On Correlation and Default Clustering in Credit Markets

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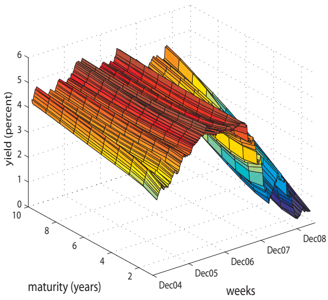
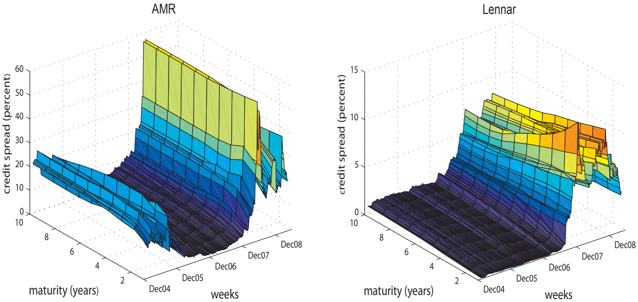
Panel A: Riskless Yield Curves



Panel B: Credit Spread Curves



Single-Name Credit Risk Pricing – What do we do?

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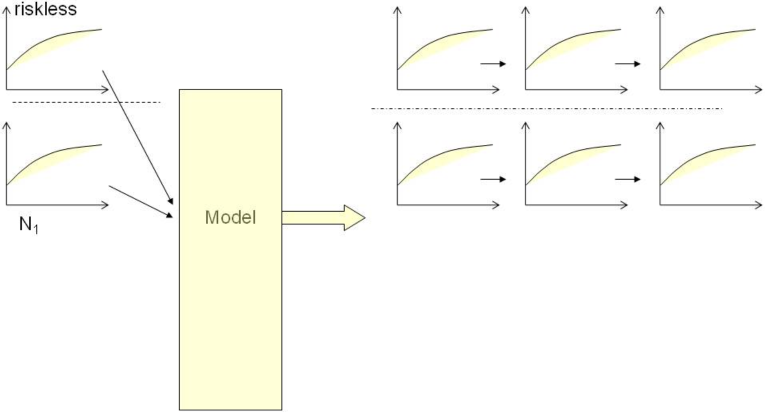
- allow different volatility structures for forward rates, that can be initialized to closely match empirical structures
- credit spreads and yield curves are represented by a finite set of state variables
- allow arbitrary interest rate-credit spread correlations
- permit shocks to the economy to impact riskless yield curves and credit spreads


riskless
N1
Model

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Extend single-name Markovian HJM models to

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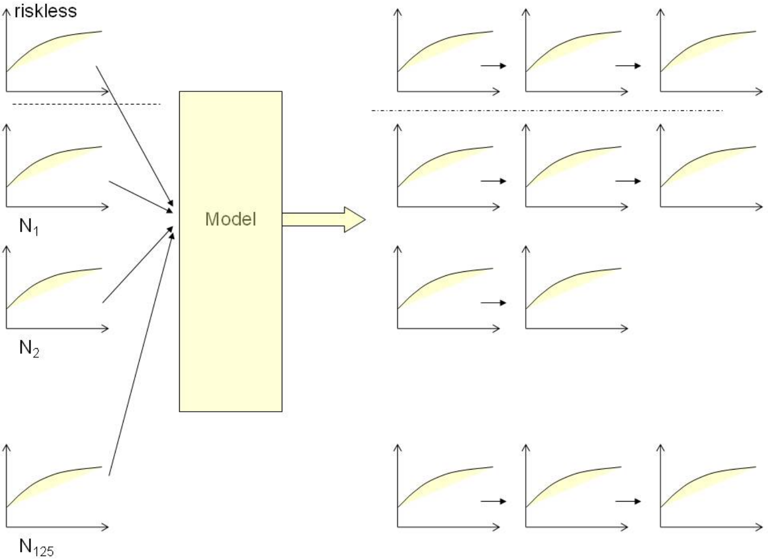
Using Kalman filter parameter estimates, we show the importance of

- interest rate-credit spread correlations
- default contagion
- the initial credit spread curve distribution


riskless
N1
N2
N125
Model

HJM Models: Riskless Dynamics

- Let $P(t, T)$ be the price at date t of a pure riskless discount bond that pays \$1 at date T:

$$P(t, T) \;=\; e^{-\int_t^T f(t,u)du},$$

where $f(t, u)$ represents the date-t forward rate for the future time increment $[u, u + dt]$.

