Tangent Lévy Models

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Problem Formulation

- Consider a *liquid market* consisting of an underlying price process $(S_t)_{t\geq 0}$ and prices of European Call options of all strikes K and maturities T:

$$\left(\{C_t(T,K)\}_{T,K>0}\right)_{t\geq 0}$$

- Want to describe a large class of *market models*: arbitrage-free stochastic models (say, given by SDE's) for time-evolution of the market, S and $\{C(T,K)\}_{T,K>0}$, such that
  1. one can start the model from "almost" any *initial condition*, which is the set of currently observed market prices;
  2. one can prescribe "almost" any *dynamics* for the model provided it doesn't contradict the no-arbitrage property.

Motivation

- Many Call Options have become liquid ⇒ need for financial models consistent with the observed option prices.

- Common *stochastic volatility* models (BS, Hull-White, Heston, etc.) are unable to reproduce the observed call prices of all strikes and maturities (fit the *implied volatility* surface).
  Local volatility models can fit option prices better.

- However, the above models have to be **recalibrated** to fit option prices at different times ⇒ they cannot be used to describe *time evolution* of call price surface.

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Preceding Results

- *E. Derman, I. Kani (1997)*: idea of "dynamic local volatility" for continuum of options.
- *P. Schönbucher, M. Schweizer, J. Wissel (1998-2008)*: consider fixed maturity and all strikes, fixed strike and all maturities, finitely many strikes and maturities (using mixture of Implied and Local Volatilities).
- *J. Jacod, P. Protter, R. Cont, J. da Fonseca, V. Durrleman (2002-2009)*: study dynamics of Implied Volatility or option prices directly.

Direct approach

- First, need a reasonable notion of "price" in the model: let's agree that pricing is linear, that is, **prices of all contingent claims are given by discounted conditional expectations of their payoffs under some measure** (assume discount rate is one).

- It seems natural to model "observables" directly under pricing measure: choose a driving *Brownian motion* B and a *Poisson random measure* N (which represent the background stochastic factors) and prescribe dynamics of (infinite-dimensional) *process of option prices* through its **semimartingale characteristics**

$$dC_t = \alpha_t dt + \beta_t \cdot dB_t + \int \gamma_t(x) \left[N(dx, dt) - \nu(dx, dt) \right]$$

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Consistency conditions

- Need to make sure these dynamics, indeed, produce option prices: each resulting $C_t(T,K)$ *should coincide with corresponding conditional expectation*.

$$\Downarrow$$

- **Consistency conditions** on $\{\alpha, \beta, \gamma\}$
- These conditions should be explicit! A perfect example is

$$F(\alpha_t, \beta_t, \gamma_t) = 0,$$

where F is known explicitly, and the above equation can be solved for some of the arguments.

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Direct approach: difficulties

Turns out, the above *direct approach* (prescribing dC_t directly) results in way too complicated consistency conditions...
Why does it happen?

- Recall that the definition of call prices as expectations implies certain **"static no-arbitrage properties"**: $C_t(T,K)$ has to be nonnegative, convex in K, converge to payoff, etc. These properties have to be preserved by the dynamics, which is reflected in the consistency conditions - hence the complexity.
- Static no-arbitrage conditions define a *manifold* in space of functions of two variables. Therefore, the "consistent" set of parameters can only be of the form

$$\alpha(C_t, t, \omega), \beta(C_t, t, \omega), \gamma(C_t, t, \omega)$$

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- Need to analyze resulting SDE in an "infinite-dimensional manifold"...

Code-books

- Let's *linearize* this manifold: find a one-to-one mapping of the set of feasible Call price surfaces (or its large enough subset) into some *open set in a linear space*. And consider dynamics in this linear space instead.
- In general, **code-book** for a given set of derivatives is a one-to-one mapping defined on a family of their feasible price sets.
- Examples of code-books include:
  - *Yield curve* for Treasury Bonds market.
  - *Implied correlation* for CDO tranches.
  - *Implied volatility* for Call options
- Recall that we require certain properties from the code-book. In particular, implied vol will not work.

