Diversity and Arbitrage in a Regulatory Breakup Model

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Bachelier World Congress
Toronto, 2010

Outline

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Market Model

- Stock capitalization (total shares×price per share) process $X_t = (X_{1,t}, \ldots, X_{n,t})$ is the unique strong solution to

$$dX_{i,t} = X_{i,t}\left[b_i(X_t)dt + \sum_{\nu=1}^{d} \sigma_{i\nu}(X_t)dW_{\nu,t}\right], \quad 1 \leq i \leq n.$$

- Money market account $B \equiv 1$, $(r \equiv 0)$.
- $d \geq n$, and the covariance matrix $\sigma(x)\sigma(x)' \in \mathbb{R}^{n\times n}$, is uniformly elliptic. That is, there exists $\kappa > 0$ such that

$$\xi'\sigma(x)\sigma(x)'\xi \geq \kappa\|\xi\|^2, \quad \forall \xi \in \mathbb{R}^n, \forall x \in (0,\infty)^n.$$

Diversity

- *Market weight process* μ:

$$\mu_{i,t} := \mu_i(X_t) := \frac{X_{i,t}}{\sum_{j=1}^n X_{j,t}}, \quad 1 \leq i \leq n.$$

- Each X_i is strictly positive, so μ lives in the simplex

$$\Delta_+^n := \left\{ (\pi_1, \ldots, \pi_n) \in (0, \infty)^n \mid \sum_i^n \pi_i = 1 \right\}.$$

- Reverse order statistics notation:

$$x_{(1)} \geq x_{(2)} \geq \ldots \geq x_{(n)}.$$

Definition

A market is *diverse* on horizon T if there exists $\delta \in (0,1)$ such that

$$\mu_{(1),t} < 1 - \delta, \; \forall t : \; 0 \leq t \leq T.$$

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Diversity Implies Arbitrage

R. Fernholz [Fer99, Fer02]: diversity and equivalent martingale measures (EMMs) are incompatible.

Standard Model Assumptions:

- Capitalizations are Itô processes (⇒continuous paths);
- covariance is uniformly elliptic;
- continuous trading;
- no transaction costs;
- no dividends;
- number of companies is constant.

Under these assumptions diversity can be maintained only via singular repulsive down-drift of $\mu_{(1)}$ [FKK05].

Such models admit relative arbitrage with respect to the market portfolio over any horizon. These arbitrages are functionally generated from μ, not requiring knowledge of b or σ to construct.

Motivating Question

Are diversity and no-arbitrage compatible if diversity is maintained by a regulator breaking up any company that becomes too large?

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Regulatory Procedure I

Confine market weights μ to U^μ by redistribution of capital via a deterministic mapping $\mathfrak{R}^\mu$ upon μ's exit from U^μ.
Assumption: Total capital is conserved.

Definition

A *regulation rule* $\mathfrak{R}^\mu$ with respect to the open, nonempty set $U^\mu \subset \Delta^n_+$ is a Borel function

$$\mathfrak{R}^\mu : \partial U^\mu \to U^\mu$$

The regulation rule is equivalently described as acting on X via

$$U^x := \mu^{-1}(U^\mu) = \{x \in (0,\infty)^n \mid \mu(x) \in U^\mu\}$$

$$\mathfrak{R}^x : \partial U^x \to U^x$$

$$\mathfrak{R}^x(x) := \left(\sum_{i=1}^n x_i\right) \mathfrak{R}^\mu(\mu(x))$$

Regulatory Procedure II

- U^x is a conic region, i.e. $x \in U^x \Rightarrow \lambda x \in U^x$, $\forall \lambda > 0$, allowing any total market value for a given $\mu \in U^\mu$.
- Regulation is first applied at the exit and stopping time

$$\alpha_1 := \inf\{t > 0 \mid \mu_t \in \partial U^\mu\} = \inf\{t > 0 \mid X_t \in \partial U^x\}$$

- After α_1 the capitalizations "reset" as if starting afresh from initial point $\mathfrak{R}^x(X_{\alpha_1})$ until exit from U^x again.
- Applying this procedure inductively defines the regulated capitalization process.
- To obtain a diverse regulated market, choose e.g.

$$U^\mu = \{\pi \in \Delta^n_+ \mid \pi_{(1)} < 1 - \delta\}.$$

Regulated Market Model

$$\tau_0 = 0, \quad W^1 := W, \quad X^1 = X, \quad \tau_1 := \alpha_1 := \inf\left\{t > 0 \mid X_t^1 \in \partial U^x\right\}.$$

By induction define the following for $k \geq 2$, on $\{\tau_{k-1} < \infty\}$,

$$W_t^k := W_{\tau_{k-1}+t} - W_{\tau_{k-1}}, \quad \forall t \geq 0, \quad \text{a B.M. on } \{\tau_{k-1} < \infty\}$$

$$dX_{i,t}^k = X_{i,t}^k \left(b_i(X_t^k)dt + \sum_{\nu=1}^{d} \sigma_{i\nu}(X_t^k) dW_t^k \right), \quad 1 \leq i \leq n,$$

$$X_0^k = \mathfrak{R}^x(X_{\alpha_{k-1}}^{k-1}),$$

$$\alpha_k := \inf\left\{t > 0 \mid X_t^k \in \partial U^x\right\}, \quad \tau_k := \sum_{j=1}^{k} \alpha_j.$$

X^k is the unique strong solution to the SDE above, on $\{\tau_{k-1} < \infty\}$ with filtration $\{\mathscr{F}_{\tau_{k-1}+t}\}_{t \geq 0}$.

