CORRELATION UNDER STRESS IN NORMAL VARIANCE MIXTURE MODELS

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Bankakademie | HfB

STRESS TESTS OF BANK PORTFOLIOS

- Stress test:
  - assessment of bank capital adequacy in a strongly adverse market environment
- Basel II; Supervisory Capital Assessment Program (SCAP; US, Q2 2009)
- Stress tests as integral part of risk management:
  - Basel Committee [BIS, 2009]
  - De Larosière Report [Larosière et al., 2009]
  - Geneva Report [Brunnermeier et al., 2009]
  - Turner Review, page 44, [Turner, 2009]:

    > [..] This implies that any use of VAR models needs to be buttressed by the application of stress test techniques which consider the impact of extreme movements beyond those which the model suggests are at all probable.

STRESS TESTS OF BANK PORTFOLIOS (II)

- Calculate risk measures (expected loss, value-at-risk, economic capital) and regulatory capital under adverse market conditions
- Crucial inputs of any portfolio model:
  - Distribution assumption on portfolio constituents, e.g.
    - normally distributed asset returns
    - fat-tailed asset returns
  - Dependence assumption among portfolio constituents, e.g. correlation
- Translate stress scenario into constraints on risk factors
  - Here: risk factors are truncated (consistency!)
- Question: How does stress testing impact correlation?

OVERVIEW

- **Normal variance mixture distribution**
- **Factor model of asset returns**
- **Stressed factor model**
- **Conditional asset correlation (normal, t)**
- **Asymptotic asset correlation in NVM model**

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NORMAL VARIANCE MIXTURE DISTRIBUTION

- Probability space $(\Omega, \mathcal{A}, \mathbb{P})$.
- Random vector $\mathbf{X}$ follows a normal variance mixture (NVM) distribution if

$$\mathbf{X} \stackrel{\mathcal{L}}{=} \mu + \sqrt{W} B\, \mathbf{Z},$$

  where

  - $\mathbf{Z} \sim \mathrm{N}_k(\mathbf{0}, I_k)$,
  - $W \geq 0$ is a random variable independent of $\mathbf{Z}$,
  - $B \in \mathbb{R}^{d \times k}$ and $\mu \in \mathbb{R}^d$.

- Observe that

$$\mathbf{X} | (W = w) \sim \mathrm{N}_d(\mu, w\Sigma),$$

  with $\Sigma = B B'$.

NORMAL VARIANCE MIXTURE DISTRIBUTION (II)

Examples of NVM distributions:

Df of W	Df of $\mathbf{X}$
constant	multivariate normal
inverse gamma	multivariate t
generalised inverse Gaussian	symmetric generalised hyperbolic

⇝ [McNeil et al., 2005], [Bingham and Kiesel, 2002]

- Normal variance mixture distribution
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FACTOR MODEL OF ASSET RETURNS

- Define

$$V := \sqrt{W} X \qquad\qquad A_i := \sqrt{W} Y_i$$

- with common factor V and asset returns $A_1, \dots, A_k$,
- where
  - $X \sim \mathrm{N}(0, 1)$
  - $\mathbf{Y} = \mathrm{N}_k(0, \rho)$
  - $\mathrm{Corr}(X, Y_i) = \rho_i$, $i = 1, \dots, k$
  - $W > 0$ with $\mathbb{E}W < \infty$, independent of X, $\mathbf{Y}$.

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STRESSED FACTOR MODEL

- Stressed common factor, stress level $C \le 0$:

$$\boxed{V \le C}$$

- Conditional distribution:

$$\boxed{\mathbb{P}^C(B)} = \mathbb{P}(B|V \le C), \quad B \in \mathcal{B}(\mathbb{R})$$

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- Multivariate normal or t-distribution: determine

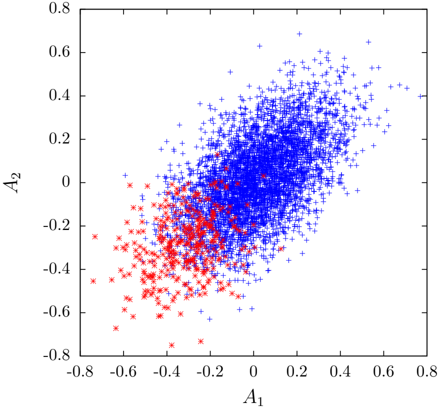
$$\boxed{\mathrm{Corr}^C(A_i, A_j)}$$

- NVM distribution: determine

$$\boxed{\lim_{C \to -\infty} \mathrm{Corr}^C(A_i, A_j)}$$

STRESSED FACTOR MODEL (II)

