Local Volatility Pricing Models for Long-Dated FX Derivatives

G. Deelstra, G. Rayee

Université Libre de Bruxelles
grayee@ulb.ac.be

Outline of the talk

1. Introduction
2. The Model
3. The local volatility function
4. Calibration
5. Extension

Introduction

- Recent years, the long-dated (maturity > 1 year) foreign exchange (FX) option's market has grown considerably
  - Vanilla options (European Call and Put)
  - Exotic options (barriers,...)
  - Hybrid options (PRDC swaps)

Introduction

- A suitable pricing model for long-dated FX options has to take into account the risks linked to:
  - domestic and foreign interest rates
    - by using stochastic processes for both domestic and foreign interest rates

$$dr_d(t) = [\theta_d(t) - \alpha_d(t)r_d(t)]dt + \sigma_d(t)dW_d^{DRN}(t),$$
$$dr_f(t) = [\theta_f(t) - \alpha_f(t)r_f(t)]dt + \sigma_f(t)dW_d^{FRN}(t)$$

  - the volatility of the spot FX rate (Smile/Skew effect)
    - by using a local volatility $\sigma(t, S(t))$ for the FX spot

$$dS(t) = (r_d(t) - r_f(t))S(t)dt + \sigma(t, S(t))S(t)dW_S^{DRN}(t),$$

    - by using a stochastic volatility $\nu(t)$ for the FX spot

$$dS(t) = (r_d(t) - r_f(t))S(t)dt + \sqrt{\nu(t)}S(t)dW_S^{DRN}(t),$$
$$d\nu(t) = \kappa(\theta - \nu(t))dt + \xi\sqrt{\nu(t)}dW_\nu^{DRN}(t)$$

    - and/or jump

Introduction

- Stochastic volatility models with stochastic interest rates:

R. Ahlip, Foreign exchange options under stochastic volatility and stochastic interest rates, *International Journal of Theoretical and Applied Finance (IJTAF)*, vol. 11, issue 03, pages 277-294, (2008).

J. Andreasen, Closed form pricing of FX options under stochastic rates and volatility, Global Derivatives Conference, ICBI, (May 2006).

A. Antonov, M. Arneguy, and N. Audet, Markovian projection to a displaced volatility Heston model, Available at http://ssrn.com/abstract=1106223, (2008).

A. van Haastrecht, R. Lord, A. Pelsser, and D. Schrager, Pricing long-maturity equity and fx derivatives with stochastic interest rates and stochastic volatility, Available at http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1125590, (2008).

A. van Haastrecht and A. Pelsser, Generic Pricing of FX, Inflation and Stock Options Under Stochastic Interest Rates and Stochastic Volatility, Available at http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1197262, (February 2009).

- Local volatility models with stochastic interest rates:

V. Piterbarg, Smiling hybrids, *Risk*, 66-71, (May 2006).

Introduction

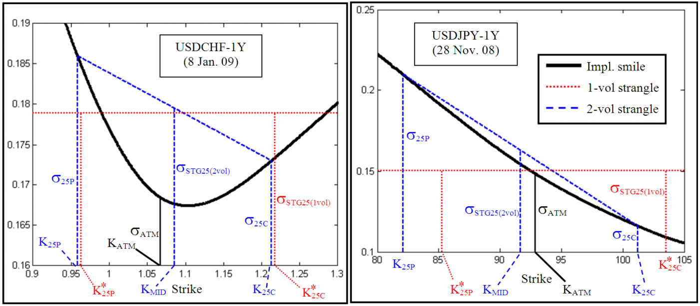
- Advantages of working with a local volatility model:
  - the local volatility $\sigma(t, S(t))$ is a deterministic function of both the FX spot and time.
    - It avoids the problem of working in incomplete markets in comparison with stochastic volatility models and is therefore more appropriate for hedging strategies
  - has the advantage to be calibrated on the complete implied volatility surface,
    - local volatility models usually capture more precisely the surface of implied volatilities than stochastic volatility models

The model

- The calibration of a model is usually done on the vanilla options market
  $\rightarrow$ local and stochastic volatility models (well calibrated) return the same price for these options.
- But calibrating a model to the vanilla market is by no mean a guarantee that all type of options will be priced correctly
  - **example:** We have compared short-dated barrier option market prices with the corresponding prices derived from either a Dupire local volatility or a Heston stochastic volatility model both calibrated on the vanilla smile/skew.