Regulated Capitalization Process

- There is a possibility of explosion, that is of $\lim_{k\to\infty}\tau_k < \infty$.

$$N_t := \sum_{k=1}^{\infty} \mathbf{1}_{\{t>\tau_k\}} \in \mathscr{F}_t, \qquad \tau_\infty := \lim_{k\to\infty} \tau_k.$$

Definition

For regulation rule $(U^\mu, \mathfrak{R}^\mu)$ and initial point $y_0 \in U^x$, the *regulated capitalization process* is defined as

$$Y_t(\omega) := y_0 \mathbf{1}_{\{0\}}(t) + \sum_{k=1}^{\infty} \mathbf{1}_{(\tau_{k-1},\tau_k]}(\omega,t) X^k_{t-\tau_{k-1}}(\omega), \quad (\omega,t) \in [0,\tau_\infty).$$

$$Y_0 = y_0 = x_0 = X^1_0$$

If $P(\tau_\infty = \infty) = 1$, then call the triple $(y_0, U^\mu, \mathfrak{R}^\mu)$ *viable*.

- The examples in this talk are viable. For the technical details, see our paper [SF10].

Split-Merge Regulation

- Split the largest company and simulataneously force the smallest two to merge.
- Let $p(i)$ return the index of the ith largest capitalization, e.g. $p(1) = i$, when x_i is the largest of $\{x_j\}_1^n$.
- For $n \geq 3$ and any open, nonempty $U^\mu \subseteq \Delta_+^n$, define $\mathfrak{R}^\mu : \partial U^\mu \to U^\mu$ via

$$\begin{aligned} \mu_{p(1)} &\mapsto \mu_{(1)}/2, \\ \mu_{p(n-1)} &\mapsto \mu_{(1)}/2, \\ \mu_{p(n)} &\mapsto \mu_{(n-1)} + \mu_{(n)}, \\ \mu_{p(i)} &\mapsto \mu_i, \quad \text{for } i \notin \{1, n-1, n\}. \end{aligned}$$

- This will be the regulatory rule used in applications, with U^μ to be specified later.

Portfolios in the Regulated Market I

- Assumption: Portfolio wealth is conserved at regulation events: $V_{\tau_k^+} = V_{\tau_k}$. Realistic for breakups and merges.
- This implies that capital gains are not given by $(H \cdot Y)_t$.
- Would like to represent the capital gains process as a stochastic integral.
- Define a net capitalization process $\hat{Y}$, reflecting only the non-regulatory movements of Y:

$$\hat{Y}_t := Y_t - \sum_{k=1}^{N_t} \Delta Y_k, \qquad \Delta Y_k := Y_{\tau_k^+} - Y_{\tau_k}.$$

- Recalling that $Y_{\tau_k^+} = \mathfrak{R}^x(Y^k_{\tau_k}) = X_0^{k+1}$ on $\{\tau_k < \infty\}$, then

$$\hat{Y}_t = X_0^1 + \sum_{k=1}^{N_t} (X^k_{\alpha_k} - X^k_0) + (X^{N_t+1}_{t-\tau_{N_t}} - X^{N_t+1}_0).$$

Portfolios in the Regulated Market II

- A wealth process V^H in the regulated model should be locally self-financing on $(\tau_{k-1}, \tau_k]$, for each $k \in \mathbb{N}$.
- This combined with the assumption of wealth-conservation at $\{\tau_k\}_1^\infty$ leads to the following definitions.

Definition

Admissible trading strategies are predictable processes H which are $\hat{Y}$-integrable, and for which there exists a constant $K > 0$:

$$(H \cdot \hat{Y})_t \geq -K, \quad \text{a.s.}, \forall t \geq 0.$$

A *self-financing* wealth processes in the regulated model is any V^H which satisfies:

$$V_t^H = V_0^H + (H \cdot \hat{Y})_t \qquad \forall t \geq 0.$$

ELMMs in the Regulated Model I

FTAP

NFLVR for $\hat{Y}$ is equivalent to existence of an equivalent local martingale measure (ELMM) for $\hat{Y}$.