- $$(A_1, A_2) \sim \mathrm{N}\left[0, \begin{pmatrix} 0.2 & 0.12 \\ 0.12 & 0.2 \end{pmatrix}\right]$$
- $V \leq -0.3$ (red)
- $\rho_1 = 0.8$, $\rho_2 = 0.7$



Proposition

Let $\mathbb{E}^C$ and Var^C be the expectation and variance under $\mathbb{P}^C$, respectively. Then, in the NVM factor model,

$$Corr^C(A_i, A_j) = \frac{\rho_i\,\rho_j\,\frac{\mathrm{Var}^C(V)}{\mathbb{E}^C(W)} + (\rho_{ij} - \rho_i\,\rho_j)}{\sqrt{\left(\rho_i^2\,\frac{\mathrm{Var}^C(V)}{\mathbb{E}^C(W)} + (1-\rho_i^2)\right)\left(\rho_j^2\,\frac{\mathrm{Var}^C(V)}{\mathbb{E}^C(W)} + (1-\rho_j^2)\right)}}.$$

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Must calculate $\frac{\mathrm{Var}^C(V)}{\mathbb{E}^C(W)}$ $\rightsquigarrow$ depends only on V and W.

- Normal variance mixture distribution
- Factor model of asset returns
- Stressed factor model
- **Conditional asset correlation (normal, t)**
- Asymptotic asset correlation in NVM model

ASSET CORRELATION IN A MULTIVARIATE NORMAL MODEL

Proposition

Let $V, A_1, \ldots, A_k$ be standard normally distributed (i.e., $W = 1$). Then,

$$\mathrm{Corr}^C(A_i, A_j) = \frac{\rho_i\, \rho_j\, \mathrm{Var}^C(V) + \rho_{ij} - \rho_i\, \rho_j}{\sqrt{(\rho_i^2\, \mathrm{Var}^C(V) + 1 - \rho_i^2)(\rho_j^2\, \mathrm{Var}^C(V) + 1 - \rho_j^2)}},$$

with

$$\mathrm{Var}^C(V) = 1 - \frac{C\,\phi(C)}{\mathrm{N}(C)} - \frac{(\phi(C))^2}{(\mathrm{N}(C))^2},$$

where ϕ is the standard normal density and N is the standard normal distribution function.

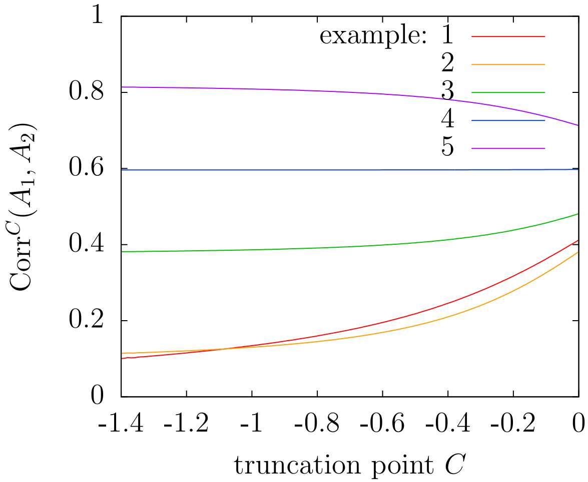
ASSET CORRELATION IN MULTIVARIATE NORMAL MODEL (II)

- 5 examples:

Example	ρ_{12}	ρ_1	ρ_2
1		1	0.6
2		0.8	0.7
3	0.6	0.6	0.6
4		0.1	0.1
5		0.7	0.02

  - Standard deviation of V: 20%

ASSET CORRELATION IN MULTIVARIATE NORMAL MODEL (III)



Examples

ASSET CORRELATION IN A T-DISTRIBUTED MODEL

Proposition

Let $V, A_1, \ldots, A_k$ follow a multivariate t-distribution with parameter $\nu > 2$ denoting the degrees of freedom. Then,

$$\frac{\mathrm{Var}^C(V)}{\mathbb{E}^C(W)} = \frac{\mathbb{E}^C(V^2) - \mathbb{E}^C(V)^2}{\mathbb{E}^C(W)} = \frac{f(\nu, C)}{g(\nu, C)},$$

