Investment, Income, and Incompleteness

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Motivation

- Apart from financial wealth, human wealth is a dominant asset for most individuals and households
- Labor income is typically not spanned by financial assets and insurance contracts offered by governments and insurance companies are far from perfect

$\hookrightarrow$ It seems impossible to find closed-form expressions for the strategies maximizing the life-time utility of an investor

Contributions

- Consideration of a continuous time life-cycle optimization problem of an investor receiving uncertain and unspanned labor income until retirement
- Suggestion of an easy procedure for finding a simple consumption and investment strategy which is near-optimal
- Testing the strategy and checking the robustness of the results
- Extension of the model to endogenous labor supply

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Financial Assets

- Available assets: bank account with constant risk-free interest rate r and a single stock
- Bank account

$$dM_t = M_t r \, dt$$

- Stock

$$dS_t = S_t \left[(r + \sigma_S \lambda_S) \, dt + \sigma_S \, dW_t \right]$$

- $W = (W_t)$ standard Brownian motion
- For simplicity, let λ_S, σ_S be constants

Income

- Exogenously given labor income rate until retirement date $\tilde{T}$

$$dY_t = Y_t \left[\alpha \, dt + \beta \left(\rho \, dW_t + \sqrt{1-\rho^2} \, d\tilde{W}_t\right)\right], \quad 0 \leq t \leq \tilde{T}$$

- $\tilde{W} = (\tilde{W}_t)$ another Brownian motion, independent of W
- Assume α, β, ρ to be constants

Wealth

- Choice of consumption strategy $c = (c_t)$ and investment strategy $\pi_S = (\pi_{St})$
- Financial wealth at time t: X_t

$$dX_t = X_t \left[(r + \pi_{St}\sigma_S\lambda_S)\, dt + \pi_{St}\sigma_S\, dW_t \right] + \left(\mathbf{1}_{\{t \le \tilde{T}\}} Y_t - c_t \right) dt$$

- Strategy (c, π_S) admissible, if it is adapted and $X_T \geq 0$

Optimization Problem of the Investor

- An admissible strategy generates the expected utility

$$J(t,x,y;c,\pi_S) = \mathrm{E}_t\left[\int_t^T e^{-\delta(s-t)}U(c_s)\,ds + \varepsilon e^{-\delta(T-t)}U(X_T)\right]$$

- δ: subjective time preference rate; conditioned on $X_t = x$ and $Y_t = y$

Indirect Utility

The indirect utility function is given by

$$J(t,x,y) = \max_{(c,\pi_S)\in\mathcal{A}_t} J(t,x,y;c,\pi_S)$$

Utility function of CRRA type with $\gamma > 1$

Main Problem

- Assumption: income is spanned, i.e. $|\rho| = 1$

$\hookrightarrow$ indirect utility function is given by

$$J^{\text{com}}(t, x, y) = \frac{1}{1-\gamma}(g^{\text{com}}(t))^{\gamma}(x + yF^{\text{com}}(t))^{1-\gamma} \qquad (1)$$

- A separation like (1) does not hold in the incomplete market
- Resort to numerical methods

A Way out of this Problem

- Karatzas, Lehoczky, Shreve, and Xu (1991) and Cvitanić and Karatzas (1992):
  Solution to the incomplete market identical to the least favorable of solutions in artificially completed markets

Our Approach

1. Augment the market by adding an additional asset
2. Look at this subset of artificially completed markets where simple closed-form solutions exist
3. By ignoring the investment in the hypothetical asset, we obtain strategies in the true incomplete market
4. Utility maximization over this family of strategies

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Completing the Market: Shiller Contract

- Until $\tilde{T}$ the individual can trade in a hypothetical asset I_t:

$$dI_t = I_t\left[(r + \lambda_I)\,dt + d\tilde{W}_t\right]$$

- Market price of risk λ_I ⇒ family of complete markets
- Fraction of wealth invested in Shiller contract: π_{It}