Introduction

- A FX market characterized by a mild skew (USDCHF) exhibits mainly a stochastic volatility behavior,
- A FX market characterized by a dominantly skewed implied volatility (USDJPY) exhibit a stronger local volatility component.



Introduction

- The market dynamics could be better approximated by a hybrid volatility model that contains both stochastic volatility dynamics and local volatility ones.
- example:

$$\begin{cases} dS(t) = (r_d(t) - r_f(t))S(t)dt + \sigma_{LOC2}(t, S(t))\sqrt{\nu(t)}S(t)dW_S^{DRN}(t), \\ dr_d(t) = [\theta_d(t) - \alpha_d(t)r_d(t)]dt + \sigma_d(t)dW_d^{DRN}(t), \\ dr_f(t) = [\theta_f(t) - \alpha_f(t)r_f(t)]dt + \sigma_f(t)dW_f^{FRN}(t), \\ d\nu(t) = \kappa(\theta - \nu(t))dt + \xi\sqrt{\nu(t)}dW_\nu^{DRN}(t). \end{cases}$$

- The local volatility function $\sigma_{LOC2}(t, S(t))$ can be calibrated from the local volatility that we have in a pure local volatility model!

The three-factor model with local volatility

- The spot FX rate S is governed by the following dynamics

$$dS(t) = (r_d(t) - r_f(t))S(t)dt + \sigma(t, S(t))S(t)dW_S^{DRN}(t), \quad (1)$$

  - domestic and foreign interest rates, r_d and r_f follow a Hull-White one factor Gaussian model defined by the Ornstein-Uhlenbeck processes

$$\begin{cases} dr_d(t) = [\theta_d(t) - \alpha_d(t)r_d(t)]dt + \sigma_d(t)dW_d^{DRN}(t), & (2) \\ dr_f(t) = [\theta_f(t) - \alpha_f(t)r_f(t) - \rho_{fS}\sigma_f(t)\sigma(t, S(t))]dt + \sigma_f(t)dW_f^{DRN}(t), & (3) \end{cases}$$

    - $\theta_d(t), \alpha_d(t), \sigma_d(t), \theta_f(t), \alpha_f(t), \sigma_f(t)$ are deterministic functions of time.
    - Equations (1), (2) and (3) are expressed in the domestic risk-neutral measure (DRN).
    - $(W_S^{DRN}(t), W_d^{DRN}(t), W_f^{DRN}(t))$ is a Brownian motion under the domestic risk-neutral measure Q_d with the correlation matrix

$$\begin{pmatrix} 1 & \rho_{Sd} & \rho_{Sf} \\ \rho_{Sd} & 1 & \rho_{df} \\ \rho_{Sf} & \rho_{df} & 1 \end{pmatrix}.$$

The local volatility derivation : first approach

The local volatility derivation : first approach

- Consider the forward call price $\widetilde{C}(K,t)$ of strike K and maturity t, defined (under the t-forward measure Q_t) by

$$\widetilde{C}(K,t) = \frac{C(K,t)}{P_d(0,t)} = \mathbf{E}^{Q_t}[(S(t)-K)^+] = \int\int\int_K^{+\infty}(S(t)-K)\phi_F(S,r_d,r_f,t)dSdr_d dr_f.$$

  - Differentiating it with respect to the maturity t leads to

$$\frac{\partial\widetilde{C}(K,t)}{\partial t} = \int\int\int_K^{+\infty}(S(t)-K)\frac{\partial\phi_F(S,r_d,r_f,t)}{\partial t}dSdr_d dr_f$$

