at saddle point $\Rightarrow$ need large α
- Tails negligible $\Rightarrow$ need small α
- Can take $\frac{2}{3} < \alpha < \frac{3}{4}$

Local expansion

- Recall Mellin transform

$$M(u) = \exp(\phi(u-1,t) + v_0\psi(u-1,t))$$

- Determine singular expansions of ϕ and ψ from Riccati equations
- Abbreviation $L := \log x$
- Local expansion of the integrand:

$$x^{-u}M(u) = Cx^{-u^*}\exp\left(2\beta L^{1/2} + \frac{a}{c^2}\log L - \beta^{-1}L^{3/2}y^2 + o(1)\right)$$

Local expansion

- Gaussian integral

$$\int_{-L^{-\alpha}}^{L^{-\alpha}} \exp(-\beta^{-1} L^{3/2} y^2) dy$$

$$= \beta^{1/2} L^{-3/4} \int_{-\beta^{-1/2} L^{3/4-\alpha}}^{\beta^{-1/2} L^{3/4-\alpha}} \exp(-w^2) dw$$

$$\sim \beta^{1/2} L^{-3/4} \int_{-\infty}^{\infty} \exp(-w^2) dw = \sqrt{\pi} \beta^{1/2} L^{-3/4}$$

Tail estimate

- Finding saddle point + local expansion fairly routine
- Problem: Verify concentration
- Needs some insight into behaviour of function away from saddle point
- Show exponential decay by ODE comparison

Result of saddle point method

- Density asymptotics for $x \to \infty$

$$D_T(x) = A_1 x^{-A_3} e^{A_2\sqrt{\log x}} (\log x)^{-3/4+a/c^2} \left(1+O((\log x)^{-1/2})\right)$$

- Constants in terms of critical moment and critical slope:

$$A_3 = u^* = s^* + 1 \qquad \text{and} \qquad A_2 = 2\frac{\sqrt{2v_0}}{c\sqrt{\sigma}}$$

- Easily extended to full asymptotic expansion

Explicit expression for constant factor

- From closed form of ϕ and ψ:

$$A_1 = \frac{1}{2\sqrt{\pi}} (2v_0)^{1/4-a/c^2} \, c^{2a/c^2-1/2} \sigma^{-a/c^2-1/4}$$

$$\times \exp\left(-v_0\left(\frac{b+s^*\rho c}{c^2}+\frac{\kappa}{c^2\sigma^2}\right)-\frac{aT}{c^2}(b+c\rho s^*)\right)$$

$$\times \left(\frac{2\sqrt{b^2+2bc\rho s^*+c^2s^*(1-(1-\rho^2)s^*)}}{c^2s^*(s^*-1)\sinh\frac{1}{2}\sqrt{b^2+2bc\rho s^*+c^2s^*(1-(1-\rho^2)s^*)}}\right)^{2a/c^2}$$

Call prices and Smile asymptotics

- Gulisashvili (2009): Assumes that density of spot varies regularly at infinity

$$D_T(x) = x^{-\gamma} h(x),$$

  h varies slowly at infinity, $\gamma > 2$
- Expansions of call prices and implied volatility
- Similarly for left tail

Smile asymptotics

- Implied volatility for log-strike $k \to \infty$

$$\sigma_{BS}(k,T)\sqrt{T} = \beta_1 k^{1/2} + \beta_2 + \beta_3 \frac{\log k}{k^{1/2}} + O\left(\frac{\varphi(k)}{k^{1/2}}\right)$$

- Constants

$$\begin{aligned}
\beta_1 &= \sqrt{2}\left(\sqrt{A_3-1} - \sqrt{A_3-2}\right), \\
\beta_2 &= \frac{A_2}{\sqrt{2}}\left(\frac{1}{\sqrt{A_3-2}} - \frac{1}{\sqrt{A_3-1}}\right), \\
\beta_3 &= \frac{1}{\sqrt{2}}\left(\frac{1}{4} - \frac{a}{c^2}\right)\left(\frac{1}{\sqrt{A_3-1}} - \frac{1}{\sqrt{A_3-2}}\right)
\end{aligned}$$

Call prices

- Call price for strike $K \to \infty$

$$C(K) = \frac{A_1}{(-A_3+1)(-A_3+2)} K^{-A_3+2} e^{A_2\sqrt{\log K}} (\log K)^{-\frac{3}{4}+\frac{a}{c^2}} \times \left(1 + O\left((\log K)^{-\frac{1}{4}}\right)\right)$$

Smile asymptotics

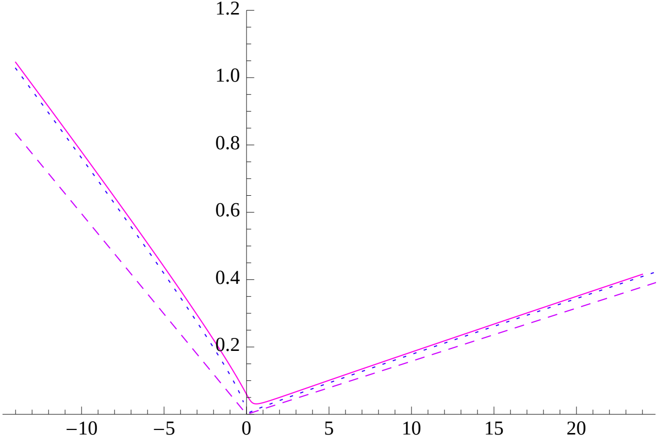


Figure: Implied variance $\sigma(k,1)^2$ in terms of log-strikes compared to the first order (dashed) and third order (dotted) approximations.

References

- P. Friz, S. Gerhold, A. Gulisashvili, S. Sturm: *On refined volatility smile expansion in the Heston model*, 2010, submitted.
- A. Gulisashvili: *Asymptotic formulas with error estimates for call pricing functions and the implied volatility at extreme strikes*, 2009
- A. Gulisashvili, E. M. Stein: *Asymptotic behavior of distribution densities in stochastic volatility models*. To appear in Applied Mathematics and Optimization, 2010.
- A. D. Drăgulescu, V. M. Yakovenko: *Probability distribution of returns in the Heston model with stochastic volatility*, Quantitative Finance 2002.
- R. W. Lee: *The moment formula for implied volatility at extreme strikes*, Math. Finance 2004.