# The Evaluation of Barrier Option Prices Under Stochastic Volatility

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# Plan of Talk

- Barrier Options: Discretely and Continously monitored
- PDE setup for Barrier option prices under Heston Model
- Allow for early exercise
- Method of Lines Approach
- Numerical Examples
- Conclusion

# 1 Barrier Option

- Path-dependent options, very popular in foreign exchange markets. The purchaser uses them to hedge very specific cash flows with similar properties but pays a cheaper price than regular options.
- Payoff is dependent on the realized asset path via its level. See Figure 1 for up-and-out call option payoff.
- Apart from "out" options, there are also "in" options which only receive a payoff if a certain level is reached, otherwise they expire worthless.
- In-Out-Parity for Barrier options: Knock-in $+$ Knock-out = Vanilla.
- Put-Call-Symmetry for Barrier Options: Up-and-out Call$(\boldsymbol{S}, \boldsymbol{K}, \boldsymbol{H}, \boldsymbol{r}, \boldsymbol{q}, \boldsymbol{\rho}) =$ Down-and-out Put$(\boldsymbol{K}, \boldsymbol{S}, \boldsymbol{SK}/\boldsymbol{H}, \boldsymbol{q}, \boldsymbol{r}, -\boldsymbol{\rho})$.
- We consider up-and-out call options in the following.

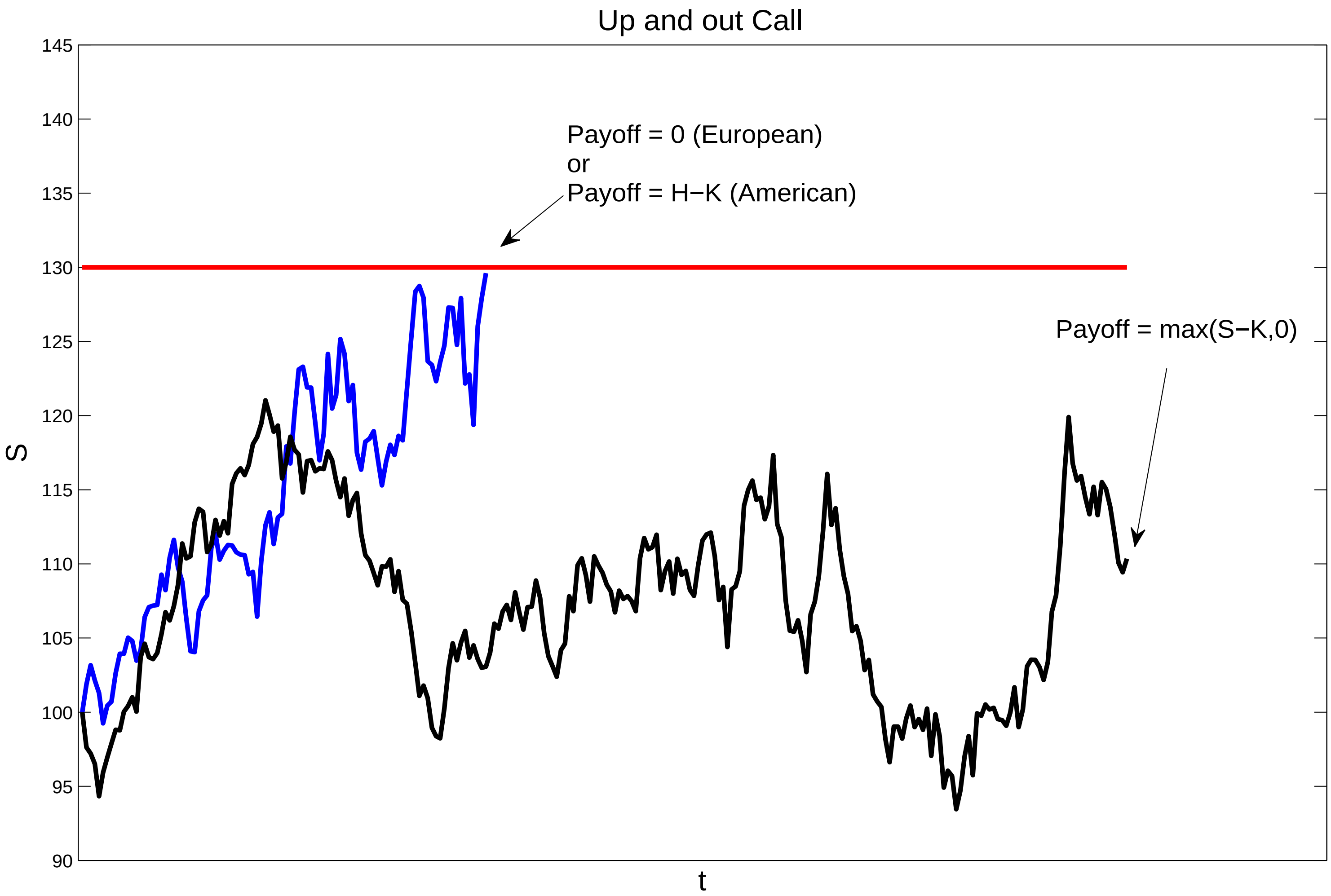


Figure 1: Diagram: Payoff of an up and out call options.

## 2 Literature Review

- Merton (1973): the derivation of the pricing formula for barrier options;
- Rich (1994) and Wong & Kwok (2003): a list of pricing formulas for one and multi-asset barrier options both under the GBM framework;
- Gao, Huang & Subrahmanyam (2000): option contracts under GBM with both knock-out barrier and American early exercise features;
- Zvan, Vetzal & Forsyth (2000): discuss the oscillatory behavior of the Crank - Nicolson method for pricing barrier options. The backward Euler method is applied to avoid unwanted oscillations;
- Griebsch (2008): discusses evaluation of barrier option prices under the Heston model with Fourier transform approach;
- Yousuf (2008, 2009): develops a higher order smoothing scheme for pricing barrier options under SV without early exercise.

# 3 Barrier Options - Evaluation under SV

- We follow Heston (1993) assuming the dynamics for $S$ under **RN measure** governed by

$$
\begin{aligned}
dS &= (r-q)Sdt + \sqrt{v}SdZ_1, \\
dv &= (\kappa_v\theta_v - (\kappa_v + \lambda)v)dt + \sigma\sqrt{v}dZ_2.
\end{aligned}
$$

- Here $S$ and $v$ are correlated with $E(dZ_1 dZ_2) = \rho dt$.
- Assumes market price of vol. risk $= \lambda\sqrt{v}$.

