Pricing Variance Swaps on Time-Changed Lévy Processes

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Introduction: Approaches and Assumptions

Variance Swaps and Lévy Processes

Time-changed Lévy processes

Approaches for Arb-free pricing of derivative securities

Underlying is a scalar stochastic process S evolving in cont. time.

Some derivative contract pays $Z_T = F(\text{path of } S)$ at fixed time T.

If no rates & no arb., price at $Z_0 = \mathbb{E}Z_T$, for some mgl. measure.

Three approaches to finding $Z_0 = \mathbb{E}Z_T$:

- 1) Parametric: Specify a set of parameters Θ governing the dynamics of S. Compute $Z_0 = f(\Theta)$
- 2) Non-parametric: If possible, make only enough assumptions so that there exists a function g of just S such that $Z_0 = \mathbb{E}g(S_T)$. Observe $\mathbb{E}g(S_T)$ from the market prices of options on S_T.
- 3) Semi-parametric (ours): If possible, generalize the non-parametric approach by letting g also depend on parameters:$Z_0 = \mathbb{E}g(\Theta, S_T)$.

Assumptions

- Zero interest rates. This is not essential.
- Work in $(\Omega, \mathcal{F}, \{\mathcal{F}_u\}, \mathbb{Q})$, where $\mathbb{Q}$ is an equiv. mgl. measure.
- S is a positive martingale share price.
- $Y_t := \log(S_t/S_0)$
- $[Y]$ denotes the quadratic variation of Y.
- (The floating leg of) a *continuously-sampled variance swap* pays $[Y]_T$ at expiry T.
- (The floating leg of) a *discretely-sampled variance swap* pays $\sum_{n=0}^{N-1}(Y_{t_{n+1}} - Y_{t_n})^2$ where $0 = t_0 < t_1 < \cdots < t_N = T$.
- Other volatility derivatives pay functions of $[Y]_T$.

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Standard variance swap valuation approach

Neuberger (90), Dupire (92), Carr-Madan (98), Derman et al (99):

- Let a *log contract* pay $-Y_T = -\log(S_T/S_0)$.

  The sign convention makes log contracts have nonnegative value.

- Assume S is *continuous*, i.e. it never jumps.

- Then the floating leg of a continuously sampled variance swap has the same value as *two log contracts*:

$$\mathbb{E}[Y]_T = 2\mathbb{E}(-Y_T).$$

Standard variance swap valuation approach

- Widely influential as a reference point for variance swap dealers
- Has been since 2003 the basis for the CBOE's VIX index, the most popular indicator of options-implied expectations of realized variance.

  The VIX is computed as the square root of twice the value of a synthetic log contract (synthesized from calls and puts).
- Also the basis for the VXN and the VDAX-NEW.
- Robust ("model-free"). Assumes only that $S > 0$ and that S is continuous.
- But what if S can *jump*?

Lévy processes

An adapted process $(X_u)_{u\geq 0}$ with $X_0 = 0$ is a Lévy process if:

- $X_v - X_u$ is independent of $\mathcal{F}_u$ for $0 \leq u < v$
- $X_v - X_u$ has same distribution as X_{v-u} for $0 \leq u < v$
- $X_v \to X_u$ in probability, as $v \to u$

Lévy -Khintchine Theorem: There exists a constant $a \in \mathbb{R}$, a second constant $\sigma \geq 0$, and a *Lévy measure* $\nu(A)$, with $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(1 \wedge x^2)\nu(\mathrm{d}x) < \infty$, such that the *characteristic function* of the Lévy process X_t

$$\mathbb{E}e^{izX_t} = e^{t\psi(z)}$$

where the *characteristic exponent* is

$$\psi(z) := iaz - \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}}(e^{izx} - 1 - izx\mathbb{I}_{|x|\leq 1})\nu(\mathrm{d}x).$$

Interpreting the Characteristic Exponent

- Recall that if X is a (scalar-valued) Lévy process, then the Lévy -Khintchine theorem states that there exist $a \in \mathbb{R}$, $\sigma \geq 0$, and a Lévy measure $\nu(A)$, with $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} (1 \wedge x^2) \nu(\mathrm{d}x) < \infty$, such that $\mathbb{E}e^{izX_t} = e^{t\psi(z)}$, where the characteristic exponent is:

$$\psi(z) := iaz - \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} (e^{izx} - 1 - izx\mathbb{I}_{|x|\leq 1}) \nu(\mathrm{d}x)$$

- The Lévy measure
  $\nu(A) = \mathbb{E}(\text{number of jumps of size} \in A, \text{ per unit time})$.
- Probabilistic Interpretation: $X_t = at + \sigma B_t + L_t^j + M_t^j$, where the constant a adds drift, the constant σ scales the S.B.M. B, L^j has Poisson arrival of only large jumps (of size > 1), and M^j is a martingale having only small jumps (of size ≤ 1).