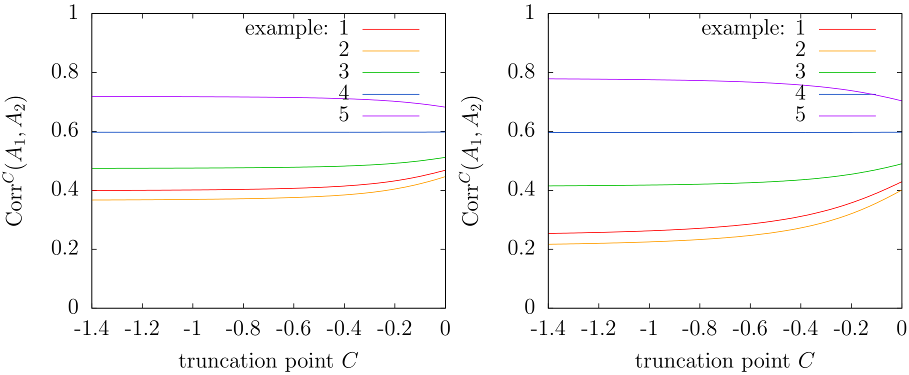
with

$$f(\nu, C) := B\left(\frac{\nu}{C^2+\nu}; \frac{\nu-2}{2}, \frac{3}{2}\right) - \frac{4\left(\frac{\nu}{C^2+\nu}\right)^{\nu-1}}{(\nu-1)^2 B\left(\frac{\nu}{C^2+\nu}; \frac{\nu}{2}, \frac{1}{2}\right)}$$

$$g(\nu, C) := \frac{B\left(\frac{1}{2}, \frac{\nu}{2}\right)}{\nu-2} - \frac{\left(B\left(\frac{\nu-2}{2}, \frac{1}{2}\right) - B\left(\frac{\nu}{\nu+C^2}; \frac{\nu-2}{2}, \frac{1}{2}\right)\right)}{\nu-1},$$

where $B(y; a, b) := \int_0^y t^{a-1}(1-t)^{b-1}\,\mathrm{d}t$ is the incomplete beta function and $B(a, b) := B(1; a, b)$ is the beta function.

ASSET CORRELATION IN *T*-DISTRIBUTED MODEL (II)



Left: $\nu = 4$; right: $\nu = 10$.

Examples

- Normal variance mixture distribution
- Factor model of asset returns
- Stressed factor model
- Conditional asset correlation (normal, t)
- **Asymptotic asset correlation in NVM model**

ASYMPTOTIC ASSET CORRELATION IN NVM MODEL

- Asymptotic asset correlation depends on distribution tail

$\rightsquigarrow$ Extreme Value Theory

EXTREME VALUE THEORY

Theorem (Fisher-Tippett Theorem)

Let (X_n) be a sequence of iid random variables, and let $M_n = \max(X_1, \dots, X_n)$. If there exist norming constants $c_n > 0$, $d_n \in \mathbb{R}$ and some non-degenerate distribution function H such that

$$\frac{M_n - d_n}{c_n} \xrightarrow{\mathcal{L}} H, \qquad \text{as } n \to \infty, \tag{1}$$

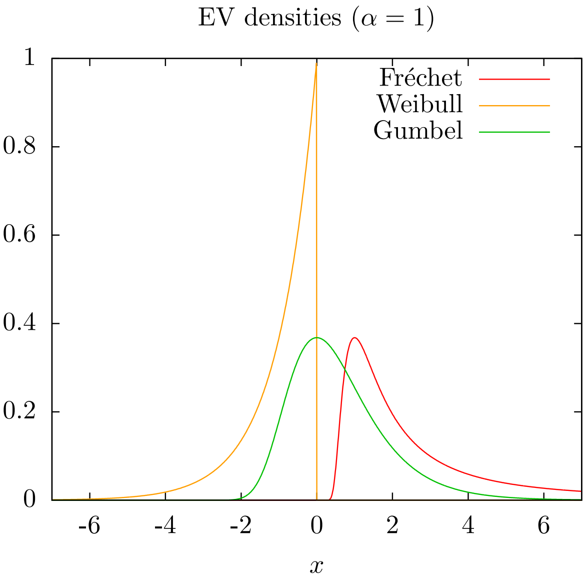
then H belongs to the type of one of the following three distribution functions:

Fréchet: $$\Phi_\alpha(x) = \begin{cases} 0, & x \leq 0 \\ \exp\{-x^{-\alpha}\}, & x > 0 \end{cases} \qquad \alpha > 0.$$

Weibull: $$\Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^{\alpha}\}, & x \leq 0 \\ 1, & x > 0 \end{cases} \qquad \alpha > 0.$$

Gumbel: $$\Lambda(x) = \exp\{-\mathbf{e}^{-x}\}, \quad x \in \mathbb{R}.$$

EXTREME VALUE THEORY (II)



Definition (Maximum domain of attraction)

A random variable X with distribution function F belongs to the maximum domain of attraction (MDA) of H if there exist constants $c_n > 0$, $d_n \in \mathbb{R}$ such that Equation (1) holds, written $X \in \text{MDA}(H)$ and $F \in \text{MDA}(H)$.

V IN THE FRÉCHET MDA

Lemma

If W is in MDA$(\Phi_{\alpha/2})$, then V is in MDA(Φ_α).

Proposition

Let $W \in$ MDA$(\Phi_{\alpha/2})$, $\alpha > 2$. Then

$$\lim_{C \to -\infty} \frac{\mathrm{Var}^C(V)}{\mathbb{E}^C(W)} = \frac{1}{\alpha - 1}.$$

V IN THE FRÉCHET MDA (II)

Sketch of proof:

(i) (short)

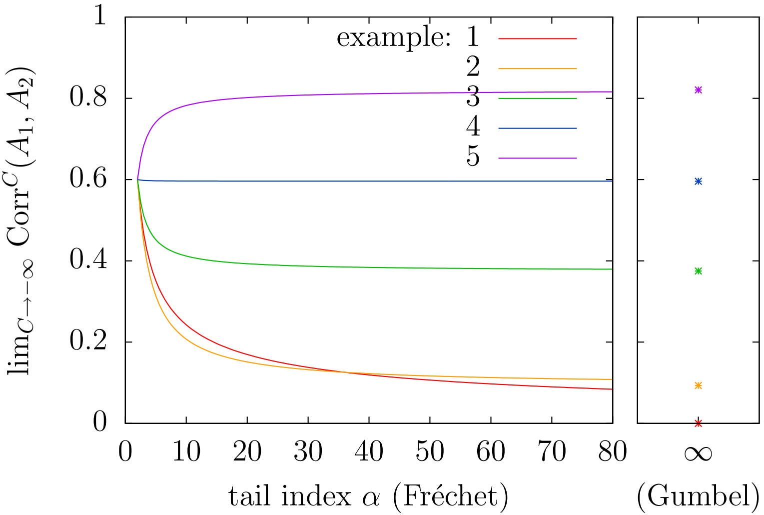
$$\lim_{C \to -\infty} \frac{\mathrm{Var}^C(V)}{C^2} = \frac{\alpha}{(\alpha-2)(\alpha-1)^2}.$$

(ii) (lengthy)

$$\lim_{C \to -\infty} \frac{\mathbb{E}^C(W)}{C^2} = \frac{\alpha}{(\alpha-2)(\alpha-1)}.$$

(Regular variation, Karamata Theorem)

V in the Fréchet MDA (III)



▸ Correlation formula ▸ Examples

V in the Gumbel MDA

Proposition

Let $V \in MDA(\Lambda)$. *Then*

$$\lim_{C \to -\infty} \frac{\mathrm{Var}^C(V)}{\mathbb{E}^C(W)} = 0.$$

V in the Gumbel MDA

Proposition

Let $V \in MDA(\Lambda)$. Then

$$\lim_{C \to -\infty} \frac{\mathrm{Var}^C(V)}{\mathbb{E}^C(W)} = 0.$$

Sketch of proof: Show that

$$\lim_{C \to -\infty} \frac{\mathrm{Var}(V|V \leq C)}{\mathbb{E}(V^2 - C^2|V \leq C)} = 0.$$

and that

$$\lim_{C \to -\infty} \frac{\mathbb{E}(V^2 - C^2|V \leq C)}{2\mathbb{E}(W|V \leq C)} = 1.$$

(Representation Theorem, mean excess function)

Example

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Thank you for your attention!

