Optimal Portfolio Selection under Disappointment Averse Utility

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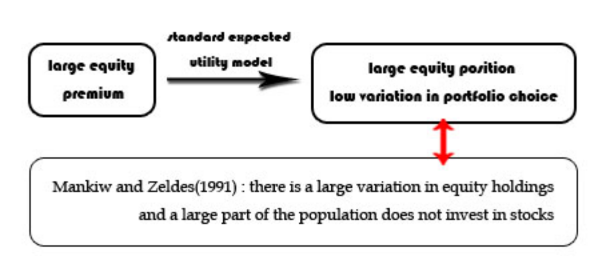
Introduction



One Solution is :

The heterogeneous preferences, especially **Gul (1991)'s disappointment averse utility.**

Utility Functions

We can define a Utility Function by assuming (1) a *certainty equivalent* and (2) a *smooth utility function*.

- Certainty Equivalent $\mu(F)$: the value which, if received with certainty, would be indifferent to receive random outcome with distribution F.

$$\mathrm{Value}(\delta_{\mu(F)}) = \mathrm{Value}(F)$$

  where $\delta_{\mu(F)}(z) = \mathbf{I}_{\{z \geq \mu(F)\}}$ for $z \in \mathbb{R}$.

- Smooth Utility Function u : a real-valued function of the certainty equivalent $\mu(F) \in [0, \infty)$.

Expected Utility

u is strictly increasing, strictly concave, continuously differentiable and $u'(0+) = \infty, u'(\infty) = 0$, and $\mu(X)$ is defined by

$$u(\mu(X)) = E\left[u(X)\right]$$

for any random variable X.

Gul's Disappointment Averse Utility

For $u : (0, \infty) \to [0, \infty)$, *a certainty equivalent* μ *associated with* u is implicitly defined by

$$u(\mu(X)) = E\left[u(X) + (\frac{1}{A} - 1)(u(X) - u(\mu(X)))\mathrm{I}_{\{X<\mu(X)\}}\right]$$

Here, $A \in (0, 1]$ is the *disappointment aversion coefficient*.

Disappointment Averse Utility

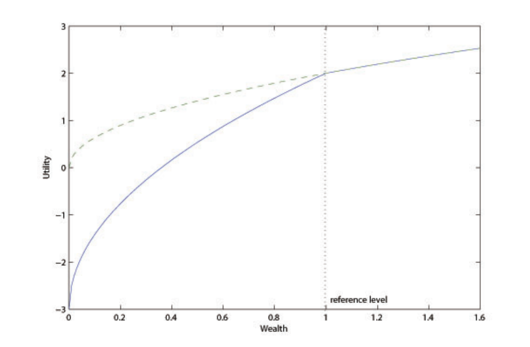


Figure: **Utility functions for wealth level**. The solid line is for the disappointment averse utility with $A = 0.4, \theta = 1$ and $u(x) = \frac{1}{0.5}x^{0.5}$ and the dashed line is for the smooth utility $u(x) = \frac{1}{0.5}x^{0.5}$.

DA Utility vs Loss-Averse(LA) Utility

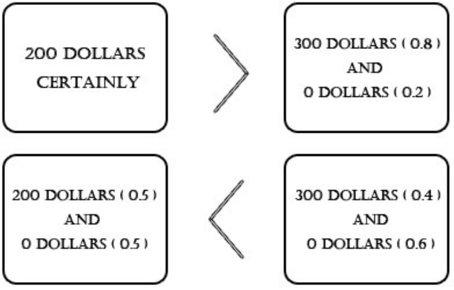
Loss Averse Utility :

- *Endogenous* Certainty Equivalent(DA) vs *Exogenous* Reference Level(LA)
- Kahneman and Tversky (1979), Berkelaar *et al.* (2004), Gomes (2005)

- What is the Reference Level for LA Utility?
- Ang *et al.* (2005) : An optimal portfolio may not exist with LA utility.
- DA utility is derived from the Theoretical Axioms (Betweenness Axiom), while LA utility is not.

Why DA Utility ?