- The price of a barrier option $C(S, v, \tau)$ at time to maturity $\tau$ is the solution to a partial differential equation (PDE) problem.
- We need to solve the PDE

$$\frac{\partial C}{\partial \tau} = \mathcal{K}C - rC,$$

on the interval $0 \leq \tau \leq T$, where the Kolmogorov operator $\mathcal{K}$

$$\begin{aligned} \mathcal{K} &= \frac{vS^2}{2}\frac{\partial^2}{\partial S^2} + \rho\sigma vS\frac{\partial^2}{\partial S \partial v} + \frac{\sigma^2 v}{2}\frac{\partial^2}{\partial v^2} + (r - q)\, S\frac{\partial}{\partial S} + \\ &\quad (\kappa_v(\theta_v - v) - \lambda v)\frac{\partial}{\partial v}. \end{aligned}$$

## 3.1 Continuously monitored barrier options

- Continuously monitored barrier option $C(S, v, \tau)$: an option which is monitored all the time between the current time $t$ and the maturity of the option at time $T$. Note that $\tau = T - t$.
- The option, has the terminal condition

$$C(S, v, 0) = (S - K)^+.$$

- The domain for the up and out call option is

$$0 \leq S \leq H,\ 0 < v < \infty,\ 0 \leq \tau \leq T.$$

- The boundary conditions for the barrier option without the early exercise features are:

$$C(0, v, \tau) = 0;\ C(H, v, \tau) = 0;\ \lim_{v \to \infty} C_v(S, v, \tau) = 0.$$

- The option with early exercise features has the free boundary condition

$$C(b(v,\tau),v,\tau) = b(v,\tau) - K, \text{ when } b(v,\tau) < H$$

where $S = b(v,\tau)$ is the early exercise boundary for the barrier option at time to maturity $\tau$ and variance $v$.

- There also hold the smooth-pasting conditions

$$\lim_{S\to b(v,\tau)} \frac{\partial C}{\partial S} = 1, \quad \lim_{S\to b(v,\tau)} \frac{\partial C}{\partial v} = 0.$$

- In the above case,

$$C(S,v,\tau) = S - K, \ \forall\, b(v,\tau) < S < H.$$

- However, if we cannot find a $b(v,\tau) < H$ then

$$C(H,v,\tau) = H - K.$$

- Technically, for the knock-out event and the exercise date to be well defined,
  - the option contract is defined in a way such that when the asset price first touches the barrier, the option holder has the option to either exercise or let the option be knocked out.
- Since in this paper we assume the rebate is equal to zero, the option should be exercised once the asset price touches the barrier.

## 3.2 Discretely monitored barrier options

- A discretely monitored barrier option is an option which is monitored only at discrete dates $t \le t_1 < t_2 < \cdots < t_N \le T$.
- The option has the terminal condition

$$C(S, v, 0) = (S - K)^{+}.$$

- The domain for the up and out call option is:

$$S \in \begin{cases} (0, H), & \tau \in \{T - t_N, T - t_{N-1}, \cdots, T - t_1\}, \\ (0, \infty), & \text{otherwise}, \end{cases}$$

and

$$0 < v < \infty, \; 0 < \tau < T.$$

- The boundary conditions for the barrier option without early exercise features are:

$$\begin{aligned} C(0, v, \tau) &= 0; \\ C(H, v, \tau) &= 0, \ \forall \tau \in \{T - t_N, \cdots, T - t_1\}; \\ \lim_{S \to \infty} C(S, v, \tau) &= 0, \ \forall \tau \notin \{T - t_N, \cdots, T - t_1\}; \\ \lim_{v \to \infty} C_v(S, v, \tau) &= 0. \end{aligned}$$

- A discretely monitored barrier option with the early exercise feature, at the monitoring times $\tau \in \{T - t_N, \cdots, T - t_1\}$, has the free (early exercise) boundary condition

$$C(b(v, \tau), v, \tau) = b(v, \tau) - K, \text{ when } b(v, \tau) < H.$$

- Here $b(v,\tau)$ is the early exercise boundary for the barrier option at time to maturity $\tau$ and variance $v$, and satisfies the smooth-pasting conditions

$$\lim_{S \to b(v,\tau)} \frac{\partial C}{\partial S} = 1, \quad \lim_{S \to b(v,\tau)} \frac{\partial C}{\partial v} = 0.$$

- In the above case, we have

$$C(S,v,\tau) = S - K, \ \forall\, b(v,\tau) < S < H$$

so that $C(S,v,\tau)$ is known over $0 < S < H$.

- If there is no such $b(v,\tau)$ then for the same reason as the case for the continuously monitored option, $C(S,v,\tau)$ must satisfy

$$C(H,v,\tau) = H - K.$$

- At all other times $\tau \notin \{T - t_N, \cdots, T - t_1\}$, standard American option free boundary conditions apply.

# 4 Method of Lines (MOL) Approach

- The method of lines has several strengths when dealing with Barrier options, especially when allowing early exercise features:
  - The **price**, **free boundary**, **delta** and **gamma** are all found as part of the computation.
  - The method discretises the PDE in an intuitive manner, and is readily adapted to be **second order accurate in time**.
- The key idea behind the method of lines is to replace a PDE with an equivalent system of one-dimensional ODEs.
- The **system of ODEs** is developed by discretising the time derivative and the derivative terms involving the variance, $v$.