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EMPIRICAL EXAMPLE

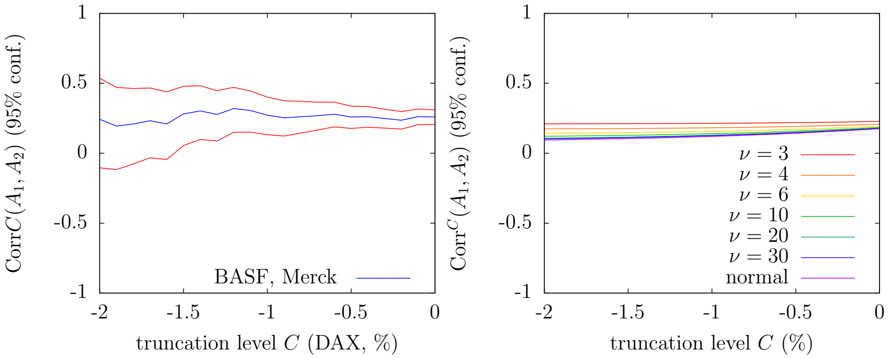
- DAX data, daily log returns, 5.4.1991-6.11.2009 (Source: Reuters)
- Empirical correlations of asset pairs:

Assets A_i, A_j	ρ_i	ρ_j	ρ_{ij}
BASF, Merck	0.73	0.36	0.30
Daimler, Telekom	0.76	0.68	0.43
Daimler, Kali & Salz	0.76	0.41	0.37

- Sample sizes of truncated DAX returns:

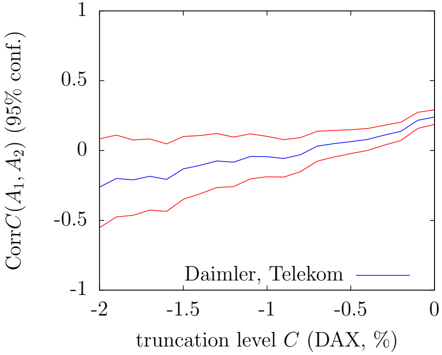
Truncation level C	0%	-0.5%	-1%	-1.5%	-2%
No. of samples	1258	526	183	74	34

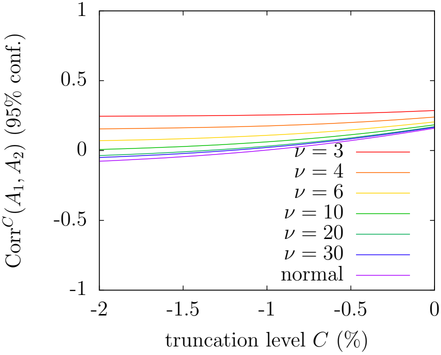
EMPIRICAL EXAMPLE (II)



- DAX data (Source: Reuters), daily log returns, 5.4.1991-6.11.2009
- 95% confidence intervals computed using Fisher z-transform

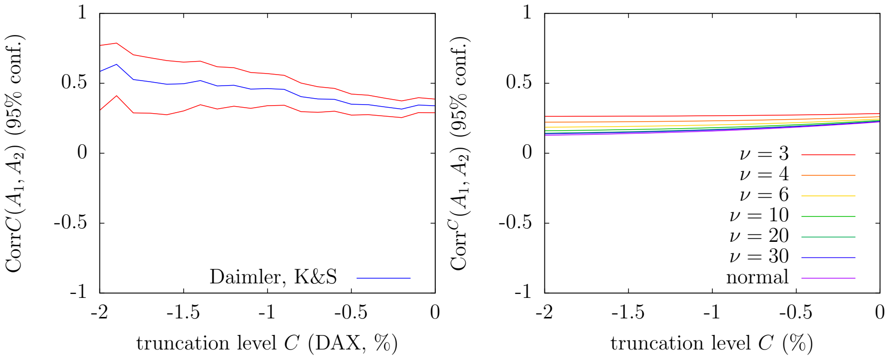
EMPIRICAL EXAMPLE (III)





- DAX data (Source: Reuters), daily log returns, 5.4.1991-6.11.2009
- 95% confidence intervals computed using Fisher z-transform

EMPIRICAL EXAMPLE (IV)



- DAX data (Source: Reuters), daily log returns, 5.4.1991-6.11.2009
- 95% confidence intervals computed using Fisher *z*-transform