- Allais Paradox of the standard expected utility model

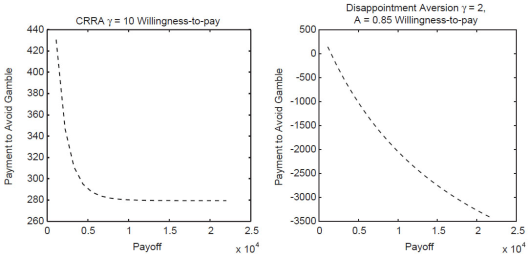


The disappointment averse utility is consistent with Allais type behavior from replacing the independence axiom by a weaker axiom called the betweenness axiom.

Why DA Utility ?

- Rabin's Gamble Problem (Ang et al.(2005))

Reject (-100/+110) gamble, then Reject (-1000/+any amount) gamble.



- Empirical Evidence based on market data :

Epstein and Zin(2001), Camerer and Ho(1994), Morone and Schmidt(2008)

History

- Ang, Bekaert and Liu(2005)
  - Find solutions of both static and dynamic portfolio choice under DA utility associated with the constant relative risk averse utility (CRRA utility)
  - Provide the explicit form of non-participation region of investors.
- Epstein and Zin(2001)
  - Consider an infinite time horizon asset pricing model with recursive framework of DA utility.
  - Using actual market data, find that DA utility provides a substantial improvement in the empirical performance of a representative agent.
- However, in my knowledge, no literature give a treatment of portfolio choice problem in the continuous time economy.

The Purpose

Optimal Portfolio Choice Problem

Provide an analytic method to solve the optimal portfolio choice problem when a disappointment averse investor want to maximize her utility on terminal wealth.

Implications

The numerical results for a special utility function, for example, the disappointment averse utility function associated with the constant relative risk averse(CRRA) utility.

- lower investment on a risky asset which partially explain the portfolio puzzle.
- the change of strategies among the time horizon.

Economy

- Complete market during finite time horizon $[0, T]$.
- risk-free asset, $S_0(t)$ and risky asset, $S(t)$

$$\begin{cases} dS_0(t) &= rS_0(t)dt \\ dS(t) &= \mu S(t)dt + \sigma S(t)dB_t \end{cases}$$

- $\mu > r > 0, \sigma > 0$.
- $\gamma = \dfrac{\mu - r}{\sigma}$: the market price of risk
- $H_t = \exp\left\{-(r + \frac{1}{2}\gamma^2)t - \gamma B_t\right\}$: the pricing kernel process
- π_t : proportion of wealth invested in risky asset
- X_t : wealth process of the investor with initial wealth $X_0 = x > 0$

$$dX_t = rX_t dt + (\mu - r)\pi_t X_t dt + \sigma \pi_t X_t dB_t$$

General Problem

Problem

When an investor has the utility $u(x)$ on $\mu(X_T)$ which is the certainty equivalent of her terminal wealth X_T, the optimal investment problem is to find the admissible portfolio strategy π_t such that

$$\max_{\pi_t} \quad u(\mu(X_T))$$

subject to

$$dX_t = rX_t dt + (\mu - r)\pi_t X_t dt + \sigma \pi_t X_t dB_t, \quad X_0 = x$$
$$X(t) \geq 0, \quad \text{for all } t \in [0, T]$$

Expected Utility Case

Using the Martingale Method, we can derive the formulas for optimal wealth process and optimal portfolio process for our problem.

Solution for Expected Utility case

Let $I(y)$ be the inverse of the marginal utility function $u'(x)$ and λ be the unique solution of

$$E\left[H_T I(\lambda H_T)\right] = x$$

The optimal terminal wealth and intermediate wealth are

$$X_T^* = I(\lambda H_T), \quad X_t^* = \frac{1}{H_t} E\left[H_T I(\lambda H_T) | \mathcal{F}_t\right]$$

if there exists a portfolio process π^* satisfying that $X_t^{x,\pi^*} = X_t^*$.

A sufficient condition to exist π^* is provided by Karatzas and Wang(2001).

Disappointment Averse Utility Case

$$\max_{\pi_t} u(\mu(X)) = \max_{\pi_t} E\left[u(X) + (\frac{1}{A} - 1)(u(X) - u(\mu(X)))\mathrm{I}_{\{X<\mu(X)\}}\right]$$

Step 1

Solve the following problem given $\mu(X_T) = \theta$,

$$\max_{\pi_t} \quad v(x, \pi; \theta) = \max_{\pi_t} \quad E\left[u(X_T) + (\frac{1}{A} - 1)(u(X_T) - u(\theta))\mathrm{I}_{\{X_T<\theta\}}\right]$$

subject to

$$dX_t = rX_t dt + (\mu - r)\pi_t X_t dt + \sigma\pi_t X_t dB_t, \quad X_0 = x$$
$$X_t \geq 0, \quad \text{for all } t \in [0, T]$$

By applying the Martingale Method, we can solve this step.