- **The PDE** to be solved is

$$\frac{\partial C}{\partial \tau} = \frac{vS^2}{2}\frac{\partial^2 C}{\partial S^2} + \rho\sigma v S\frac{\partial^2 C}{\partial S \partial v} + \frac{\sigma^2 v}{2}\frac{\partial^2 C}{\partial v^2} + (r-q)S\frac{\partial C}{\partial S} + (\kappa_v(\theta_v - v) - \lambda v)\frac{\partial C}{\partial v}.$$

- The computational domain for the problem will depend on the specific Barrier option, for example,
  - for a continuously monitored up and out call option, we would have:

$$0 < S_0 < S < H,\ 0 < v < \infty,\ 0 < \tau < T.$$

- We discretise according to $\tau_n = n\Delta\tau$ and $v_m = m\Delta v$, where $n = 1, \ldots, N; m = 1, \ldots, M.$

- $C(S, v_m, \tau_n) = C_m^n(S)$,

$$V(S, v_m, \tau_n) \equiv \frac{\partial C(S, v_m, \tau_n)}{\partial S} = V_m^n(S).$$

- We use the **standard central difference scheme**

$$\frac{\partial^2 C}{\partial v^2} = \frac{C_{m+1}^n - 2C_m^n + C_{m-1}^n}{(\Delta v)^2}, \quad \frac{\partial^2 C}{\partial S \partial v} = \frac{V_{m+1}^n - V_{m-1}^n}{2\Delta v}.$$

- We use an **upwinding finite difference scheme** for the first order derivative term

$$\frac{\partial C}{\partial v} = \begin{cases} \frac{C_{m+1}^n - C_m^n}{\Delta v} & \text{if} \quad v \leq \frac{\alpha}{\beta}, \\ \frac{C_m^n - C_{m-1}^n}{\Delta v} & \text{if} \quad v > \frac{\alpha}{\beta}. \end{cases}$$

- A **second order approximation for the time derivative**,

$$\frac{\partial C}{\partial \tau} = \frac{3}{2}\frac{C_m^n - C_m^{n-1}}{\Delta\tau} + \frac{1}{2}\frac{C_m^{n-1} - C_m^{n-2}}{\Delta\tau}.$$

- After taking the boundary conditions into consideration, we must solve a system of $(M-1)$ second order ODEs in $S$ along the line segment $(v_m, \tau_n), S \in [S_0, H]$ or $S \in [S_0, S_{max}]$ depending on the properties of the barrier option for $m = 1, .., M-1$ and fixed $\tau_n$.

- We then solve the ODEs for increasing values of $v$, using the latest available estimates for $C_{m+1}^n$, $C_{m-1}^n$, $V_{m+1}^n$ and $V_{m-1}^n$.

- We iterate until the price profile converges to a desired level of accuracy.

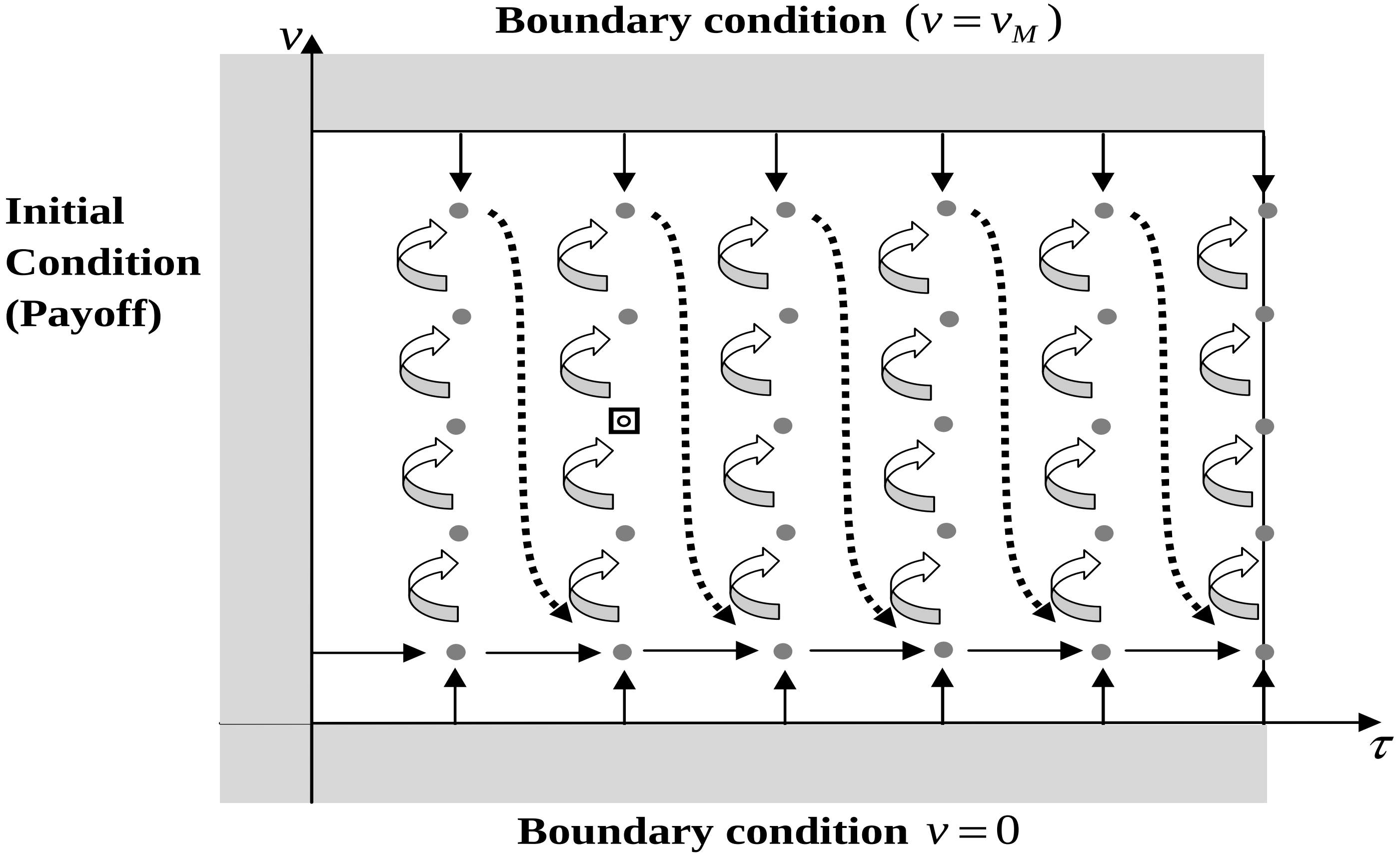


Figure 2: One sweep of the solution scheme on the $v - \tau$ grid. The stencil for the typical point $\boxed{\text{o}}$ is displayed in Figure 3.