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- We assume

$$df(t, T) = \mu_f(t, T)dt + \sigma_f(t, T)dz_f(t) + c_f(t, T)dN_f(t),$$

given $f(0, T)$. $N_f(t)$ is independent Poisson process with intensity η_f.

Dynamics of Riskless Bond Prices

Apply Ito's lemma for jump-diffusion processes to obtain

$$\frac{dP(t,T)}{P(t,T)} = \left(r(t) + \frac{1}{2}\sigma_p(t,T)\sigma_p'(t,T) - \int_t^T \mu_f(t,u)du \right) dt$$
$$-\sigma_p(t,T)\,dz_f(t) + \left(e^{-K_p(t,T)} - 1 \right) dN_f(t),$$

where

$$\sigma_p(t,T) = \int_t^T \sigma_f(t,u)\,du,$$
$$K_p(t,T) = \int_t^T c_f(t,u)\,du.$$

HJM Models: Risky Debt

- For firm A that has not defaulted prior to date t, we have

$$dY_A(t) = \begin{cases} 1 & \text{with probability} \quad \eta_A(X_t)dt \\ 0 & \text{with probability} \quad 1 - \eta_A(X_t)dt, \end{cases}$$

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- The date-t price of a bond issued by A is given by

$$\Pi_A(t, T) = V_A(t, T) 1_{\tau_A > t},$$

where

$$\begin{aligned} V_A(t, T) &= e^{-\int_t^T (f(t,u) + \lambda_A(t,u))\, du} \\ &= P(t, T)\, S_A(t, T). \end{aligned}$$

$\lambda_A(t) = \eta_A(t)\ell_A(t)$ is firm A's forward credit spread, $\eta_A(t)$ is the default arrival intensity, and $\ell_A(t)$ denotes LGD.

Credit Spreads Dynamics

We assume

$$d\lambda_A(t,T) \;=\; \mu_A(t,T)\,dt + \sigma_A(t,T)\,dz_A(t) + c_{fA}(t,T)\,dN_f(t),\ t \le \tau_A,$$

where

- correlation with diffusive riskless term structure:

$$E(dz_f(t)dz'_A(t)) = \Sigma^A_{m\times n}\,dt = \left(\rho^A_{ij}\right)\,dt$$

- a jump in riskless rates could transmit to shocks in the credit spreads
- $\sigma_A(t,T)$ is predictable
- $c_{fA}(t,T)$ is a deterministic function of time to maturity, $T-t$

Proposition 1: HJM Restrictions on the Drift Terms

No arbitrage implies

$$\begin{aligned}
\mu_f(t,T) &= \sigma_p(t,T)\sigma_f'(t,T) - c_f(t,T)e^{-K_p(t,T)}\eta_f \\
\mu_A(t,T) &= \sigma_{S_A}(t,T)\sigma_A'(t,T) + \sigma_f(t,T)\Sigma^A\sigma_{S_A}'(t,T) \\
&\quad +\sigma_p(t,T)\Sigma^A\sigma_A'(t,T) + g_A(t,T),
\end{aligned}$$

where

$$g_A(t,T) = \eta_f\left(c_f(t,T)e^{-K_p(t,T)} - (c_f(t,T) + c_{fA}(t,T))e^{-(K_p(t,T)+K_{fA}(t,T))}\right).$$

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Problem with HJM models:

- In general, the dynamics are not Markovian in a small number of state variables
- To overcome this issue, we curtail the volatility structures

Markovian HJM: Volatilities and Jump-Impact Factors

- **Volatilities** are given by

$$\begin{aligned} \sigma_{f_i}(t,T) &= h_{f_i}(t)\, e^{-\kappa_{f_i}(T-t)}, \\ \sigma_{A_j}(t,T) &= h_{A_j}(t)\, e^{-\kappa_{A_j}(T-t)}. \end{aligned}$$

where $h_{f_j}(t)$ and $h_{A_j}(t)$ are predictable functions.