Disappointment Averse Utility Case

Step2

Let $X_t^*(\theta)$ and $\pi_t^*(\theta)$ be the optimal wealth process and the optimal portfolio process when $\mu(X_T) = \theta$ is given. Find the fixed point θ, in that the solution of the equation

$$u(\theta) = v(x, \pi^*(\theta); \theta) = E\left[u(X_T^*(\theta)) + (\frac{1}{A} - 1)(u(X_T^*(\theta)) - u(\theta))\mathrm{I}_{\{X_T^*(\theta) < \theta\}}\right]$$

Example : DA Utility associated with CRRA

- The CRRA utility function is defined by for any $x \in [0, \infty)$,

$$u(x) := \frac{1}{\alpha} x^{\alpha} \quad \text{if } \alpha \leq 1 \text{ and } \alpha \neq 0$$

where $(1 - \alpha)$ is the relative risk aversion of an investor.

- The DA utility associated CRRA utility
The certainty equivalent is defined implicitly by

$$\frac{1}{\alpha}\mu(X)^{\alpha} = E\left[\frac{1}{\alpha}X^{\alpha} + (\frac{1}{A} - 1)(\frac{1}{\alpha}X^{\alpha} - \frac{1}{\alpha}\mu(X)^{\alpha})\mathrm{I}_{\{X<\mu(X)\}}\right]$$

where $A \in (0, 1]$ is her disappointment aversion coefficient. The utility of the investor on her wealth X is $\frac{1}{\alpha}\mu(X)^{\alpha}$.

CRRA Utility

Theorem 4.1 (Optimal Portfolio and Optimal Wealth)

When an investor has CRRA utility with risk aversion $(1-\alpha)$ and her initial endowment is $x > 0$, her optimal portfolio is

$$\pi_t^* = \frac{\mu - r}{\sigma^2(1-\alpha)}, \quad \text{for } t \in [0, T]$$

and her optimal wealth process is

$$X_t^{x,\pi^*} = xH_t^{\frac{1}{\alpha-1}} \exp\{\frac{\alpha}{\alpha-1}(r - \frac{\gamma^2}{2(\alpha-1)})t\}, \quad \text{for } t \in [0, T]$$

DA Utility associated with CRRA Utility

Theorem 4.2 (Optimal Terminal Wealth)

Under the given value θ for the certainty equivalent of terminal wealth, the optimal wealth at terminal time T corresponding to θ is

$$X_T^{x,\pi^*(\theta)} = \begin{cases} (\lambda^*(\theta)H_T)^{\frac{1}{\alpha-1}} & \text{if } \lambda^*(\theta)H_T < \theta^{\alpha-1} \\ \theta & \text{if } \theta^{\alpha-1} \leq \lambda^*(\theta)H_T \leq \dfrac{1}{A}\theta^{\alpha-1} \\ (A\lambda^*(\theta)H_T)^{\frac{1}{\alpha-1}} & \text{if } \dfrac{1}{A}\lambda^*(\theta)H_T < \theta^{\alpha-1} \end{cases}$$

Theorem 4.2 - Continue

where $\lambda^*(\theta)$ is the unique solution of the following equation.

$$
\begin{aligned}
x &= \theta e^{-rT}\Big(N\big(\frac{1}{\gamma\sqrt{T}}\ln(\lambda)+\gamma\sqrt{T}+B_1(\theta)\big)-N\big(\frac{1}{\gamma\sqrt{T}}\ln(\lambda)+\gamma\sqrt{T}+B_2(\theta)\big)\Big)\\
&+ \lambda^{\frac{1}{\alpha-1}}\exp\{-\frac{\alpha}{\alpha-1}(r-\frac{\gamma^2}{2(\alpha-1)})T\}N\big(-\frac{\alpha\gamma\sqrt{T}}{\alpha-1}-\frac{1}{\gamma\sqrt{T}}\ln(\lambda)-B_1(\theta)\big)\\
&+ (A\lambda)^{\frac{1}{\alpha-1}}\exp\{-\frac{\alpha}{\alpha-1}(r-\frac{\gamma^2}{2(\alpha-1)})T\}N\big(\frac{\alpha\gamma\sqrt{T}}{\alpha-1}+\frac{1}{\gamma\sqrt{T}}\ln(\lambda)+B_2(\theta)\big)
\end{aligned}
$$