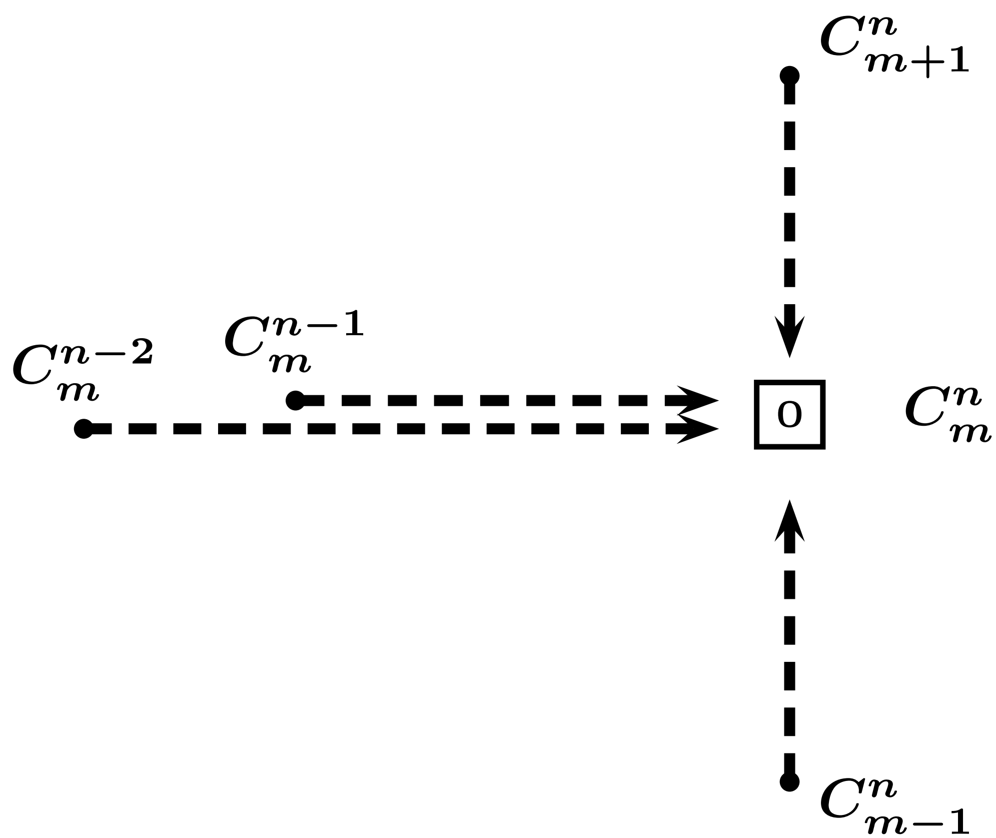


Figure 3: Stencil for the typical grid point o of Figure 2. The stencil for $C^n_m$ depends on $(C^n_{m-1}, C^n_m, C^n_{m+1}, C^{n-1}_m, C^{n-2}_m)$.

- **The generic first order form of the ODE**

**Delta** $$\frac{dC_m^n}{dS} = V_m^n,$$

**Gamma** $$\frac{dV_m^n}{dS} = A_m(S)C_m^n + B_m(S)V_m^n + P_m^n(S),$$

where $P_m^n(S)$ is also a function of $C_{m+1}^n$, $C_{m-1}^n$, $V_{m+1}^n$, $V_{m-1}^n$, $C_m^{n-1}, C_m^{n-2}$.

- We solve the above system using the Riccati transform.

- **The Riccati transformation**

$$C_m^n(S) = R_m(S)V_m^n(S) + W_m^n(S).$$

- Where $R$ and $W$ are solutions to the initial value problems

$$\frac{dR_m}{dS} = 1 - B_m(S)R_m - A_m(S)(R_m)^2, \quad R_m(S_0) = 0,$$
$$\frac{dW_m^n}{dS} = -A_m(S)R_m(S)W_m^n - R_m(S)P_m^n(S), \quad W_m^n(S_0) = 0.$$

- Given $R$ and $W$ we try to find $V_m^n$ by solving

$$\frac{dV_m^n}{dS} = A_m(S)(R_m(S)V_m^n + W_m^n(S)) + B_m(S)V_m^n + P_m^n(S),$$

backward subject to an terminal condition which depends on the properties and the specifications of the barrier options.