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- Example: $h_f(t) = \min\left(|\tilde{h}_f(t)|, \bar{h}_f\right)$, where $\bar{h}_f$ is a large yet finite constant, and

$$\tilde{h}_f(t) = \sigma_f\, r(t)$$

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- **Jump-impact factors** are of the form

$$c_f(t,T) = c_f\, e^{-\gamma_f(T-t)},$$
$$c_{fA}(t,T) = c_{fA}\, e^{-\gamma_{fA}(T-t)}$$

Proposition 2: Markovian Models for Riskless Debt

Under volatility restrictions, we get exponential affine riskless bond prices:

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left(-\sum_{i=1}^{2}\sum_{j=1}^{m} H_{ij}(t,T)\psi_{ij}(t) - H_3(t,T)\psi_3(t) + H_J(t,T)\right),$$

where

$$\begin{aligned}
H_{1j}(t,T) &= 1/\kappa_{f_j}^2 \left(1 - e^{-\kappa_{f_j}(T-t)}\right), \quad \text{for } j = 1,\ldots,m \\
H_{2j}(t,T) &= -1/(2\kappa_{f_j}^2) \left(1 - e^{-2\kappa_{f_j}(T-t)}\right), \quad \text{for } j = 1,\ldots,m \\
H_3(t,T) &= c_f/\gamma_f \left(1 - e^{-\gamma_f(T-t)}\right), \\
H_J(t,T) &= \eta_f e^{-\frac{c_f}{\gamma_f}} \int_t^T \left(e^{\frac{c_f}{\gamma_f} e^{-\gamma_f(u-t)}} - e^{\frac{c_f}{\gamma_f} e^{-\gamma_f u}}\right) du.
\end{aligned}$$

The dynamics of the state variables are

$$\begin{aligned}
d\psi_{1j}(t) &= (h_{f_j}^2(t) - \kappa_{f_j}\psi_{1j}(t))dt + \kappa_{f_j} h_{f_j}(t) dz_{f_j}(t) \\
d\psi_{2j}(t) &= (h_{f_j}^2(t) - 2\kappa_{f_j}\psi_{2j}(t))dt \\
d\psi_3(t) &= -\gamma_f \psi_3(t)dt + dN_f(t).
\end{aligned}$$

Proposition 2: Markovian Models for Risky Debt

Under volatility restrictions, and assuming RMV, the risky bond price at t is $\Pi_A(t,T) = P(t,T)S_A(t,T)1_{\tau_A > t}$, where $S_A(t,T)$ is exponential affine:

$$
\begin{aligned}
S_A(t,T) \quad = \quad & \frac{S_A(0,T)}{S_A(0,t)} \exp[-A_0(t,T) - \sum_{j=1}^{n} (K_{0,j}(t,T)\xi_{0,j} - K_{1,j}(t,T)\xi_{1,j}) \\
& + \sum_{i=1}^{m} \sum_{j=1}^{n} (K_{2,ij}(t,T)\xi_{2,ij} - K_{3,ij}(t,T)\xi_{3,ij} - K_{4,ij}(t,T)\xi_{4,ij}) \\
& - K_5(t,T)\xi_5(t)].
\end{aligned}
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d\xi_{0,j}(t) &= (h^2_{A_j}(t) - \kappa_{A_j}\xi_{0j}(t))\,dt + \kappa_{A_j} h_{A_j}(t)\,dz_{A_j}(t) \\
d\xi_{1,j}(t) &= (h^2_{A_j}(t) - 2\kappa_{A_j}\xi_{1j}(t))\,dt \\
d\xi_{2,ij}(t) &= (h_{f_i}(t)h_{A_j}(t) - (\kappa_{A_j} + \kappa_{f_i})\xi_{2,ij}(t))\,dt \\
d\xi_{3,ij}(t) &= (h_{f_i}(t)h_{A_j}(t) - \kappa_{f_i}\xi_{3,ij}(t))\,dt \\
d\xi_{4,ij}(t) &= (h_{f_i}(t)h_{A_j}(t) - \kappa_{A_j}\xi_{4,ij}(t))\,dt \\
d\xi_5(t) &= -\gamma_{fA}\xi_5(t) + dN_f(t).
\end{aligned}
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Stochastic Drivers vs State Variables