and

$$
B_1(\theta)=-\frac{1}{\gamma\sqrt{T}}\big((r+\frac{1}{2}\gamma^2)T+(\alpha-1)\ln(\theta)\big),\quad B_2(\theta)=B_1(\theta)-\ln(A)
$$

Theorem 4.3 (Optimal Wealth Process)

Under same assumptions of Theorem 4.2, the investor's optimal wealth at time $t \in [0, T)$ is

$$\begin{aligned} X_t^{x,\pi^*(\theta)} &= \theta e^{-r(T-t)} \left(N(\frac{K_1}{\gamma\sqrt{T-t}} + \gamma\sqrt{T-t}) - N(\frac{K_2}{\gamma\sqrt{T-t}} + \gamma\sqrt{T-t}) \right) \\ &+ (\lambda^*(\theta)H_t)^{\frac{1}{\alpha-1}} e^{-\frac{\alpha}{\alpha-1}(r-\frac{\gamma^2}{2(\alpha-1)})(T-t)} N(-\frac{\alpha\gamma}{\alpha-1}\sqrt{T-t} - \frac{K_1}{\gamma\sqrt{T-t}}) \\ &+ (A\lambda^*(\theta)H_t)^{\frac{1}{\alpha-1}} e^{-\frac{\alpha}{\alpha-1}(r-\frac{\gamma^2}{2(\alpha-1)})(T-t)} N(\frac{\alpha\gamma}{\alpha-1}\sqrt{T-t} + \frac{K_2}{\gamma\sqrt{T-t}}) \end{aligned}$$

where

$$K_1 = \ln H_t + \ln \lambda^*(\theta) - (\alpha - 1)\ln\theta - (r + \frac{1}{2}\gamma^2)(T-t), \quad K_2 = K_1 + \ln A$$

Theorem 4.3 (Optimal Prtfolio Process)

Under same assumptions of Theorem 4.2, her optimal portfolio weight at time t is

$$\pi_t^*(\theta) = \frac{1}{\sigma X_t^{x,\pi^*(\theta)}} \Bigg[\frac{\gamma}{1-\alpha} (\lambda^*(\theta) H_t)^{\frac{1}{\alpha-1}} e^{-\frac{\alpha}{\alpha-1}(r - \frac{\gamma^2}{2(\alpha-1)})(T-t)}$$
$$\times \left(N(-\frac{K_1}{\gamma\sqrt{T-t}} - \frac{\alpha\gamma}{\alpha-1}\sqrt{T-t}) + A^{\frac{1}{\alpha-1}} N(\frac{K_2}{\gamma\sqrt{T-t}} + \frac{\alpha\gamma}{\alpha-1}\sqrt{T-t}) \right)$$
$$- \frac{\theta}{\sqrt{2\pi(T-t)}} e^{-(r+\frac{1}{2}\gamma^2)(T-t)} (e^{-K_1 - \frac{K_1^2}{2\gamma^2(T-t)}} - e^{-K_2 - \frac{K_2^2}{2\gamma^2(T-t)}})$$
$$+ \frac{(\lambda^*(\theta) H_t)^{\frac{1}{\alpha-1}}}{\sqrt{2\pi(T-t)}} e^{-\frac{\alpha}{\alpha-1}(r+\frac{1}{2}\gamma^2)(T-t)} (e^{-\frac{\alpha}{\alpha-1}K_1 - \frac{K_1^2}{2\gamma^2(T-t)}} - A^{\frac{1}{\alpha-1}} e^{-\frac{\alpha}{\alpha-1}K_2 - \frac{K_2^2}{2\gamma^2(T-t)}}) \Bigg]$$

Theorem 4.4 (Level of the Certainty Equivalent)