Constructing a time-changed Lévy processes

- Let X be a Lévy process such that $\mathbb{E}e^{X_1} < \infty$.

  Let $\bar{X}_u := X_u - u \log \mathbb{E}e^{X_1}$, so that $e^{\bar{X}}$ is a martingale.

- Let the time change $\{\tau_t\}_{t\in[0,T]}$ be an increasing continuous family of stopping times.

  So τ is a stochastic "clock" that measures "business time":

  $$\text{Calendar time } t \quad \leftrightarrow \quad \text{Business time } \tau_t$$

  We do not assume that τ and X are independent.

- Assume $Y_t = \bar{X}_{\tau_t}$ and $S_t = S_0 \exp(Y_t)$.

  The time-changed Lévy process Y can exhibit stochastic volatility, stochastic jump intensity, volatility clustering, and "leverage" effects.

Examples of time-changed Lévy processes

Example of a family of TCLPs: Assume $Y_t = \bar{X}_{\tau_t}$ where

$$\bar{X}_u = B_u^x - u/2 + L_u$$

$$\mathrm{d}\tau_t = V_t \mathrm{d}t$$

$$\mathrm{d}V_t = \alpha(V_t, Y_t)\mathrm{d}t + \beta(V_t, Y_t)\mathrm{d}W_t + \gamma(V_{t-}, Y_{t-})\mathrm{d}J_t$$

where J is a Poisson process, L is a pure-jump Lévy process, and

$$B_u^v := \rho B_u^x + \sqrt{1-\rho^2} B_u$$

$$W_t = \int_0^t \frac{1}{\sqrt{V_s}} \mathrm{d}B_{\tau_s}^v$$

where B^x and B are independent Brownian motions wrt business-time filtration $\mathcal{F}$. We do not specify $V_0, \alpha, \beta, \gamma, \rho, J$.

Exponentials of TCLPs include all PC martingales

- All positive continuous (PC) martingales have the form $S_0 \exp \bar{X}_{\tau_t}$ where $\bar{X}$ is Brownian motion with drift $-1/2$.
- This follows from Dambis-Dubins-Schwarz:

  Apply DDS to the local martingale

  $$M_t := \int_0^t \frac{1}{S_\theta} \mathrm{d}S_\theta = \log(S_t/S_0) + \frac{1}{2}[\log(S_\cdot/S_0)]_t$$

  to find BM W such that $M_t = W_{\tau_t}$ where

  $$\tau_t := [M]_t = [\log(S_\cdot/S_0)]_t.$$

  Conclude that $\log(S_t/S_0) = W_{\tau_t} - \tau_t/2$.
- So the standard valuation model for variance swaps is just the Brownian special case of a time-changed Lévy process (TCLP) framework.

Variance swaps on time-changed Lévy processes

We show that:

- In general, the payoff of a variance swap can no longer be replicated (The only perfect hedge is in a Japanese garden).
- However, variance swaps still admit pricing in terms of log contracts. The floating leg of a variance swap still has value given by a positive multiple of the value of a log contract.
- The correct *multiplier* is not necessarily 2.
- The multiplier depends only on the characteristics of the driving Lévy process X, not on the stochastic clock τ that we run it on.

The multiplier

- Let X be a nonconstant Lévy process, with $\mathbb{E}e^{X_1} < \infty$ and $\mathbb{E}[X]_1 < \infty$. Define the multiplier of X as:

$$Q_X := \frac{\mathbb{E}[X]_1}{\log \mathbb{E}e^{X_1} - \mathbb{E}X_1}$$

- Proposition: The multiplier exists and satisfies

$$0 < Q_X = \frac{\mathrm{Var} X_1}{\log \mathbb{E}e^{X_1} - \mathbb{E}X_1} = \frac{\kappa''(0)}{\kappa(1) - \kappa'(0)}$$

  where $\kappa(z) := \log \mathbb{E}e^{zX_1}$ is the cumulant generating function of X.

  Proof: Define the martingale $M_u := X_u - u\mathbb{E}X_1$. Then

$$\mathrm{Var} X_1 = \mathbb{E}M_1^2 = \mathbb{E}[M]_1 = \mathbb{E}[X]_1.$$

The multiplier

- Proposition: Let X from the last slide have Lévy triple (a, σ^2, ν).