  - we have shown that the t-forward probability density ϕ_F satisfies the following forward PDE:

$$\begin{aligned}\frac{\partial\phi_F}{\partial t} &= -(r_d(t)-f_d(0,t))\,\phi_F - \frac{\partial[(r_d(t)-r_f(t))S(t)\phi_F]}{\partial x} - \frac{\partial[(\theta_d(t)-\alpha_d(t)\;r_d(t))\phi_F]}{\partial y}\\ &- \frac{\partial[(\theta_f(t)-\alpha_f(t)\;r_f(t))\phi_F]}{\partial z} + \frac{1}{2}\frac{\partial^2[\sigma^2(t,S(t))S^2(t)\phi_F]}{\partial x^2} - \frac{1}{2}\frac{\partial^2[\sigma_d^2(t)\phi_F]}{\partial y^2} + \frac{1}{2}\frac{\partial^2[\sigma_f^2(t)\phi_F]}{\partial z^2}\\ &+ \frac{\partial^2[\sigma(t,S(t))S(t)\sigma_d(t)\rho_{Sd}\phi_F]}{\partial x\partial y} + \frac{\partial^2[\sigma(t,S(t))S(t)\sigma_f(t)\rho_{Sf}\phi_F]}{\partial x\partial z} + \frac{\partial^2[\sigma_d(t)\sigma_f(t)\rho_{df}\phi_F]}{\partial y\partial z}.\end{aligned} \quad (4)$$

The local volatility derivation : first approach

- Integrating by parts several times we get

$$\begin{aligned}\frac{\partial \widetilde{C}(K,t)}{\partial t} &= f_d(0,t)\widetilde{C}(K,t) + \int\int\int_K^{+\infty} [r_d(t)K - r_f(t)S(t)]\phi_F(S,r_d,r_f,t)dSdr_ddr_f \\ &+\frac{1}{2}(\sigma(t,K)K)^2\int\int \phi_F(K,r_d,r_f,t)dr_ddr_f \\ &= f_d(0,t)\widetilde{C}(K,t) + \mathbf{E}^{Q_t}[(r_d(t)K - r_f(t)S(t))\mathbf{1}_{\{S(t)>K\}}] \\ &+\frac{1}{2}(\sigma(t,K)K)^2\frac{\partial^2\widetilde{C}(K,t)}{\partial K^2}.\end{aligned}$$

- This leads to the following expression for the local volatility surface in terms of the forward call prices $\widetilde{C}(K,t)$

$$\sigma^2(t,K) = \frac{\frac{\partial \widetilde{C}(K,t)}{\partial t} - f_d(0,t)\widetilde{C}(K,t) - \mathbf{E}^{Q_t}[(r_d(t)K - r_f(t)\ S(t))\mathbf{1}_{\{S(t)>K\}}]}{\frac{1}{2}K^2\frac{\partial^2\widetilde{C}(K,t)}{\partial K^2}}.$$

The local volatility derivation : first approach

- The (partial) derivatives of the forward call price with respect to the maturity can be rewritten as

$$
\begin{aligned}
\frac{\partial \widetilde{C}(K,t)}{\partial t} &= \frac{\partial[\frac{C(K,t)}{P_d(0,t)}]}{\partial t} = \frac{\partial C(K,t)}{\partial t}\frac{1}{P_d(0,t)} + f_d(0,t)\widetilde{C}(t,K),\\
\frac{\partial^2 \widetilde{C}(t,K)}{\partial K^2} &= \frac{\partial^2[\frac{C(K,t)}{P_d(0,t)}]}{\partial K^2} = \frac{1}{P_d(0,t)}\frac{\partial^2 C(t,K)}{\partial K^2}.
\end{aligned}
$$

- Substituting these expressions into the last equation, we obtain the expression of the local volatility $\sigma^2(t,K)$ in terms of call prices $C(K,t)$

$$
\sigma^2(t,K) = \frac{\frac{\partial C(K,t)}{\partial t} - P_d(0,t)\mathbf{E}^{Q_t}[(r_d(t)K - r_f(t)\ S(t))\mathbf{1}_{\{S(t)>K\}}\ ]}{\frac{1}{2}K^2\frac{\partial^2 C(K,t)}{\partial K^2}}.
$$

The local volatility derivation : second approach

The local volatility derivation : second approach

- Applying Tanaka's formula to the convex but non-differentiable function $e^{-\int_0^t r_d(s)ds}\,(S(t)-K)^+$ leads to

$$e^{-\int_0^t r_d(s)ds}(S(t)-K)^+ = (S(0)-K)^+ - \int_0^t r_d(u)e^{-\int_0^u r_d(s)ds}(S(u)-K)^+du + \int_0^t e^{-\int_0^u r_d(s)ds}\mathbf{1}_{\{S(u)>K\}}dS_u + \frac{1}{2}\int_0^t e^{-\int_0^u r_d(s)ds}dL_u^K(S)$$

where $L_u^K(S)$ is the local time of S defined by

$$L_t^K(S) = \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\int_0^t \mathbf{1}_{[K,K+\epsilon]}(S(s))d<S,S>_s .$$