Let $\overline{\theta}$ be a constant defined by $\overline{\theta} = x\exp\{(r - \frac{\gamma^2}{2(\alpha-1)})T\}$. The reference rate, $\mu(X_T^{x,\pi^*})$, is a solution in $(0, \overline{\theta})$ of the following equation.

$$
\begin{aligned}
\theta^\alpha = \theta^\alpha \Big(& N(\frac{1}{\gamma\sqrt{T}}\ln(\lambda^*(\theta)) + B_1(\theta)) - \frac{1}{A}N(\frac{1}{\gamma\sqrt{T}}\ln(\lambda^*(\theta)) + B_2(\theta))\Big) \\
& + (\lambda^*(\theta))^{\frac{\alpha}{\alpha-1}} e^{-\frac{\alpha}{\alpha-1}(r-\frac{\gamma^2}{2(\alpha-1)})T} N(-\frac{\alpha\gamma\sqrt{T}}{\alpha-1} - \frac{1}{\gamma\sqrt{T}}\ln(\lambda^*(\theta)) - B_1(\theta)) \\
& + A^{\frac{1}{\alpha-1}} (\lambda^*(\theta))^{\frac{\alpha}{\alpha-1}} e^{-\frac{\alpha}{\alpha-1}(r-\frac{\gamma^2}{2(\alpha-1)})T} N(\frac{\alpha\gamma\sqrt{T}}{\alpha-1} + \frac{1}{\gamma\sqrt{T}}\ln(\lambda^*(\theta)) + B_2(\theta))
\end{aligned}
$$

Moreover, the optimal wealth and portfolio weight are obtained by substituting θ by the solution $\mu(X_T^{x,\pi^*})$.

Results - Optimal Terminal Wealth

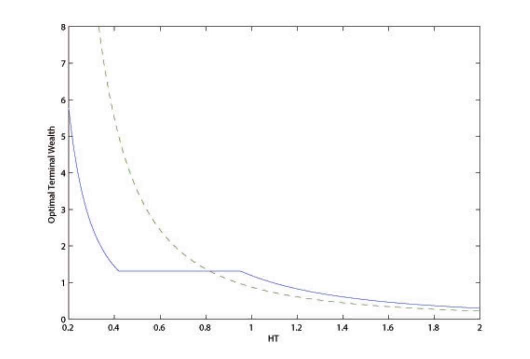


Figure: **The optimal terminal wealth among the realized value of pricing kernel** H_T. The solid line is for a disappointment averse investor with $A = 0.44$ and $\alpha = 0.5$ and the dashed line is for a CRRA type investor with $\alpha = 0.5$. The parameters are $r = 0.0408$, $\mu = 0.1063$, and $\sigma = 0.2193$. Investor's initial wealth is $x = 1$ and length of life time is $T = 1$.

Results - Optimal Intermediate Wealth

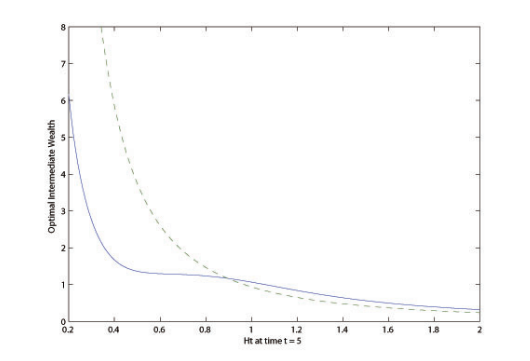


Figure: **The optimal intermediate wealth among H_t at time $t = 0.5$.** The solid line is for a disappointment averse investor with $A = 0.44$ and $\alpha = 0.5$ and the dashed line is for a CRRA type investor with $\alpha = 0.5$. The parameters are assumed as same as in figure 2.

Results - Optimal Portfolio Amount

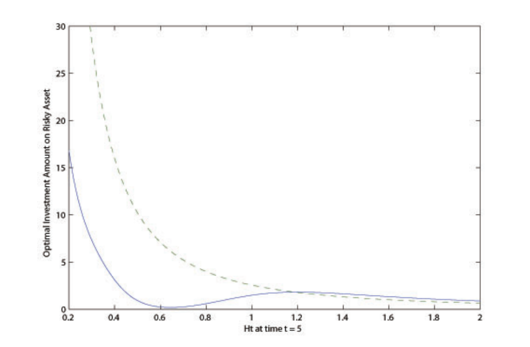


Figure: **The optimal portfolio amount among H_t at time** $t = 0.5$. The solid line is for a disappointment averse investor with $A = 0.44$ and $\alpha = 0.5$ and the dashed line is for a CRRA type investor with $\alpha = 0.5$. The parameters are assumed as same as in figure 2.