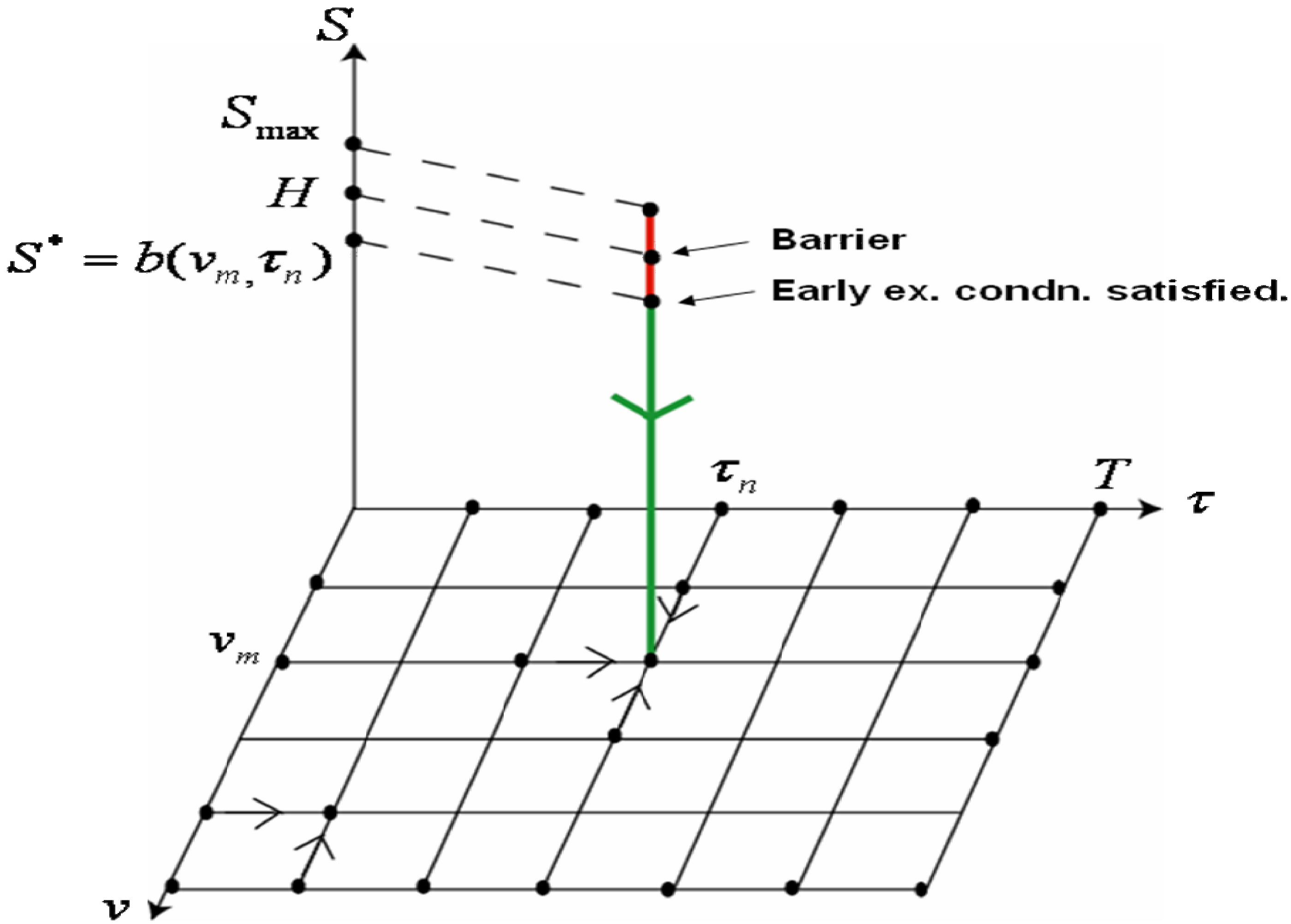


Figure 4: Solving for the option prices along a $(v_m, \tau_n)$ line.

- Continuously monitored barrier options without early exercise opportunities, using the fact that $C_m^n(H) = 0$ we obtain from the Riccati transform that the terminal condition is

$$V_m^n(H) = -\frac{W_m^n(H)}{R_m(H)},$$

  and then integrate the equation for $V_m^n$ from $S = H$ to $S = S_0$.

- Continuously monitored barrier option with early exercise opportunity we integrate the equation for $W_m^n$ and $R_m$ from $S_0$ to $S_{max}$ and monitor the function

$$\phi(S) = R_m(S) + W_m^n(S) - (S - K).$$

- If $\phi(S^*) = 0$ for some $S^* \in (S_0, H)$ then $S^*$ is the early exercise boundary $b(v_m, \tau_n) = b_m^n$ at the grid point $(v_m, \tau_n)$.
  - Once $b_m^n$ is found we integrate the equation for $V_m^n$ backward from $b_m^n$ toward $S_0$ subject to the terminal condition

$$V(b_m^n) = 1.$$

- If $\phi(S)$ has no zero in $[S_0, H)$ then there is no early exercise below the barrier and we solve the equation for $V_m^n$ subject to

$$V_m^n(H) = \frac{H - K - W_m^n(H)}{R_m(H)}.$$

  - In fact, at any time to maturity $\tau$, if the asset hits the barrier $H$, then the option will be exercised, namely, $C(H, v, \tau) = H - K$, according to the Riccati transform we have

$$C_m^n(H) = R_m(H)V_m^n(H) + W_m^n(H) = H - K.$$

# 5 Numerical Examples

| Parameter | Value | SV Parameter | Value |
|---|---|---|---|
| $\boldsymbol{r}$ | 0.03 | $\boldsymbol{\theta}$ | 0.1 |
| $\boldsymbol{q}$ | 0.05 | $\boldsymbol{\kappa_v}$ | 2.00 |
| $\boldsymbol{T}$ | 0.5 | $\boldsymbol{\sigma}$ | 0.1 |
| $\boldsymbol{K}$ | 100 | $\boldsymbol{\lambda_v}$ | 0.00 |
| $\boldsymbol{\rho}$ | $\pm\mathbf{0.50}$ | $\boldsymbol{H}$ | 130 |

Table 1: Parameter values used for the barrier option. The stochastic volatility (SV) parameters are those used in Heston's original paper.

| $\rho = -0.50, v = 0.1$ | | | $S$ | | |
| --- | --- | --- | --- | --- | --- |
| Method $(N, M, S_{\text{pts}})$ | 80 | 90 | 100 | 110 | 120 |
| MOL (50,100,1140) | 0.9045 | 1.8807 | 2.5978 | 2.4859 | 1.4858 |
| MOL (100,200,6400) | 0.9044 | 1.8781 | 2.5908 | 2.4769 | 1.4782 |
| FD $(200, 100, 200)$ | 0.9029 | 1.8778 | 2.5903 | 2.4760 | 1.4775 |
| MC $(400, 20)$ | 0.9355 | 1.9579 | 2.7407 | 2.6706 | 1.6773 |
| MC upper bound | 0.9389 | 1.9628 | 2.7464 | 2.6762 | 1.6820 |
| MC lower bound | 0.9321 | 1.9530 | 2.7351 | 2.6649 | 1.6726 |

Table 2: Prices of the continuously monitored barrier option without early exercise features computed using method of lines (MOL), finite difference (FD) and Monte Carlo simulation (MC). Parameter values are given in Table 1, with $\rho = -0.50$ and $v = 0.1$.

| $\rho = -0.50, v = 0.1$ | | | $S$ | | |
|---|---|---|---|---|---|
| Method $(N, M, S_{\text{pts}})$ | 80 | 90 | 100 | 110 | 120 |
| MOL (50,150,1140) | 1.4009 | 3.9350 | 8.2981 | 14.4015 | 21.8229 |
| MOL (100,200,2440) | 1.4012 | 3.9364 | 8.3003 | 14.4033 | 21.8219 |
| MOL (100,200,6400) | 1.4012 | 3.9363 | 8.3003 | 14.4032 | 21.8216 |
| MOL (200,400,9100) | 1.4015 | 3.9371 | 8.3014 | 14.4037 | 21.8201 |
| MC $(100, 20, 50)$ | 1.3994 | 3.9238 | 8.2302 | 14.1086 | 20.9401 |
| MC upper bound | 1.4058 | 3.9347 | 8.2454 | 14.1261 | 20.9568 |
| MC lower bound | 1.3930 | 3.9129 | 8.2151 | 14.0909 | 20.9234 |