↪ Change in wealth dynamics

$$\begin{aligned} dX_t =& X_t\Big[\Big(r + \pi_{St}\sigma_S\lambda_S + \mathbf{1}_{\{t\le\tilde{T}\}}\pi_{It}\lambda_I\Big)\,dt \\ &+ \pi_{St}\sigma_S\,dW_t + \mathbf{1}_{\{t\le\tilde{T}\}}\pi_{It}\,d\tilde{W}_t\Big] + \Big(\mathbf{1}_{\{t\le\tilde{T}\}}Y_t - c_t\Big)\,dt \end{aligned}$$

↪ Change in indirect utility

$$J^{\text{art}}(t,x,y;\lambda_I) = \max_{(c,\pi_S,\pi_I)} J(t,x,y;c,\pi_S,\pi_I)$$

Solution with Shiller Contracts

Theorem

If the investor has access to Shiller contracts with constant λ_I until retirement, then his indirect utility is given by

$$J^{\text{art}}(t,x,y;\lambda_I) = \frac{1}{1-\gamma} g^{\text{art}}(t;\lambda_I)^{\gamma}(x + yF^{\text{art}}(t;\lambda_I))^{1-\gamma}.$$

Fraction of Wealth optimally invested

$$\pi_{St}^{\text{art}} = \frac{\lambda_S}{\gamma\sigma_S}\frac{X_t + Y_t F^{\text{art}}(t;\lambda_I)}{X_t} - \frac{\beta\rho}{\sigma_S}\frac{Y_t F^{\text{art}}(t;\lambda_I)}{X_t}$$

Transform π_S:

$$\pi_{St}^{\text{art}} = \frac{\lambda_S}{\gamma\sigma_S} + \left(\frac{\lambda_S}{\gamma\sigma_S} - \frac{\beta\rho}{\sigma_S}\right)\frac{Y_t F^{\text{art}}(t;\lambda_I)}{X_t}$$

Bounds on Utilities

- For the moment only constant λ_I
- For any choice of λ_I:

$$J(t,x,y) \leq J^{\text{art}}(t,x,y;\lambda_I)$$

- Find $\bar{\lambda}_I = \arg\min_{\lambda_I} J^{\text{art}}(t,x,y;\lambda_I) \rightarrow$ upper bound for the incomplete market $\bar{J}(t,x,y) := J^{\text{art}}(t,x,y;\bar{\lambda}_I)$
- Performance of any admissible strategy in the incomplete market via percentage wealth loss L

$$J(t,x,y;c,\pi_S) = \bar{J}(t,x[1-L],y[1-L])$$

An Admissible Strategy

- Take investment and consumption strategy (c^{art}, π_S^{art}) from the artificially completed market and disregard the investment in Shiller contract I
- To assure an admissible strategy, we need to modify the strategies

Strategies

$$c_t(\lambda_I) = \frac{X_t + \mathbf{1}_{\{X_t > k\}} Y_t F^{\text{art}}(t; \lambda_I)}{g^{\text{art}}(t; \lambda_I)}$$

$$\pi_{St}(\lambda_I) = \frac{\lambda_S}{\gamma \sigma_S} \frac{X_t + \mathbf{1}_{\{X_t > k\}} Y_t F^{\text{art}}(t; \lambda_I)}{X_t} - \mathbf{1}_{\{X_t > k\}} \frac{\beta \rho}{\sigma_S} \frac{Y_t F^{\text{art}}(t; \lambda_I)}{X_t}$$

Expected Utility and Welfare Loss

- For any given λ_I, we can compute the expected utility $J(t, x, y; c(\lambda_I), \pi_S(\lambda_I))$ by MC simulation of the processes X and Y (only until $\tilde{T}$)
- Maximize over λ_I:

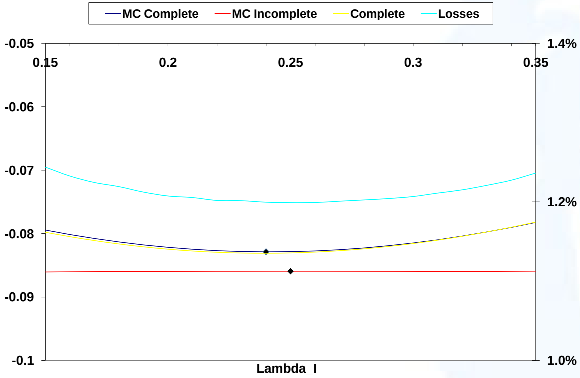
$$\hat{\lambda}_I = \arg\max_{\lambda_I} J(t, x, y; c(\lambda_I), \pi_S(\lambda_I))$$

$\hookrightarrow$

$$\left(c(\hat{\lambda}_I), \pi_S(\hat{\lambda}_I)\right) \to \hat{J}(t, x, y) \equiv J(t, x, y; \hat{c}, \hat{\pi}_S)$$

- Unknown optimal utility bounded from above and below

$$\hat{J}(t, x, y) \leq J(t, x, y) \leq \bar{J}(t, x, y)$$



Expected utilities and the welfare loss for a correlation of $\rho = 0.4$

Benchmark Parameter Values

Benchmark values are similar to those used in the existing literature

- **Investor characteristics:** $X_0 = 2$ ($\sim$ USD 20,000), $\delta = 0.03$, $\gamma = 4$, $t = 0$, $\tilde{T} = 20$, $T = 40$
- **Financial market:** $r = 0.02$, $\lambda_S = 0.25$, $\sigma_S = 0.2$
- **Labor income:** $Y_0 = 2$, $\alpha = 0.02$, $\beta = 0.1$
- **Simulation parameters:** time steps per year=250, runs=10000, $k = 0.3$

Welfare Losses

	Income-stock correlation ρ				
	0	0.2	0.4	0.6	0.8
$\epsilon = 0.1$	2.18%	1.53%	1.19%	0.86%	0.46%
$\epsilon = 1$	2.20%	1.55%	1.20%	0.86%	0.48%
$\epsilon = 10$	2.22%	1.56%	1.22%	0.88%	0.48%

Welfare loss for the near-optimal strategy with constant λ_I

An Improvement

- Can these results be further improved by time-dependent market prices of risk of the affine form?
- The closed-form solution carries over to this case with a slight modification of $g^{\text{art}}(t)$ and $F^{\text{art}}(t)$

$$\lambda_I(t) = \Lambda_1 t + \Lambda_0, \quad \Lambda_1, \Lambda_0 \in \mathbb{R}.$$

	Income-stock correlation ρ				
	0	0.2	0.4	0.6	0.8
$\bar{\Lambda}_1$	-0.0165	-0.0163	-0.0154	-0.0135	-0.0102
$\bar{\Lambda}_0$	0.4059	0.3947	0.3675	0.3207	0.2415
L	1.04%	0.36%	0.12%	0.04%	0.01%

Welfare loss for the near-optimal strategy with affine $\lambda_I(t)$

Misspecified Model

We evaluate the welfare loss from using the consumption and investment strategy derived under a complete market assumption ($|\rho| = 1$) when the labor income is really unspanned (,i.e. true market incomplete)

	Income-stock correlation ρ				
	0	0.2	0.4	0.6	0.8
$\epsilon = 0.1$	14.41%	9.95%	6.21%	3.25%	1.15%
$\epsilon = 1$	14.43%	9.93%	6.21%	3.24%	1.14%
$\epsilon = 10$	14.39%	9.94%	6.20%	3.24%	1.15%

Welfare loss for the misspecified strategy with exogenous income and constant λ_l

Extensions of the Model

- Flexible labor supply
  - individual decides on his leisure
  - additional control variable
- Stochastic Interest Rates modeled by an Vasicek process: welfare losses are of the same order

Conclusion and Future Work

- We provide and test an easy procedure for finding a simple, near-optimal consumption and investment strategy of an investor receiving an unspanned labor income stream
- We extend the model to endogenous labor supply and stochastic interest rates and provide strategies
- Can we generalize the procedure?
- Compute a numerical solution for the incomplete market