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Local Volatility as a code-book

- *B.Dupire (1994)* deduced that, if

$$d\tilde{S}_T = \tilde{S}_T a(T, \tilde{S}_T) dW_T, \quad \tilde{S}_0 = S_t, \tag{1}$$

then

$$a^2(T,K) := \frac{2\frac{\partial}{\partial T} C(T,K)}{K^2 \frac{\partial^2}{\partial K^2} C(T,K)} \tag{2}$$

- We can use (2) to recover Local Volatility "a" from market prices of Call options, and
- we can use (1) to generate a (*feasible*!) Call price surface from a given Local Vol (and current level of underlying S_t).
- Only *some regularity and nonnegativity* is required from surface $a(.,.)$!

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Other code-books

- When can we use Local Vol as a (static) code-book for Call prices?

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- Can we develop a general approach to construction of code-books?

- Local Volatility code-book can be interpreted as follows: we choose a model from the *class of diffusion models*, such that it produces the correct (market-given) call prices, and the corresponding Local Vol gives the code-book value.

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Constructing convenient code-books

- Consider a class of "simple" financial models for the underlying, parameterized by $\theta \in \Theta$

$$\mathcal{M} = \{M(\theta)\}_{\theta \in \Theta}$$

For example, $\mathcal{M}$ can be a class of *diffusion models* parameterized by Local Vol and initial value: $\theta = \left(a(.,.), \tilde{S}_0\right)$

- Each model $M(\theta)$ produces Call prices $C^\theta(T, K)$. If the mapping $\theta \mapsto C^\theta$ is invertible, we obtain a **code-book associated with** $\mathcal{M}$.

- Of course, Θ needs to be an *open set in a linear space* - but usually this can be achieved.

- We have rediscovered **calibration**, but with a proper meaning now!

Tangent models

- **Construct market model by prescribing time-evolution of θ_t**, and obtain C_t as an inverse of the code-book transform.

- Recall that "feasibility" of call prices means there is a "true" (but unknown) martingale model for underlying process S in the background.

- If at time t there exists $\theta_t \in \Theta$, such that C^{θ_t} coincides with "true" Call price surface C_t, we say that the "true" model **admits a tangent model from class $\mathcal{M}$** at time t.

- In the above notation, process $(\theta_t)_{t\geq 0}$ is **consistent** with a "true" model for S if $M(\theta_t)$ is **tangent** to this "true" model at any time t.

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Lévy-based code-book

Consider a model $M(\kappa, s)$, given by

- Exponential of a **pure jump additive (time-inhomogeneous Lévy)** process

$$\tilde{S}_T = s + \int_t^T \int_{\mathbb{R}} \tilde{S}_{u-}(e^x - 1)\left[N(dx, du) - \nu(dx, du)\right],$$

  where $N(dx, du)$ is a *Poisson random measure* associated with jumps of $\log(\tilde{S})$, given by its compensator

$$\nu(dx, du) = \kappa(u, x) dxdu$$

- equipped with its natural filtration.

Thus, we obtain the set of "simple" models $\mathcal{M} = \{M(\kappa, s)\}$, with κ changing in a space of (time-dependent) Lévy densities.

Lévy density as a code-book

- Notice that $C^{\kappa,s}(T,e^x)$ satisfies a PIDE analogous to the Dupire's equation.
- Introduce $\Delta^{\kappa,s}(T,x) = -\partial_x C^{\kappa,s}(T,e^x)$, and deduce an initial-value problem for $\Delta^{\kappa,s}$ from the PIDE for call prices.
- Take Fourier transform in "x" to obtain $\hat{\Delta}^{\kappa,s}(T,\xi)$. The initial-value problem in Fourier domain can be solved in closed form, which gives us an explicit expression for $\hat{\Delta}^{\kappa,s}$ in terms of κ and s. This expression can be inverted to obtain κ from $\hat{\Delta}^{\kappa,s}$ and s.
- Thus, given s $(= S_t)$, we have a bijection: $C^{\kappa,s} \leftrightarrow \Delta^{\kappa,s} \leftrightarrow \hat{\Delta}^{\kappa,s} \leftrightarrow \kappa$.