Table 3: Prices of the continuously monitored barrier option with early exercise features computed using method of lines (MOL) and Monte Carlo simulation (MC). Parameter values are given in Table 1, with $\rho = -0.50$ and $v = 0.1$.

| $\rho = -0.50, v = 0.1$ | | | $S$ | | |
|---|---|---|---|---|---|
| Method $(N, M, S_{\text{pts}})$ | 80 | 90 | 100 | 110 | 120 |
| MOL(50,100,1140) | 1.0764 | 2.5173 | 4.0895 | 4.9894 | 4.8291 |
| MOL (100,200,6400) | 1.0807 | 2.5289 | 4.1116 | 5.0235 | 4.8706 |
| COS $(100, 200, 100)$ | 1.0809 | 2.4871 | 4.0454 | 4.9779 | 4.8646 |
| MC $(400, 20)$ | 1.0780 | 2.5257 | 4.1033 | 5.0166 | 4.8605 |
| MC upper bound | 1.0834 | 2.5339 | 4.1135 | 5.0279 | 4.8718 |
| MC lower bound | 1.0726 | 2.5175 | 4.0930 | 5.0054 | 4.8492 |

Table 4: Prices of the discretely monitored barrier option without early exercise features computed using method of lines (MOL), Fourier Cosine expansion (COS) and Monte Carlo simulation (MC). Parameter values are given in Table 1, with $\rho = -0.50$ and $v = 0.1$.

| $\rho = -0.50, v = 0.1$ | | | $S$ | | |
|---|---|---|---|---|---|
| Method $(N, M, S_{\text{pts}})$ | 80 | 90 | 100 | 110 | 120 |
| MOL(50,100,1140) | 1.4008 | 3.9339 | 8.3010 | 14.4446 | 22.0389 |
| MOL (100,250,2400) | 1.4012 | 3.9364 | 8.3025 | 14.4182 | 21.8719 |
| MOL (150,250,6400) | 1.4014 | 3.9368 | 8.3028 | 14.4157 | 21.8615 |
| MC $(100, 20, 50)$ | 1.4002 | 3.9338 | 8.2967 | 14.4285 | 21.9274 |
| MC upper bound | 1.4066 | 3.9449 | 8.3123 | 14.4473 | 21.9459 |
| MC lower bound | 1.3938 | 3.9228 | 8.2810 | 14.4097 | 21.9089 |

Table 5: Prices of the discretely monitored barrier option with early exercise features computed using method of lines (MOL) and Monte Carlo simulation (MC). Parameter values are given in Table 1, with $\rho = -0.50$ and $v = 0.1$.


Figure 5: Price profile of a continuously monitored up-and-out call option without early exercise opportunities.

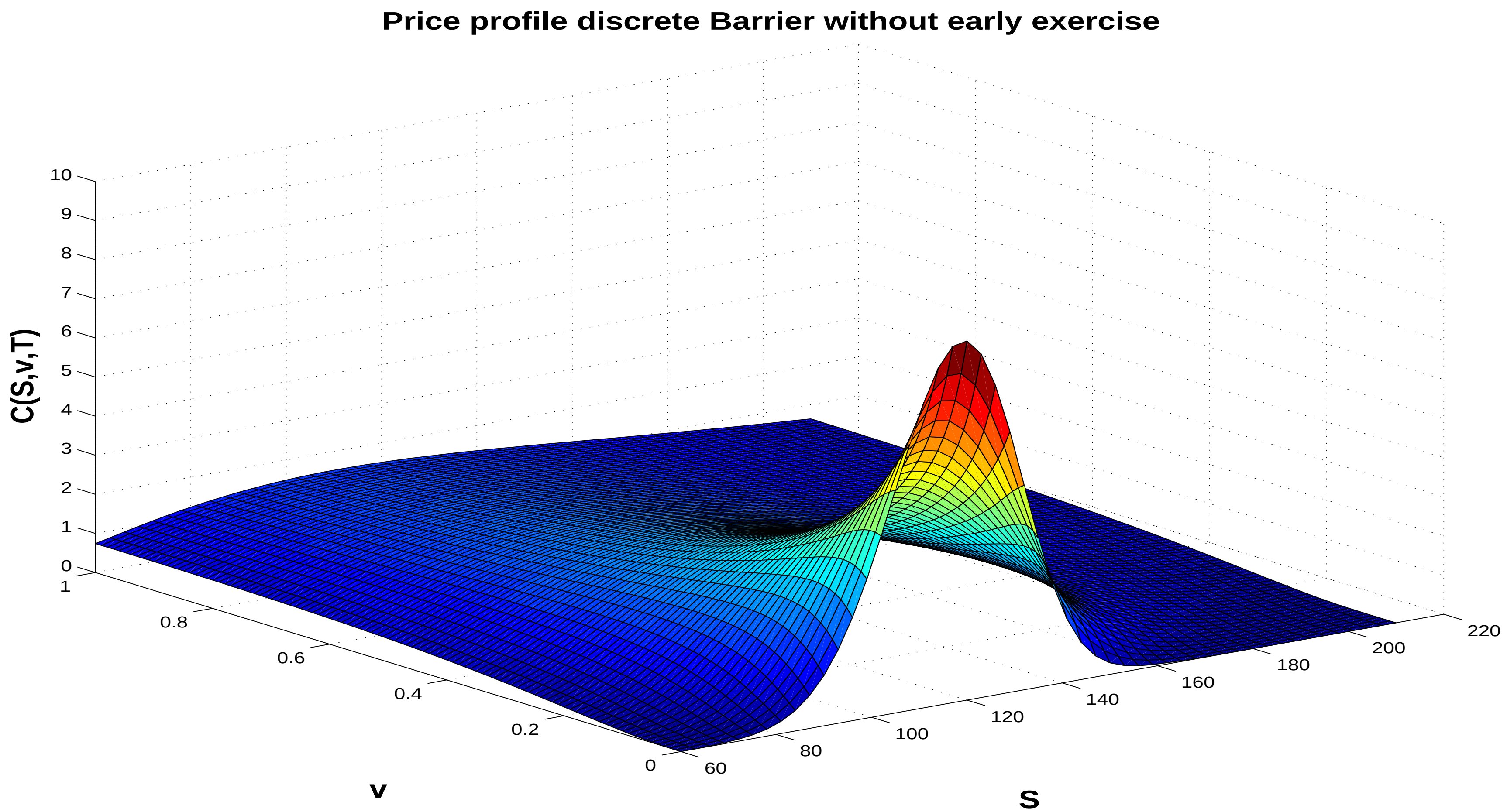


Figure 6: Price profile of a discretely monitored up-and-out call option without early exercise opportunities.