  Then
  $$Q_X = \frac{\sigma^2 + \int x^2 \nu(\mathrm{d}x)}{\sigma^2/2 + \int (e^x - 1 - x)\nu(\mathrm{d}x)}.$$
  Proof: We have $-\mathbb{E}X_1 = -a - \int_{|x|\geq 1} x\nu(\mathrm{d}x)$ and
  $$\log \mathbb{E}e^{X_1} = a + \sigma^2/2 + \int (e^x - 1 - x\mathbb{I}_{|x|\leq 1})\nu(\mathrm{d}x)$$
  Summing gives the denominator of Q_X.
- Use this formula if given the Lévy measure of X.

  Use formula on the last slide if given the characteristic function of X.

Variance swap valuation

Proposition

If $\mathbb{E}\tau_T < \infty$ *then*

$$\mathbb{E}[Y]_T = Q_X \mathbb{E}(-Y_T).$$

Hence the floating leg of a continuously sampled variance swap has the same value as Q_X *log contracts.*

Proof.

$[X]_u + Q_X \bar{X}_u$ is a Lévy martingale, so by Wald's equation

$$\mathbb{E}([X]_{\tau_T} + Q_X \bar{X}_{\tau_T}) = 0.$$

Continuity of τ implies $\mathbb{E}[Y]_T = \mathbb{E}[X_{\tau_\cdot}]_T = \mathbb{E}[X]_{\tau_T} = Q_X \mathbb{E}(-Y_T)$. $\square$

Time-change of bracket = bracket of time-change

Last step (swapping bracket and time-change) assumed continuity of τ.

In this setting, jumps arise from X jumping, not from clock jumping (although we allow the clock *rate* to jump).

Example: S is time-changed geom'c Brownian motion

Let X be standard Brownian motion. Then

$$Q_X = \frac{\mathbb{E}[X]_1}{\log \mathbb{E}e^{X_1} - \mathbb{E}X_1} = \frac{1}{1/2} = 2$$

This recovers the 2 multiplier when the underlying of the options is a positive continuous martingale.

Example: Time-changed fixed-size jump diffusion

Let X have Brownian variance σ^2 and Lévy measure

$$\lambda_1 \delta_{c_1} + \lambda_2 \delta_{c_2}$$

where δ_c denotes a point mass at c, and $c_1 > 0$ and $c_2 < 0$. Then

$$Q_X = \frac{\sigma^2 + \lambda_1 c_1^2 + \lambda_2 c_2^2}{\sigma^2/2 + \lambda_1(e^{c_1} - 1 - c_1) + \lambda_2(e^{c_2} - 1 - c_2)}.$$

Third-order Taylor expansion in (c_1, c_2) about $(0,0)$, if $\sigma \neq 0$:

$$Q_X \approx 2 - \frac{2\lambda_1}{3\sigma^2} c_1^3 + \frac{2\lambda_2}{3\sigma^2} |c_2|^3,$$

increasing in absolute down-jump size, decreasing in up-jump size.

Time-changed double-exponential jump-diffusion

Let X have Brownian variance σ^2 and Lévy density

$$\nu(x) = \lambda_1 a_1 e^{-a_1|x|}\mathbb{I}_{x>0} + \lambda_2 a_2 e^{-a_2|x|}\mathbb{I}_{x<0}$$

where $a_1 \geq 1$ and $a_2 > 0$. So up-jumps have mean size $1/a_1$, down-jumps have mean absolute size $1/a_2$. Then

$$Q_X = \frac{\sigma^2 + 2\lambda_1/a_1^2 + 2\lambda_2/a_2^2}{\sigma^2/2 + \lambda_1/(a_1-1) - \lambda_2/(a_2+1) - \lambda_1/a_1 + \lambda_2/a_2}.$$

Third-order Taylor expansion in $(1/a_1, 1/a_2)$ about $(0,0)$, if $\sigma \neq 0$:

$$Q_X \approx 2 - \frac{4\lambda_1/\sigma^2}{a_1^3} + \frac{4\lambda_2/\sigma^2}{a_2^3},$$

Example: Time-changed extended CGMY

Let X have the extended CGMY Lévy density

$$\nu(x) = \frac{C_n}{|x|^{1+Y_n}} e^{-G|x|} \mathbb{I}_{x<0} + \frac{C_p}{|x|^{1+Y_p}} e^{-M|x|} \mathbb{I}_{x>0},$$

where $C_p, C_n > 0$ and $G, M > 0$, and $Y_p, Y_n < 2$. Then $Q_X =$

$$\frac{-C_n\Gamma(2-Y_n)G^{Y_n-2} - C_p\Gamma(2-Y_p)M^{Y_p-2}}{C_n\Gamma(-Y_n)[G^{Y_n}-(G+1)^{Y_n}+Y_nG^{Y_n-1}] + C_p\Gamma(-Y_p)[M^{Y_p}-(M-1)^{Y_p}-Y_pM^{Y_p-1}]}$$