Tangent Lévy Models

We say that $(S_t)_{t\in[0,\bar{T}]}$ and $(\kappa_t)_{t\in[0,\bar{T}]}$ form a **tangent Lévy model** if the following holds under the pricing measure:

1. $C^{\kappa_t,S_t} = C_t$ at each t.
2. Process S is a *martingale*, and $\kappa_t \geq 0$.
3. S and κ evolve according to

$$\begin{cases} S_t = S_0 + \int_0^t \int_{\mathbb{R}} S_{u-}(\exp\left(\gamma(\omega,u,x)\right) - 1)(N(dx,du) - \rho(x)dxdu), \\ \kappa_t = \kappa_0 + \int_0^t \alpha_u du + \sum_{n=1}^m \int_0^t \beta_u^n dB_u^n, \end{cases}$$

where

- $B = \left(B^1,\ldots,B^m\right)$ is a m-dimensional Brownian motion,
- N is a *Poisson random measure* with compensator $\rho(x)dxdu$,
- $\gamma(\omega,t,x)$ is a *predictable random function*,
- processes α and $\{\beta^n\}_{n=1}^m$ take values in a corresponding function space.

Consistency conditions

Given that 2 and 3 hold, 1 is equivalent to the following pair of conditions:

1. *Drift restriction:*

$$\alpha_t(T,x) = \mathbf{Q}(\beta_t; T, x) :=$$
$$-e^{-x}\sum_{n=1}^{m}\int_{\mathbb{R}}\int_t^T \partial^2_{y^2}\psi^{\beta^n_t}(u;y)\,du\,\Big[\psi^{\beta^n_t}(T;x-y)$$
$$-(1-y\partial_x)\,\psi^{\beta^n_t}(T;x)\Big] - \int_t^T \psi^{\beta^n_t}(u;y)\,du\,\psi^{\beta^n_t}(T;x-y)\,dy$$

2. *Compensator specification:* $\kappa_t(t,x)dxdt = (\rho(x)\,dxdt)\circ\gamma^{-1}(t,.)$

where $\psi^{\beta^n_t}(T,x) = -e^x\int_x^{\mathrm{sign}(x)\infty}\beta^n_t(T,y)dy$

Specifications

- Choose $\rho(x) := e^{-\lambda|x|}\left(|x|^{-1-2\delta} \vee 1\right)$, with some fixed $\lambda > 1$ and $\delta \in (0,1)$.
- Consider κ of the form: $\kappa(T,x) = \rho(x)\tilde{\kappa}(T,x)$, where $\tilde{\kappa}$ is an element of the space of continuous functions, equipped with usual "sup" norm.
- Then $\tilde{\alpha}_t = \alpha_t/\rho$ and $\tilde{\beta}_t = \beta_t/\rho$, and we have

$$d\tilde{\kappa}_t = \tilde{\alpha}_t dt + \tilde{\beta}_t \cdot dB_t,$$

stopped at $\tau_0 = \inf\left\{ t \geq 0 : \ \inf_{T\in[t,\bar{T}],x\in\mathbb{R}} \tilde{\kappa}_t(T,x) \leq 0 \right\}$.

- Then, $\kappa_t := \rho\tilde{\kappa}_{t\wedge\tau_0}$ is nonnegative and **changes on an open set in a linear space**!
- There exists a (tractable) specification $\gamma(t,x) := \Gamma(\tilde{\kappa}_t; x)$ which fulfills the "compensator specification" automatically.

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Local existence

$$\begin{cases} S_t = S_0 + \int_0^t \int_{\mathbb{R}} S_{u-}(\exp(\Gamma(\tilde{\kappa}_u; x)) - 1)(N(dx, du) - \rho(x)dxdu) \\ \tilde{\kappa}_t = \tilde{\kappa}_0 + \int_0^{t\wedge\tau_0} \mathbf{Q}(\rho\tilde{\beta}_u)du + \sum_{n=1}^m \int_0^{t\wedge\tau_0} \tilde{\beta}_u^n dB_u^n \end{cases} \quad (3)$$

For any given Poisson random measure N, with compensator $\rho(x)dxdt$, any Brownian motion $B = (B^1, \ldots, B^m)$ independent of N, and any progressively measurable square integrable stochastic processes $\left\{\tilde{\beta}^n\right\}_{n=1}^m$ (with values in corresponding function space) independent of N, there exists a unique pair $(S_t, \tilde{\kappa}_t)_{t\in[0,\bar{T}]}$ of processes satisfying (3). **The pair** $(S_t, \rho\tilde{\kappa}_{t\wedge\tau_0})_{t\in[0,\bar{T}]}$ **forms a tangent Lévy model**.