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- Number of **state variables**: $3mn + 2(m+n+1)$

- Number of state variables can sometimes be reduced:
  - $m = n = 1$: 8
  - $m = n = 1$, no jumps and constant $h(\cdot)$ functions: 2
  - $m = n = 1$, no jumps and no correlations between interest rates and credit spreads: 4

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- Standard HJM:
  - Forward rates of $30 \times 12 = 360$ monthly interest rates and credit spreads need to be tracked.
  - As such, the model is Markovian in 720 state variables.
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- Markovian HJM:
  - A maximum of **8** state variables need to be maintained, no matter what the partition.

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- This can easily be accommodated in our multifactor models ($m > 1$, $n > 1$).
- Proposition 2 can be generalized to enable humped volatility structures even when $m = n = 1$.
- In fact, we are able to establish arbitrary shapes:

$$\sigma_f(t, T) \quad = \quad h_f(t) \sum_{j=1}^{k} a_j \, e^{-\kappa_j (T-t)}, \quad k > 1.$$

Relationship with Duffie-Kan Affine Models

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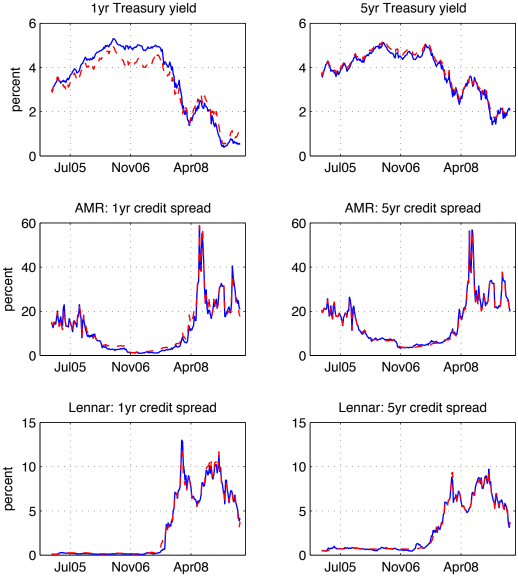
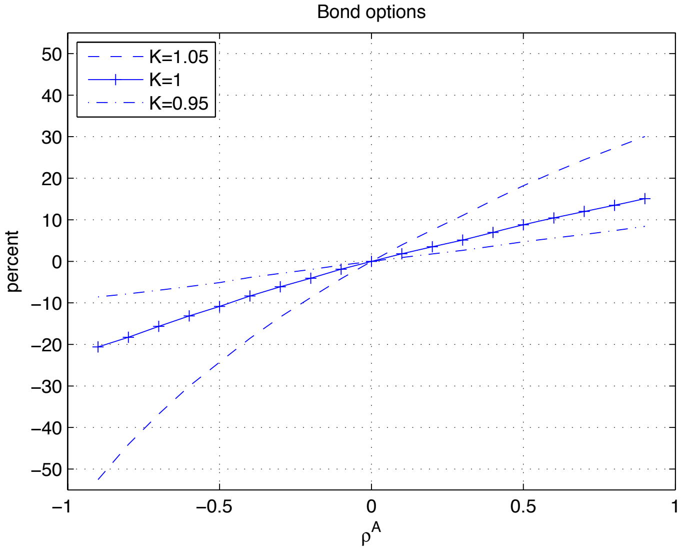
- As a result, the family of models we have established are very rich in structure, yet are easy to implement.
- In that sense, our analysis complements Duffie-Kan '96.