Expanding the denominator in $1/G$ and $1/M$,

$$Q_X \approx 2 \times \frac{G^{Y_n-2} + \rho M^{Y_p-2}}{G^{Y_n-2}(1 - \frac{2-Y_n}{3G} + \ldots) + \rho M^{Y_p-2}(1 + \frac{2-Y_p}{3M} + \ldots)}.$$

where $\rho := C_p\Gamma(2-Y_p)/(C_n\Gamma(2-Y_n))$.

Example: Time-changed basic CGMY

The basic CGMY model takes $C_p = C_n = C$ and $Y_p = Y_n = Y$.

$$\nu(x) = \frac{C}{|x|^{1+Y}} e^{-G|x|} \mathbb{I}_{x<0} + \frac{C}{|x|^{1+Y}} e^{-M|x|} \mathbb{I}_{x>0}$$

Its multiplier is

$$\begin{aligned} Q_X &= \frac{Y(1-Y)(G^{Y-2} + M^{Y-2})}{G^Y - (G+1)^Y + YG^{Y-1} + M^Y - (M-1)^Y - YM^{Y-1}} \\ &\approx 2 \times \frac{G^{Y-2} + M^{Y-2}}{G^{Y-2}(1 - \frac{2-Y}{3G} + \ldots) + M^{Y-2}(1 + \frac{2-Y}{3M} + \ldots)}. \end{aligned}$$

Note the sign asymmetry between the $-\frac{2-Y}{3G}$ and the $+\frac{2-Y}{3M}$.

(Small G, big M) gives a bigger multiplier than (big G, small M).

Example: Time-changed VG

The Variance Gamma model takes $Y = 0$.

$$\nu(x) = \frac{C}{|x|} e^{-G|x|} \mathbb{I}_{x<0} + \frac{C}{|x|} e^{-M|x|} \mathbb{I}_{x>0}$$

Its multiplier is

$$\begin{aligned} Q_X &= \frac{1/G^2 + 1/M^2}{-\log(1+1/G) + 1/G - \log(1-1/M) - 1/M} \\ &\approx 2 \times \frac{G^{-2} + M^{-2}}{G^{-2}(1 - \frac{2}{3G} + \dots) + M^{-2}(1 + \frac{2}{3M} + \dots)}. \end{aligned}$$

Note the sign asymmetry between the $-\frac{2}{3G}$ and the $+\frac{2}{3M}$.

Example: Time-changed normal inverse Gaussian (NIG)

Let X have no Brownian component. Let X have Lévy density

$$\nu(x) = \frac{\delta\alpha}{\pi}\frac{\exp(\beta x)K_1(\alpha|x|)}{|x|},$$

where $\delta > 0$, $\alpha > 0$, $|\beta| < \alpha$, and K_1 denotes the modified Bessel function of the second kind and order 1. Then X has CGF

$$\kappa(z) = \log \mathbb{E}e^{zX_1} = \gamma z + \delta\left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + z)^2}\right),$$

for some γ that we need not specify. Then

$$Q_X = \frac{\kappa''(0)}{\kappa(1) - \kappa'(0)} = \frac{\alpha^2/(\alpha^2 - \beta^2)}{\alpha^2 - \beta^2 - \beta - \sqrt{(\alpha^2 - \beta^2)(\alpha^2 - (\beta + 1)^2)}}.$$

Small jump-size limit: take $\alpha \to \infty$. Expanding in $1/\alpha$,

$$Q_X \approx 2 - \frac{4\beta + 1}{2\alpha^2}.$$

which is decreasing in β, the parameter which controls the “tilt”.