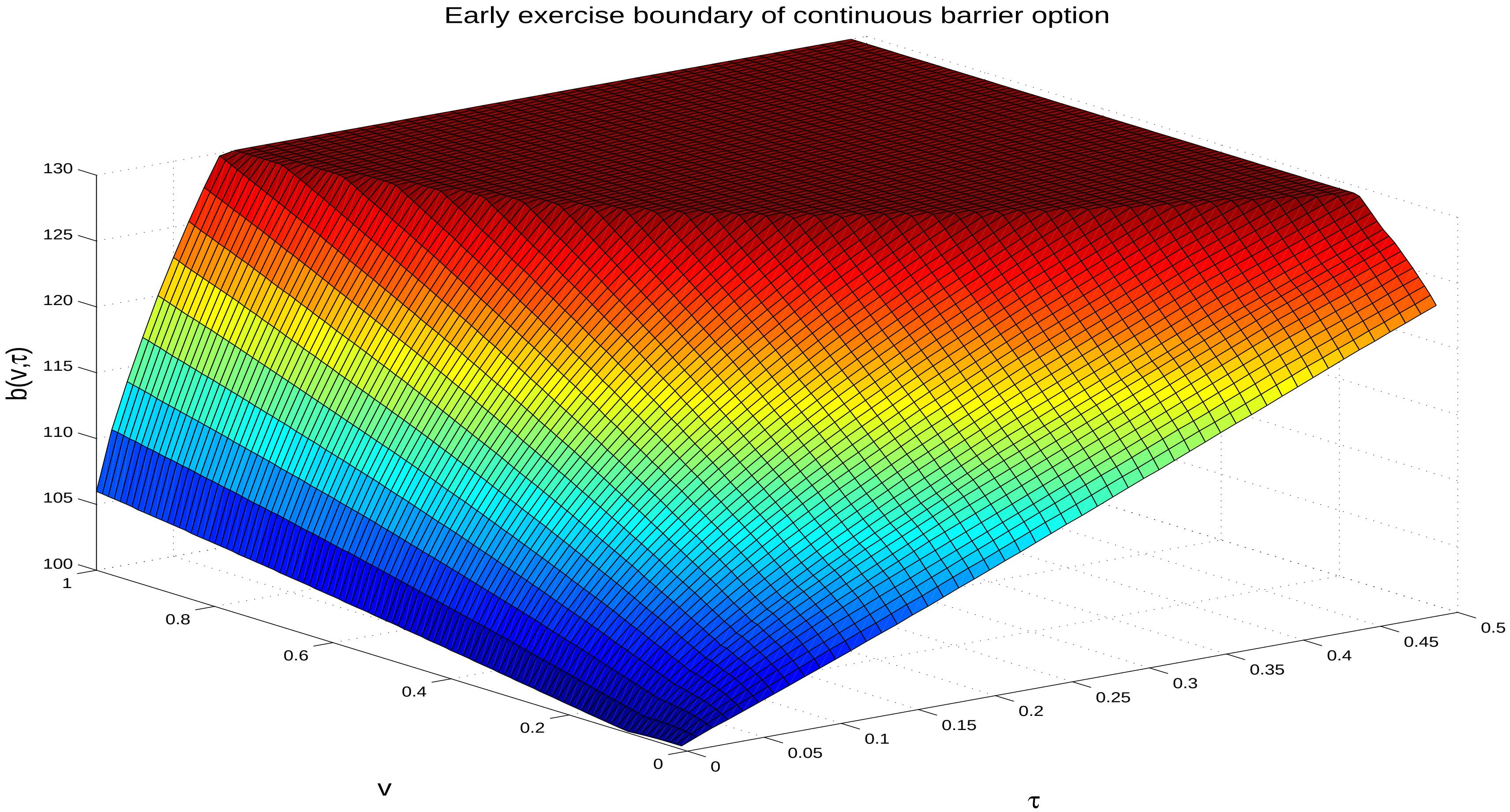


Figure 7: Early exercise boundary of a continuously monitored up-and-out call option.