Empirical Evidence: Using Kalman Filter



Importance of Interest Rate-Credit Spread Correlations



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- The model already allows market-wide events to affect the riskless yield curve and credit spread curves.
- We now allow a primary firm's default to impact the credit spread of surviving (secondary) firms.
- Examples: Secondary firm
  - could carry significant debt of the primary firm,
  - may sell much of its goods to a primary firm,
  - may be in competition with the primary firm.

Credit Spread Dynamics: Secondary Firms

We assume

$$
\begin{aligned}
d\lambda_B(t,T) &= \mu_B(t,T)dt + \sigma_B(t,T)dz_B(t) + c_{fB}(t,T)dN_f(t) \\
&\quad + \sum_{i=1}^{m_B} c_{A_iB}(1 - Y_{A_i}(t))dY_{A_i}(t), \ \forall t \le \tau_B,
\end{aligned}
$$

where

- correlation with diffusive riskless term structure:

$$E(dz_f(t)dz'_B(t)) = \Sigma^B_{m\times n}\,dt = \left(\rho^B_{ij}\right)\,dt$$

- correlation with firm A's diffusive term:

$$E(dz_A(t)dz'_B(t)) = \Sigma^{AB}_{n\times n}\,dt = \left(\rho^{AB}_{ij}\right)\,dt$$

- volatility structures are curtailed

Proposition 3: Pricing Risky Debt of Secondary Firms

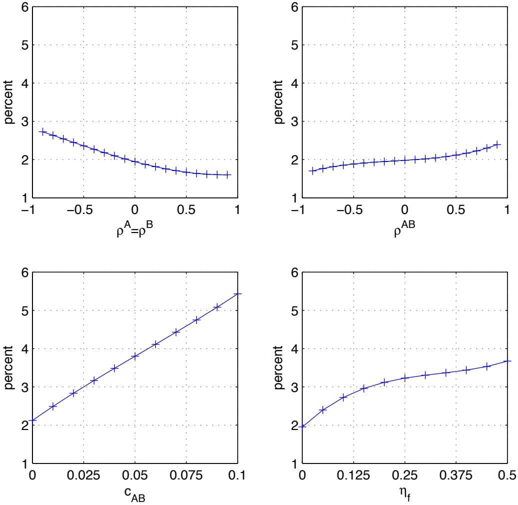
The price of a risky bond issued by a secondary firm is given by $\Pi_B(t,T) = V_B(t,T)1_{\tau_B > t}$, where $V_B(t,T) = P(t,T)S_B(t,T)$ and

$$
\begin{aligned}
S_B(t,T) &= \frac{S_B(0,T)}{S_B(0,t)} e^{-B_0(t,T) - \sum_{j=1}^{n}(K^B_{0,j}(t,T)\xi^B_{0,j} - K^B_{1,j}(t,T)\xi^B_{1,j})} \\
&\quad \times e^{\sum_{i=1}^{m}\sum_{j=1}^{n}\left(K^B_{2,ij}(t,T)\xi^B_{2,ij} - K^B_{3,ij}(t,T)\xi^B_{3,ij} - K^B_{4,ij}(t,T)\xi^B_{4,ij}\right)} \\
&\quad \times e^{-K^B_5(t,T)\xi^B_5(t)} \\
&\quad \times e^{\sum_{i=1}^{m_B}\left(\left(1 - e^{-c_{A_iB}(T-t)}\right)U_{A_iB}(t) - c_{A_iB}(T-t)Y_{A_i}(t)\right)}.
\end{aligned}
$$