Results - Optimal Portfolio Weight

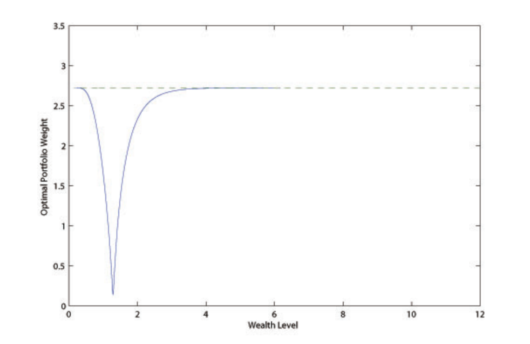


Figure: **The optimal portfolio weight among investor's wealth level at time** $t = 0.5$. The solid line is for a disappointment averse investor with $A = 0.44$ and $\alpha = 0.5$ and the dashed line is for a CRRA type investor with $\alpha = 0.5$. The parameters are assumed as same as in figure 2.

Results - Reference Level (Certainty Equivalent)

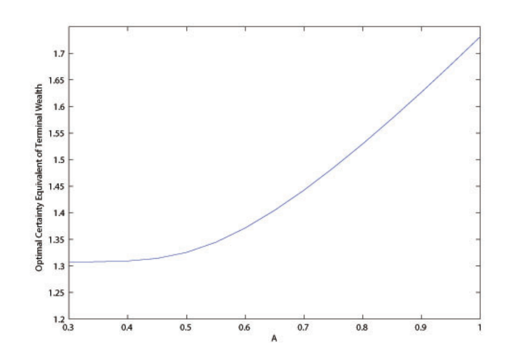


Figure: **The reference level $\mu(X_T^*)$ among disappointment aversion coefficient A.** We assume that all parameters except for A are as same as in figure 2.

Results - Portfolio Weight among DA

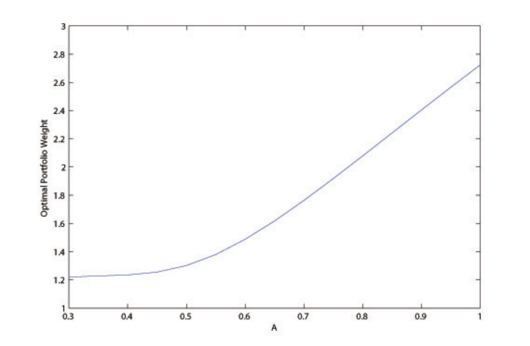


Figure: **The optimal initial portfolio wight among disappointment aversion coefficient** A. Initial wealth level is $x = 1$ and all parameters except for A are assumed as same as in figure 2.

Results - Portfolio Weight among Time Horizon

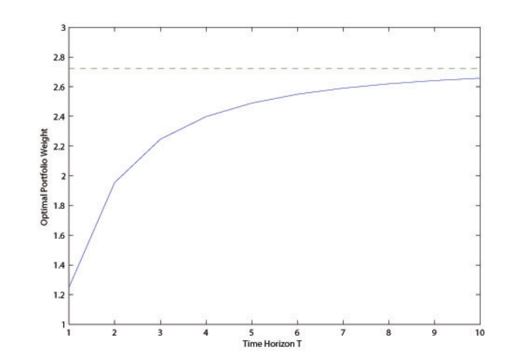


Figure: **The optimal initial portfolio weight among the length of life time T.** The disappointment aversion coefficient is $A = 0.44$ and relative risk aversion is $\alpha = 0.5$. The parameters are $r = 0.0408$, $\mu = 0.1063$, and $\sigma = 0.2193$. Investor's initial wealth is $x = 1$.

Conclusion

- We provide an analytic method to solve the optimal investment problem for disappointment averse investors in a continuous time economy.
- The portfolio weight invested on a risk asset for a disappointment averse investor is lower than the portfolio weight for an investor who has the standard expected utility.
- The portfolio weight under the disappointment aversion model is changed among the time horizon, while it is constant under the CRRA utility model.