Intuition: Up-jumps vs down-jumps

First-order effects: Multiplier is decreasing in up-jump size, but increasing in down-jump size. From Itô's formula for semi-martingales:

$$
\begin{aligned}
-2\log(S_T/S_0) &= \int_{0+}^{T} \frac{-2}{S_{t-}} \mathrm{d}S_t + \frac{1}{2}\int_{0+}^{T} \frac{2}{S_{t-}^2} \mathrm{d}[S]_t^c \\
&\quad + \sum_{0<t\le T} \Big(-2\Delta \log S_t - \frac{-2}{S_{t-}} \Delta S_t \Big) \\
&= \int_{0+}^{T} \frac{-2}{S_{t-}} \mathrm{d}S_t + [Y]_T + \sum_{0<t\le T} \Big(\frac{2}{S_{t-}} \Delta S_t - 2\Delta Y_t - (\Delta Y_t)^2 \Big).
\end{aligned}
$$

So 2 log contracts, plus zero-expectation share-trading P&L, pay

$$
[Y]_T + \sum_{0<t\le T} \Big(2e^{\Delta Y_t} - 2 - 2\Delta Y_t - (\Delta Y_t)^2 \Big) \;\approx\; [Y]_T + \sum_{0<t\le T} \frac{1}{3} (\Delta Y_t)^3.
$$

So $2\mathbb{E}(-Y_T) > \mathbb{E}[Y]_T$ in presence of up-jumps $\Delta Y_t > 0$.

Skewness

- For a Lévy measure ν such that $\int_{|x|>1} e^x \nu(\mathrm{d}x) < \infty$ and $\int_{|x|>1} x^2 \nu(\mathrm{d}x) < \infty$, define the *exponential skewness* of ν by

$$6 \int (e^x - 1 - x - x^2/2)\nu(\mathrm{d}x).$$

  which $= \int (x^3 + x^4/4 + x^5/20 + \cdots)\nu(\mathrm{d}x)$

  so leading term is third moment.

- Proposition: for any X with Lévy measure ν, we have $Q_X > 2$ if and only if ν has negative exponential skewness.

  Proof: Exponential skewness is negative if and only if

$$\sigma^2/2 + \int (e^x - 1 - x)\nu(\mathrm{d}x) < \sigma^2/2 + \int (x^2/2)\nu(\mathrm{d}x)$$

  and $Q_X = 2 \times \mathrm{RHS}/\mathrm{LHS}$.

Multipliers of empirically estimated processes

CGMY's calibration. S&P 500 options in 2000. CIR clock.

X	Data	Lévy parameters	Q_X
CGMY	Jun	$(G, M, Y_p, Y_n) = (0.423, 24.6, -4.51, 1.67)$	2.37
VG	Jun	$G = 11.0, M = 30.1$	2.10
NIG	Jun	$\alpha = 69.7, \beta = -62.1$	2.12
CGMY	Sep	$(G, M, Y_p, Y_n) = (1.64, 16.9, -2.90, 1.54)$	2.17
VG	Sep	$G = 12.4, M = 33.6$	2.09
NIG	Sep	$\alpha = 99.8, \beta = -91.1$	2.11
CGMY	Dec	$(G, M, Y_p, Y_n) = (3.68, 52.9, -2.12, 1.22)$	2.13
VG	Dec	$G = 11.7, M = 42.7$	2.10
NIG	Dec	$\alpha = 274.8, \beta = -265.4$	2.10

Conclusion

- As in the standard variance swap (VS) pricing theory, we assumed continuous monitoring and that the log contract paying $-\log(S/S_0)$ can be priced off vanillas on S.
- However, we generalize the standard VS pricing theory by assuming that S is a time-changed exponential Lévy process.
- We showed that the floating leg of a VS has the same value as Q_X log contracts.
- When the underlying price process is a positive continuous martingale, the multiplier Q_X is 2. With jumps, the multiplier depends *only* on the Lévy process. Empirical calibrations imply 2.1 to 2.4.

An Empirical Puzzle

- Since variance swaps have become liquid, it is feasible to collect market quotes on variance swap rates.
- In joint work with Liuren Wu, we obtained quotes from a dealer and compared them with values of log contracts synthesized from index options.
- Our basic finding is that the empirical multiplier hovers about 2. Apparently, dealers are ignoring the effect of negatively skewed jumps when pricing variance swaps.
- It can be that negative jump skewness is priced in, but that other effects cheapen variance swaps, eg stochastic dividends, stochastic interest rates, counterparty credit risk, feedback effects from dynamic hedging, and of course, supply and demand.

Completed Extensions

- In reality, all variance swaps have discrete sampling. Roger Lee and I have investigated the effects of discrete sampling when the VS is written on a time-changed Lévy process (TCLP). We find that discrete sampling increases variance swap rates under an independence condition. The premium is bounded below by squared spreads of log contracts.
- Roger and I have further generalized the analysis to contracts paying *functions of* the quadratic variation of a TCLP, eg. hockey sticks or square roots. If the time-change is independent of the Lévy driver, then these general contracts can be priced off vanillas on S.