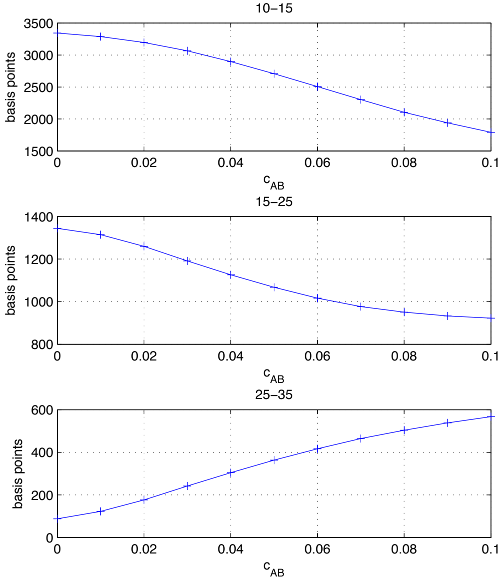
Here, $B_0(t,T) = \int_t^T \int_0^t g_B(v,u)\,dv\,du$. The K^B coefficients and ξ^B state variables are defined as in Proposition 2, and

$$U_{A_iB}(t) = \int_0^{t\wedge\tau_{A_i}} \eta_{A_i}(u)e^{-c_{A_iB}(t-u)}du, \quad \text{for } i = 1, \dots, m_B.$$

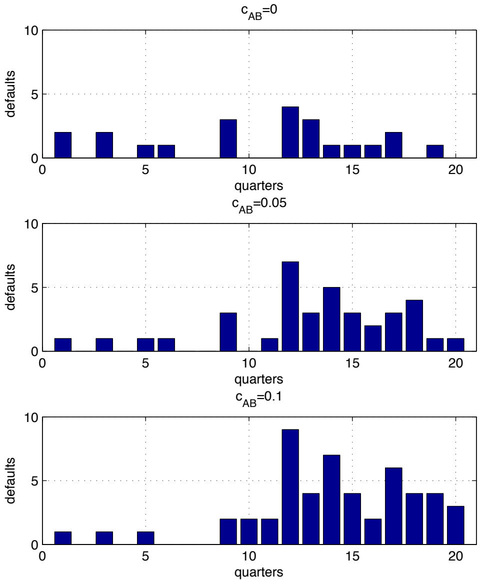
Importance of Default Contagion: Counterparty Risk in Insurance Contracts



Importance of Default Contagion: CDS Index Tranches



Generating Default Clustering



Importance of the Initial Credit Spread Curve Distr.

Distribution of initial credit spread curves	Tranche spreads		
	10–15	15–25	25–35
$\lambda(0,t) = 0.05$	3451	1450	111
$\lambda(0,0) \sim$ Uniform$(0.025, 0.075)$ and $\lambda(0,t) = \lambda(0,0)$	3523	1528	128
$\lambda(0,0) \sim$ Uniform$(0, 0.1)$ and $\lambda(0,t) = \lambda(0,0)$	3528	1528	121
$\lambda(0,t)$ incr from $\lambda(0,t) = 0.0125$ to $\lambda(0,t) = 0.0875$	2698	1340	107
$\lambda(0,t)$ incr from $\lambda(0,t) = 0.025$ to $\lambda(0,t) = 0.075$	2883	1366	108
$\lambda(0,t)$ incr from $\lambda(0,t) = 0.0375$ to $\lambda(0,t) = 0.0625$	3130	1405	110
$\lambda(0,t) = 0.05$	3451	1450	111
$\lambda(0,t)$ decr from $\lambda(0,t) = 0.0625$ to $\lambda(0,t) = 0.0375$	3857	1514	115
$\lambda(0,t)$ decr from $\lambda(0,t) = 0.075$ to $\lambda(0,t) = 0.025$	4345	1601	120
$\lambda(0,t)$ decr from $\lambda(0,t) = 0.0875$ to $\lambda(0,t) = 0.0125$	4947	1772	141

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- Fairly easy to implement
- Models have exponentially affine representations for riskless and risky bond prices
  - Yet the variance structures need not be affine
  - The number of state variables is decoupled from the number of stochastic drivers
  - Allow flexible specification of correlations between interest rates and credit spreads
  - Permit default clustering through a variety of channels (diffusive correlations, jumps, contagion effects)