$\hat{Y}$ obeys the SDE

$$d\hat{Y}_{i,t} = Y_{i,t}\left(b_i(Y_t)dt + \sum_{v=1}^{d}\sigma_{iv}(Y_t)dW_t^k\right), \quad 1 \le i \le n.$$

Since $\sigma(\cdot)$ is uniformly elliptic, there exists a market price of risk, $\theta := \sigma_t'(\sigma_t\sigma_t')^{-1}b_t$. When

$$\int_0^T |\theta(Y_t)|^2 dt < \infty, \qquad \text{a.s., } \forall T > 0$$

then we may define the local martingale and supermartingale,

$$Z_t := \mathscr{E}(-(\theta(Y)\cdot W))_t = \exp\left\{-\left(\int_0^t \theta(Y_s)dW_s + \frac{1}{2}\int_0^t |\theta(Y_s)|^2 ds\right)\right\}$$

ELMMs in the Regulated Model II

Theorem

If Z is a martingale, then the measure Q generated from $\frac{dQ}{dP} := Z_T$ is a local martingale measure for $\hat{Y}$ on $[0, T]$.

The usual tools, e.g. the Kazamaki and Novikov criteria, provide sufficient conditions for Z to be a martingale.

Proposition

If Q is an ELMM for $\hat{Y}$ and σ is bounded, then Q is an EMM and there is no relative arbitrage with respect to the market portfolio.

In particular, this can rule out functionally-generated relative arbitrages with respect to the market.

Standard vs Regulated: Compare/Contrast

Standard Model

$$dX_t = X_t \star [b(X_t)dt + \sigma(X_t)dW_t]$$

- $\mu_t \in \Delta_+^n$
- $X_t \in (0,\infty)^n$
- $V_t^H = V_0 + H \cdot X$
- ELMM if $\theta(X)$ is well-behaved
- Diversity can be maintained only through b
- Diversity and no-arbitrage not compatible

Regulated Model

$$d\hat{Y}_t = Y_t \star [b(Y_t)dt + \sigma(Y_t)dW_t]$$

- $\mu_t \in U^\mu \subseteq \Delta_+^n$
- $Y_t \in U^x \subseteq (0,\infty)^n$
- $V_t^H = V_0 + H \cdot \hat{Y}$
- ELMM if $\theta(Y)$ is well-behaved
- σ an b may both be constant and Y be diverse
- Diversity and no-arbitrage compatible

Outline

Regulated and Diverse GBM

- Take a geometric Brownian motion model

$$dX_{i,t} = X_{i,t}\left[b_i dt + \sum_{\nu=1}^{n} \sigma_{i\nu} dW_{\nu,t}\right].$$

- Impose diversity by choosing the regulatory region

$$U^{\mu} := \{\pi \in \Delta_+^n \mid \pi_{(1)} < 1 - \delta\}.$$

- Choose $\mathfrak{R}^{\mu}$ to be the split-merge rule.
- The resulting regulated market is viable.
- $\theta = \sigma^{-1} b$ is a constant, so Z is a martingale and NFLVR and no relative arbitrage hold.

Log-Pole Market

- A diverse market where each company behaves like a geometric Brownian motion when it is not the largest [FKK05].
- The volatility σ is constant. The drift $b(\cdot)$ is given by

$$b_i(x) := g_i 1_{\mathcal{Q}_i^c}(x) - \frac{c}{\delta} \frac{1_{\mathcal{Q}_i}(x)}{\log\left(\left(1-\delta\right)/\mu_i(x)\right)}, \qquad 1 \le i \le n,$$

  where $\{g_i\}_1^n$ are non-negative numbers, c is a positive number, and when $x \in \mathcal{Q}_i$, then x_i is the largest of the $\{x_j\}_1^n$ with ties going to the smaller index.
- The largest company is repulsed away from the log-pole-type singularity in its drift at $1-\delta$ in μ-space.
- The market is diverse and has constant volatility, so over any horizon there are long-only relative arbitrage portfolios that are functionally generated from the market portfolio.

Regulated Log-Pole Market

- Blocking access to the singularity removes the arbitrage.
- Choose $\delta' \in (\delta, \frac{n-1}{n+1})$ and

$$U^\mu := \{\pi \in \Delta^n_+ \mid \pi_{(1)} < 1 - \delta'\}.$$

- Set $\mathfrak{R}^\mu$ as the split-merge rule. Then the regulated market is viable and $b \restriction_{U^\times} (\cdot)$ is bounded.
- This implies that θ is bounded, the Novikov condition is satisfied, and so Z is a true martingale.
- The regulated market is diverse, satisfies NFLVR and no relative arbitrage.

Summary and Outlook

- EMMs (with respect to $\hat{Y}$) and diversity (with respect to Y) are compatible in this regulatory breakup model.
- The key condition here is that $\theta \restriction_{U^\times} (\cdot)$ be well-behaved.
- When companies may split, diversity no longer imposes constraints on b.
- The assumption of constant number of companies, i.e. splits and merges occurring simultaneously, may be eliminated. No arbitrage and diversity remain compatible.
- Future work: Incorporate more stylized facts into equity market models. This will lead to further insights and clarifications regarding the feasibility of relative arbitrage with respect to the market portfolio.

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