- Using $dS(t) = (r_d(t)-r_f(t))S(t)dt + \sigma(t,S(t))S(t)dW_S^{DRN}(t)$, taking the domestic risk neutral expectation of each side and finally differentiating,

$$dC(K,t) = \mathbf{E}^{Q_d}[e^{-\int_0^t r_d(s)ds}(Kr_d(t)-r_f(t)S(t))\mathbf{1}_{\{S(t)>K\}}]dt + \frac{1}{2}\lim_{\epsilon\downarrow 0}\mathbf{E}^{Q_d}[\frac{1}{\epsilon}\mathbf{1}_{[K,K+\epsilon]}(S(t))e^{-\int_0^t r_d(s)ds}\sigma^2(t,S(t))\ S^2(t)]dt.$$

The local volatility derivation : second approach

- Using conditional expectation properties, the last term can be rewritten as follows

$$\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\mathbf{E}^{Q_d}[\mathbf{1}_{[K,K+\epsilon]}(S(t))e^{-\int_0^t r_d(s)ds}\sigma^2(t,S(t))S^2(t)]$$
$$=\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\mathbf{E}^{Q_d}[\mathbf{E}^{Q_d}[e^{-\int_0^t r_d(s)ds}\mid S(t)]\mathbf{1}_{[K,K+\epsilon]}(S(t))\sigma^2(t,S(t))S^2(t)]$$
$$=\underbrace{\mathbf{E}^{Q_d}[e^{-\int_0^t r_d(s)ds}\mid S(t)=K]p_d(K,t)}_{\frac{\partial^2 C(K,t)}{\partial K^2}}\sigma^2(t,K)\ K^2$$

where $p_d(K,t)=\int\int\phi_d(K,r_d,r_f,t)$ is the domestic risk neutral density of $S(t)$ in K.

- This leads to the local volatility expression where the expectation is expressed under the domestic risk neutral measure Q_d

$$\sigma^2(t,K)=\frac{\frac{\partial C(K,t)}{\partial t}-\mathbf{E}^{Q_d}[e^{-\int_0^t r_d(s)ds}(Kr_d(t)-r_f(t)S(t))\mathbf{1}_{\{S(t)>K\}}]}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}. \quad (5)$$

The local volatility derivation : second approach

- Making the well known change of measure : $\frac{dQ_T}{dQ_d} = \frac{e^{-\int_0^t r_d(s)ds} P_d(t,T)}{P_d(0,T)}$, you get the expression with the expectation expressed into the t-forward measure Q_t

$$\sigma^2(t,K) = \frac{\frac{\partial C(K,t)}{\partial t} - P_d(0,t)\mathbf{E}^{Q_t}[(Kr_d(t) - r_f(t)S(t))\mathbf{1}_{\{S(t)>K\}}]}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}.$$

Calibration

Calibration

- Before pricing any derivatives with a model, it is usual to calibrate it on the vanilla market,
  - determine all parameters present in the different stochastic processes which define the model in such a way that all European option prices derived in the model are as consistent as possible with the corresponding market ones.

Calibration

- The calibration procedure for the three-factor model with local volatility can be decomposed in three steps:
  1. Parameters present in the Hull-White one-factor dynamics for the domestic and foreign interest rates, $\theta_d(t)$, $\alpha_d(t)$, $\sigma_d(t)$, $\theta_f(t)$, $\alpha_f(t)$, $\sigma_f(t)$, are chosen to match European swaption / cap-floors values in their respective currencies.
  2. The three correlation coefficients of the model, ρ_{Sd}, ρ_{Sf} and ρ_{df} are usually estimated from historical data.
  3. After these two steps, the calibration problem consists in finding the local volatility function of the spot FX rate which is consistent with an implied volatility surface.