Example of a tangent Lévy model

- Choose $m = 1$, and $\tilde{\beta}_t(T,x) = \xi_t C(x)$, where $C(x)$ is some fixed function (satisfying some technical conditions), and

$$\xi_t = \xi(\tilde{\kappa}_t) = \frac{\sigma}{\epsilon}\left(\inf_{T\in[t,\bar{T}],x\in\mathbb{R}} \tilde{\kappa}_t(T,x) \wedge \epsilon\right)$$

- Then "drift restriction" simplifies to

$$\mathbf{Q}(\rho\tilde{\beta}_t;T,x) = -\frac{e^{-x}}{\rho(x)}\int_{\mathbb{R}}\int_t^T \partial_y\psi^{\rho\tilde{\beta}_t}(u,y)\,du\,\partial_x\psi^{\rho\tilde{\beta}_t}(T,x-y)$$
$$-\int_t^T \psi^{\rho\tilde{\beta}_t}(u,y)\,du\,\psi^{\rho\tilde{\beta}_t}(T,x-y)\,dy = \xi^2(\tilde{\kappa}_t)\,(T-t\wedge T)\,A(x)$$

and

$$\tilde{\kappa}_t(T,x) = \tilde{\kappa}_0(T,x) + (T-t\wedge T)\,A(x)\int_0^t \xi^2(\tilde{\kappa}_u)du + C(x)\int_0^t \xi^2(\tilde{\kappa}_u)dB_u$$

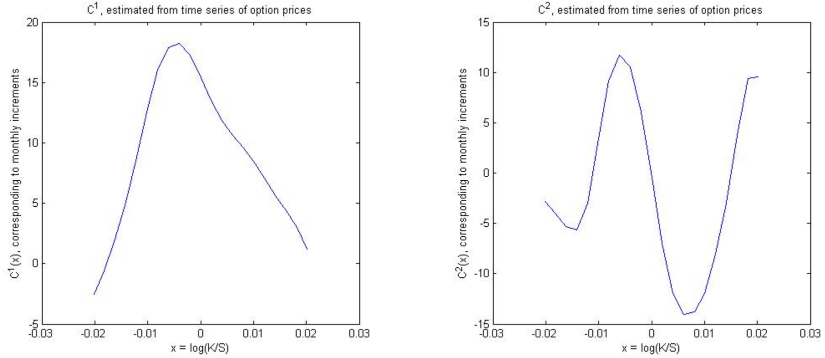
Conclusions

- We have described a general approach to constructing **market models** for Call options: find the right **code-book** by choosing a space of **tangent models**, prescribe time-evolution of the code-book value via its semimartingale characteristics and analyze **consistency** of resulting dynamics.
- This approach was illustrated by **"Tangent Lévy Models"** - *a large class of market models, explicitly constructed and parameterized by* $\tilde{\beta}$!
- Proposed *market models* allow one to start with observed call price surface and model explicitly its future values under the risk-neutral measure. For example, they provide a flexible framework for simulating the (arbitrage-free) evolution of *implied volatility surface*.

Further extensions

- One needs to consider $\tilde{\beta}_t = \tilde{\beta}(\tilde{\kappa}_t)$ and solve the resulting SDE for $\tilde{\kappa}_t$, as shown in the example, in order to **ensure that $\tilde{\kappa}$ stays positive**.
- There exists an extension of the Lévy-based code-book, the pair **("Lévy density", "instantaneous volatility")**, which allows the "true" underlying to have a non-trivial continuous martingale component.

Estimation of TL Model



Estimated coefficients C^1 and C^2, as functions of $x = \log(K/S)$.