Calibration

$$\sigma^2(t,K) = \frac{\frac{\partial C(K,t)}{\partial t} - P_d(0,t)\mathbf{E}^{Q_t}[(Kr_d(t) - r_f(t)S(t))\mathbf{1}_{\{S(t)>K\}}]}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}.$$

Difficult because of $\mathbf{E}^{Q_t}[(Kr_d(t) - r_f(t)S(t))\mathbf{1}_{\{S(t)>K\}}]$:

- there exists no closed form solution
- it is not directly related to European call prices or other liquid products.
- Its calculation can obviously be done by using numerical methods but you have to solve (numerically) a three-dimensional PDE:

$$\begin{aligned} 0 \quad = \quad & \frac{\partial \phi_F}{\partial t} + (r_d(t) - f_d(0,t))\,\phi_F + \frac{\partial[(r_d(t) - r_f(t))S(t)\phi_F]}{\partial x} + \frac{\partial[(\theta_d(t) - \alpha_d(t)\ r_d(t))\phi_F]}{\partial y} \\ & + \frac{\partial[(\theta_f(t) - \alpha_f(t)\ r_f(t))\phi_F]}{\partial z} - \frac{1}{2}\frac{\partial^2[\sigma^2(t,S(t))S^2(t)\phi_F]}{\partial x^2} - \frac{1}{2}\frac{\partial^2[\sigma_d^2(t)\phi_F]}{\partial y^2} - \frac{1}{2}\frac{\partial^2[\sigma_f^2(t)\phi_F]}{\partial z^2} \\ & - \frac{\partial^2[\sigma(t,S(t))S(t)\sigma_d(t)\rho_{Sd}\phi_F]}{\partial x \partial y} - \frac{\partial^2[\sigma(t,S(t))S(t)\sigma_f(t)\rho_{Sf}\phi_F]}{\partial x \partial z} - \frac{\partial^2[\sigma_d(t)\sigma_f(t)\rho_{df}\phi_F]}{\partial y \partial z}. \end{aligned} \quad (6)$$

First method : by adjusting the Dupire formula

Calibration : Comparison between local volatility with and without stochastic interest rates

- In a deterministic interest rates framework, the local volatility function $\sigma_{1f}(t,K)$ is given by the well-known Dupire formula:

$$\sigma_{1f}^2(t,K) = \frac{\frac{\partial C(K,t)}{\partial t} + K(f_d(0,t) - f_f(0,t))\frac{\partial C(K,t)}{\partial K} + f_f(0,t)C(K,t)}{\frac{1}{2}K^2\frac{\partial^2 C(K,t)}{\partial K^2}}.$$

- If we consider the three-factor model with stochastic interest rates, the local volatility function is given by

$$\sigma_{3f}^2(t,K) \quad = \quad \frac{\frac{\partial C(K,t)}{\partial t} - P_d(0,t)\mathbf{E}^{Q_t}[(Kr_d(t) - r_f(t)S(t))\mathbf{1}_{\{S(t)>K\}}]}{\frac{1}{2}K^2\frac{\partial^2 C(K,t)}{\partial K^2}}.$$

- We can derive the following interesting relation between the simple Dupire formula and its extension

$$\sigma_{3f}^2(t,K) - \sigma_{1f}^2(t,K) = \frac{KP_d(0,t)\{\mathbf{Cov}^{Q_t}[r_f(t) - r_d(t), \mathbf{1}_{\{S(t)>K\}}] + \frac{1}{K}\mathbf{Cov}^{Q_t}[r_f(t), (S(t)-K)^+]\}}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}. \tag{7}$$

Second method : by mimicking stochastic volatility models

Calibrating the local volatility by mimicking stochastic volatility models

- Consider the following domestic risk neutral dynamics for the spot FX rate

$$dS(t) = (r_d(t) - r_f(t))\ S(t)\ dt + \gamma(t,\nu(t))\ S(t)\ dW_S^{DRN}(t)$$

  - $\nu(t)$ is a stochastic variable which provides the stochastic perturbation for the spot FX rate volatility.
  - Common choices:
    1. $\gamma(t,\nu(t)) = \nu(t)$
    2. $\gamma(t,\nu(t)) = exp(\sqrt{\nu(t)})$
    3. $\gamma(t,\nu(t)) = \sqrt{\nu(t)}$
  - The stochastic variable $\nu(t)$ is generally modelled by
    - a Cox-Ingersoll-Ross (CIR) process as for example the Heston stochastic volatility model:

$$d\nu(t) = \kappa(\theta - \nu(t))dt + \xi\sqrt{\nu(t)}dW_\nu^{DRN}(t)$$

    - a Ornstein-Uhlenbeck process (OU) as for example the Schöbel and Zhu stochastic volatility model:

$$d\nu(t) = k[\lambda - \nu(t)]dt + \xi dW_\nu^{DRN}(t)$$

Calibrating the local volatility by mimicking stochastic volatility models

- Applying Tanaka's formula to the non-differentiable function $e^{-\int_0^t r_d(s)ds}(S(t)-K)^+$, where $dS(t) = (r_d(t) - r_f(t))\ S(t)\ dt + \gamma(t,\nu(t))\ S(t)\ dW_S^{DRN}(t)$

$$dC(K,t) = \mathbf{E}^{Q_d}[e^{-\int_0^t r_d(s)ds}(Kr_d(t) - r_f(t)S(t))\mathbf{1}_{\{S(t)>K\}}]dt + \frac{1}{2}\lim_{\epsilon\downarrow 0}\mathbf{E}^{Q_d}[\frac{1}{\epsilon}\mathbf{1}_{[K,K+\epsilon]}(S(t))e^{-\int_0^t r_d(s)ds}\gamma^2(t,\nu(t))S^2(t)]dt.$$

Here, the last term can be rewritten as

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\mathbf{E}^{Q_d}[\mathbf{1}_{[K,K+\epsilon]}(S(t))e^{-\int_0^t r_d(s)ds}\gamma^2(t,\nu(t))S^2(t)] \\
&= \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\mathbf{E}^{Q_d}[\mathbf{E}^{Q_d}[\gamma^2(t,\nu(t))e^{-\int_0^t r_d(s)ds} \mid S(t)]\mathbf{1}_{[K,K+\epsilon]}(S(t))S^2(t)] \\
&= \mathbf{E}^{Q_d}[\gamma^2(t,\nu(t))e^{-\int_0^t r_d(s)ds} \mid S(t)=K]p_d(K,t)K^2 \\
&= \frac{\mathbf{E}^{Q_d}[\gamma^2(t,\nu(t))e^{-\int_0^t r_d(s)ds} \mid S(t)=K]}{\mathbf{E}^{Q_d}[e^{-\int_0^t r_d(s)ds} \mid S(t)=K]}\frac{\partial^2 C(K,t)}{\partial K^2}K^2. \qquad (8)
\end{aligned}$$

Calibrating the local volatility by mimicking stochastic volatility models

$$\frac{\mathbf{E}^{Q_d}[\gamma^2(t,\nu(t))e^{-\int_0^t r_d(s)ds} \mid S(t)=K]}{\mathbf{E}^{Q_d}[e^{-\int_0^t r_d(s)ds} \mid S(t)=K]} = \underbrace{\frac{\frac{\partial C}{\partial t} - \mathbf{E}^{Q_d}[e^{-\int_0^t r_d(s)ds}(Kr_d(t)-r_f(t)S(t))\mathbf{1}_{\{S(t)>K\}}]}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}}_{\sigma^2(t,K)}$$

- Therefore, if there exists a local volatility such that the one-dimensional probability distribution of the spot FX rate with the diffusion

$$dS(t) = (r_d(t) - r_f(t))\ S(t)\ dt + \sigma(t,S(t))\ S(t)\ dW_S^{DRN}(t),$$

is the same as the one of the spot FX rate with dynamics

$$dS(t) = (r_d(t) - r_f(t))\ S(t)\ dt + \gamma(t,\nu(t))\ S(t)\ dW_S^{DRN}(t)$$

for every time t, then this local volatility function has to satisfy

$$\begin{aligned}\sigma^2(t,K) &= \frac{\mathbf{E}^{Q_d}[\gamma^2(t,\nu(t))e^{-\int_0^t r_d(s)ds} \mid S(t)=K]}{\mathbf{E}^{Q_d}[e^{-\int_0^t r_d(s)ds} \mid S(t)=K]}\\ &= \mathbf{E}^{Q_t}[\gamma^2(t,\nu(t)) \mid S(t)=K].\end{aligned}$$

A particular case with closed form solution

- Consider the three-factor model with local volatility

$$\begin{cases} dS(t) = (r_d(t) - r_f(t))S(t)dt + \sigma(t, S(t))S(t)dW_S^{DRN}(t), \\ dr_d(t) = [\theta_d(t) - \alpha_d r_d(t)]dt + \sigma_d dW_d^{DRN}(t), \\ dr_f(t) = [\theta_f(t) - \alpha_f r_f(t) - \rho_{fS}\sigma_f \nu(t)]dt + \sigma_f dW_f^{DRN}(t), \end{cases}$$

- Calibration by mimicking a Schöbel and Zhu-Hull and White stochastic volatility model

$$\begin{cases} dS(t) = (r_d(t) - r_f(t))S(t)dt + \nu(t)S(t)dW_S^{DRN}(t), \\ dr_d(t) = [\theta_d(t) - \alpha_d r_d(t)]dt + \sigma_d dW_d^{DRN}(t), \\ dr_f(t) = [\theta_f(t) - \alpha_f r_f(t) - \rho_{fS}\sigma_f \nu(t)]dt + \sigma_f dW_f^{DRN}(t), \\ d\nu(t) = k[\lambda - \nu(t)]\ dt + \xi dW_\nu^{DRN}(t), \end{cases}$$

- The local volatility function is given by:

$$\begin{aligned} \sigma^2(T, K) &= \mathbf{E}^{Q_T}[\nu^2(T)|S(T) = K] \\ &= \mathbf{E}^{Q_T}[\nu^2(T)] \quad \text{if we assume independence between } S \text{ and } \nu \\ &= (\mathbf{E}^{Q_T}[\nu(T)])^2 + \mathbf{Var}^{Q_T}[\nu(T)] \end{aligned}$$

A particular case with closed form solution

- Under the T-Forward measure:

$$d\nu(t) = [k(\lambda - \nu(t)) - \rho_{d\nu}\sigma_d b_d(t,T)\xi]dt + \xi\ dW_\nu^{TF}(t)$$

$$\nu(T) = \nu(t)e^{-k(T-t)} + \int_t^T k(\lambda - \frac{\rho_{d\nu}\sigma_d b_d(u,T)\xi}{k})e^{-k(T-u)}du + \int_t^T \xi e^{-k(T-t)}dW_\nu^{TF}(u)$$

$$\text{where } b_d(t,T) = \frac{1}{\alpha_d}(1 - e^{-\alpha_d(T-t)})$$

- so that $\nu(T)$ conditional on $\mathcal{F}_t$ is normally distributed with mean and variance given respectively by

$$\mathbf{E}^{Q_T}[\nu(T)|\mathcal{F}_t] = \nu(t)e^{-k(T-t)} + (\lambda - \frac{\rho_{d\nu}\sigma_d\xi}{\alpha_d k})(1 - e^{-k(T-t)}) + \frac{\rho_{d\nu}\sigma_d\xi}{\alpha_d(\alpha_d + k))}(1 - e^{-(\alpha_d+k)(T-t)})$$

$$\mathbf{Var}^{Q_T}[\nu(T)|\mathcal{F}_t] = \frac{\xi^2}{2k}(1 - e^{-2k(T-t)})$$

A particular case with closed form solution

$$
\begin{aligned}
\sigma^2(T,K) &= (\mathbf{E}^{Q_T}[\nu(T)])^2 + \mathbf{Var}^{Q_T}[\nu(T)] \\
&= \left(\nu(t)e^{-kT} + (\lambda - \frac{\rho_{d\nu}\sigma_d\xi}{\alpha_d k})(1-e^{-kT}) + \frac{\rho_{d\nu}\sigma_d\xi}{\alpha_d(\alpha_d+k))}(1-e^{-(\alpha_d+k)T})\right)^2 \\
&\quad + \frac{\xi^2}{2k}(1-e^{-2kT}) \\
&= \sigma^2(T)
\end{aligned}
$$

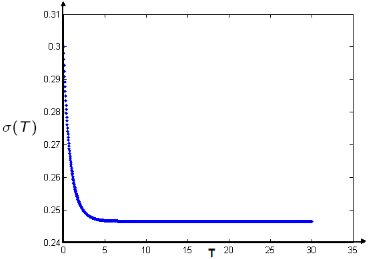


Figure: $\xi = 20\%$, $k = 50\%$, $\alpha_d = 5\%$, $\nu(0) = 10\%$, $\sigma_d = 1\%$, $\lambda = 20\%$, $\rho_{d\nu} = 1\%$

Extension : Hybrid volatility model

Hybrid volatility model

- Here we consider an extension of the three-factor model with local volatility that incorporates a stochastic component to the FX spot volatility by multiplying the local volatility with a stochastic volatility.

Hybrid volatility model

1. Consider a hybrid volatility model where the volatility for the spot FX rate corresponds to a local volatility $\sigma_{LOC2}(t, S(t))$ multiplied by a stochastic volatility $\gamma(t, \nu(t))$ where $\nu(t)$ is a stochastic variable,

$$\begin{cases} dS(t) = (r_d(t) - r_f(t))S(t)dt + \sigma_{LOC2}(t, S(t))\gamma(t, \nu(t))S(t)dW_S^{DRN}(t), \\ dr_d(t) = [\theta_d(t) - \alpha_d(t)r_d(t)]dt + \sigma_d(t)dW_d^{DRN}(t), \\ dr_f(t) = [\theta_f(t) - \alpha_f(t)r_f(t) - \rho_{fS}\sigma_f(t)\sigma_{LOC2}(t, S(t))\gamma(t, \nu(t))]dt + \sigma_f(t)dW_f^{DRN}(t), \\ d\nu(t) = \alpha(t, \nu(t))dt + \vartheta(t, \nu(t))dW_\nu^{DRN}(t). \end{cases}$$

2. Consider the three-factor model where the volatility of the spot FX rate is modelled by a local volatility denoted by $\sigma_{LOC1}(t, S(t))$,

$$\begin{cases} dS(t) = (r_d(t) - r_f(t))S(t)dt + \sigma_{LOC1}(t, S(t))S(t)dW_S^{DRN}(t), \\ dr_d(t) = [\theta_d(t) - \alpha_d(t)r_d(t)]dt + \sigma_d(t)dW_d^{DRN}(t), \\ dr_f(t) = [\theta_f(t) - \alpha_f(t)r_f(t) - \rho_{fS}\sigma_f(t)\sigma_{LOC1}(t, S(t))]dt + \sigma_f(t)dW_f^{DRN}(t). \end{cases}$$

Hybrid volatility model

Gyöngy's result

- Consider a general n-dimensional Itô process ξ_t of the form:

$$d\xi_t = \delta(t,w)dW(t) + \beta(t,w)dt$$

where $W(t)$ is a k-dimensional Wiener process on a probability space $(\Omega, \mathcal{F}, P)$, $\delta \in \mathbb{R}^{n \times k}$ and $\beta \in \mathbb{R}^n$ are bounded $\mathcal{F}_t$-nonanticipative processes such that $\delta\delta^T$ is uniformly positive definite.

- This process gives rise to marginal distributions of the random variables ξ_t for each t.
- Gyöngy then shows that there is a Markov process $x(t)$ with the same marginal distributions.
- The explicit construction is given by:

$$dx_t = \sigma(t,x_t)dW(t) + b(t,x_t)dt$$

where:

$$\begin{aligned} \sigma(t,x) &= (\mathbb{E}[\delta(t,w)\delta^T(t,w)|\xi_t = x])^{\frac{1}{2}} \\ b(t,x) &= \mathbb{E}[\beta(t,w)|\xi_t = x] \end{aligned}$$

Hybrid volatility model

$$\sigma_{LOC2}(t,K) = \frac{\sigma_{LOC1}(t,K)}{\mathbb{E}^{Q_d}[\gamma(t,\nu(t))|r_d(t)=x, r_f(t)=y, S(t)=K]}$$

where the conditional expectation is by definition given by

$$\begin{aligned}
&\mathbb{E}^{Q_d}[\gamma(t,\nu(t))|r_d(t)=x, r_f(t)=y, S(t)=K] \\
&= \frac{\int_0^\infty \gamma(t,\nu(t))\phi_d(S(t)=K, r_d(t)=x, r_f(t)=y, \nu(t), t)d\nu}{\int_0^\infty \phi_d(S(t)=K, r_d(t)=x, r_f(t)=y, \nu(t), t)d\nu}.
\end{aligned}$$

Thank you for your attention