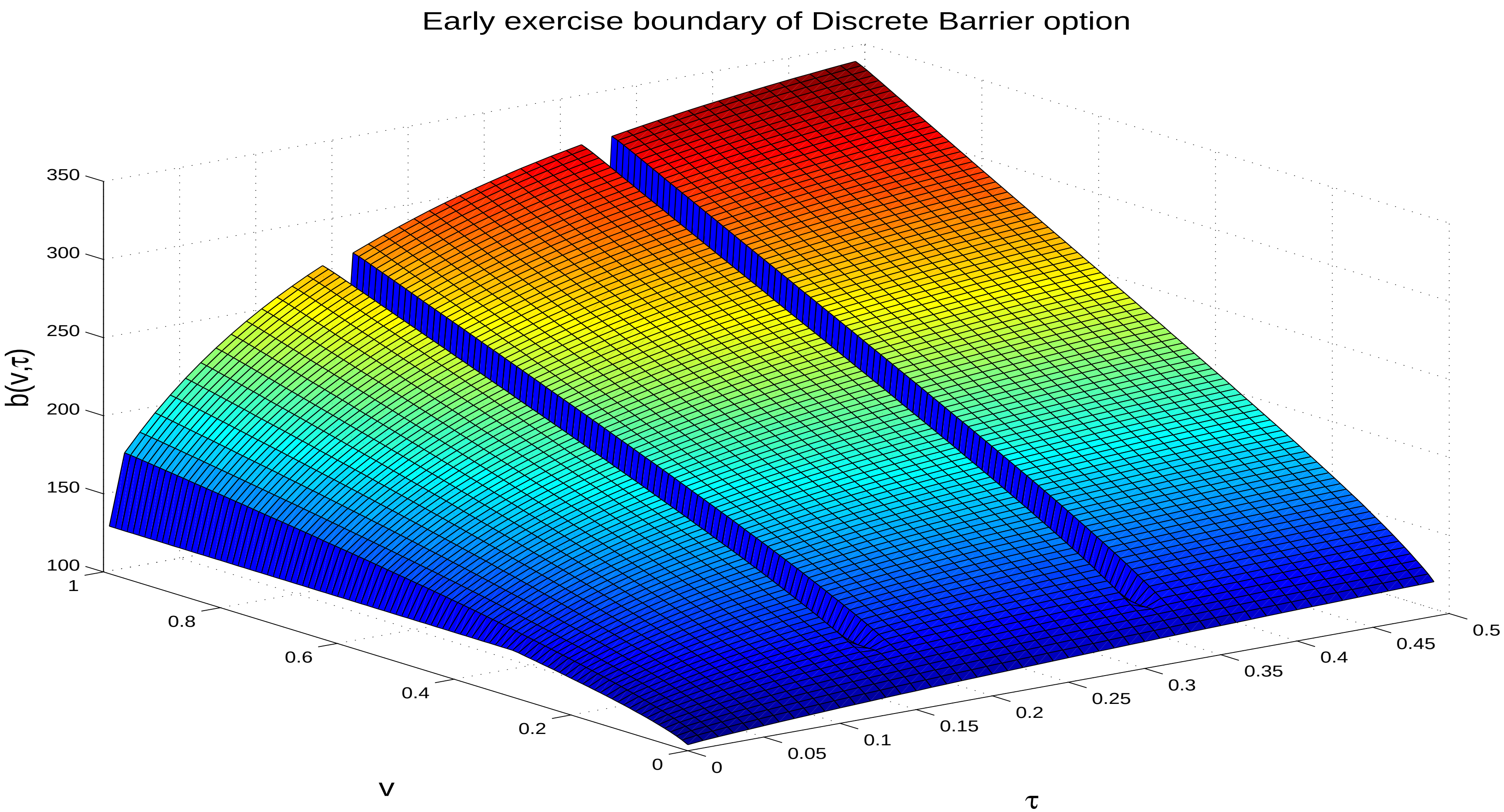


Figure 8: Early exercise boundary of a discretely monitored up-and-out call option.

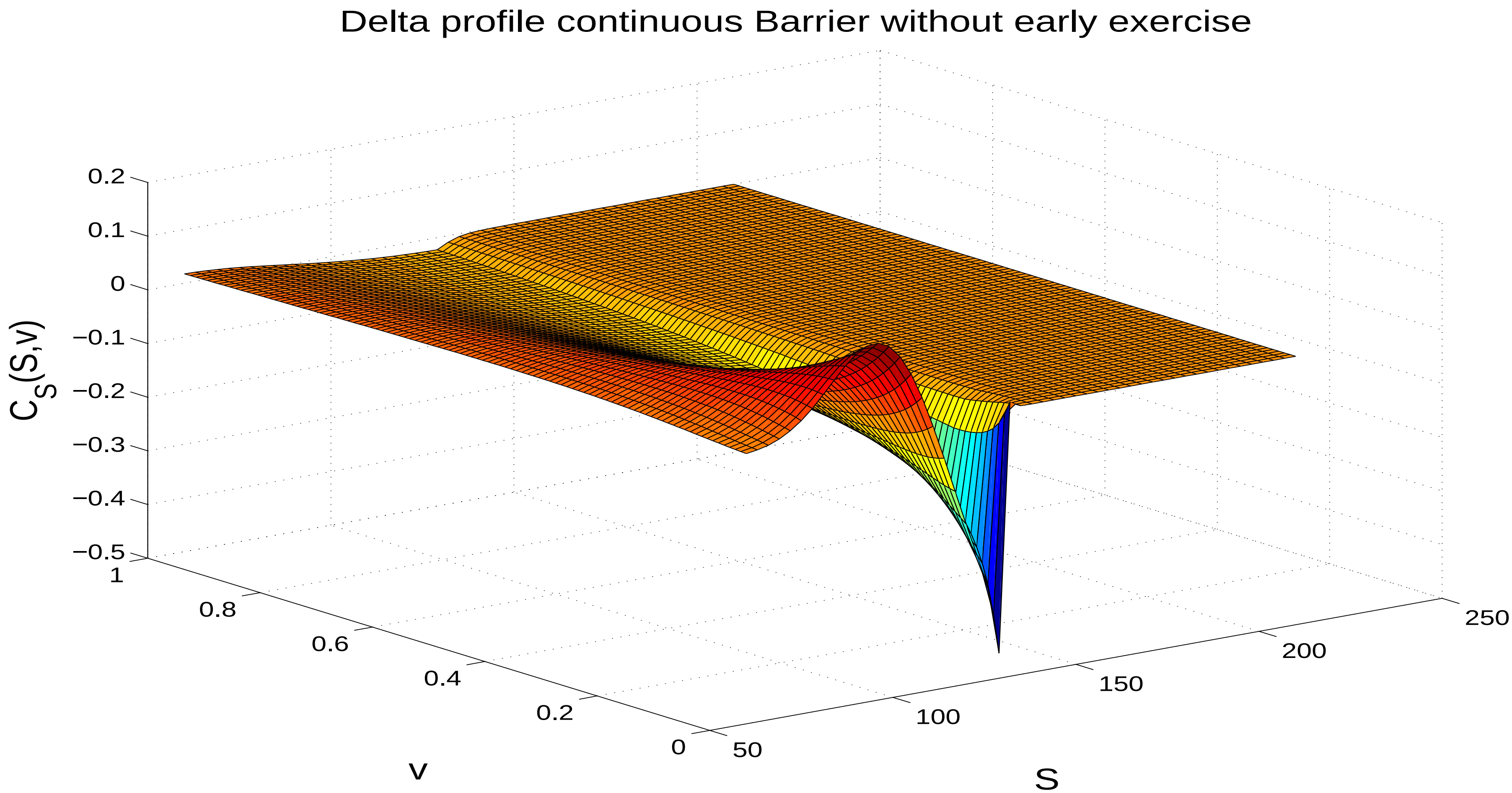


Figure 9: Delta profile of a continuously monitored up-and-out call option without early exercise opportunities.

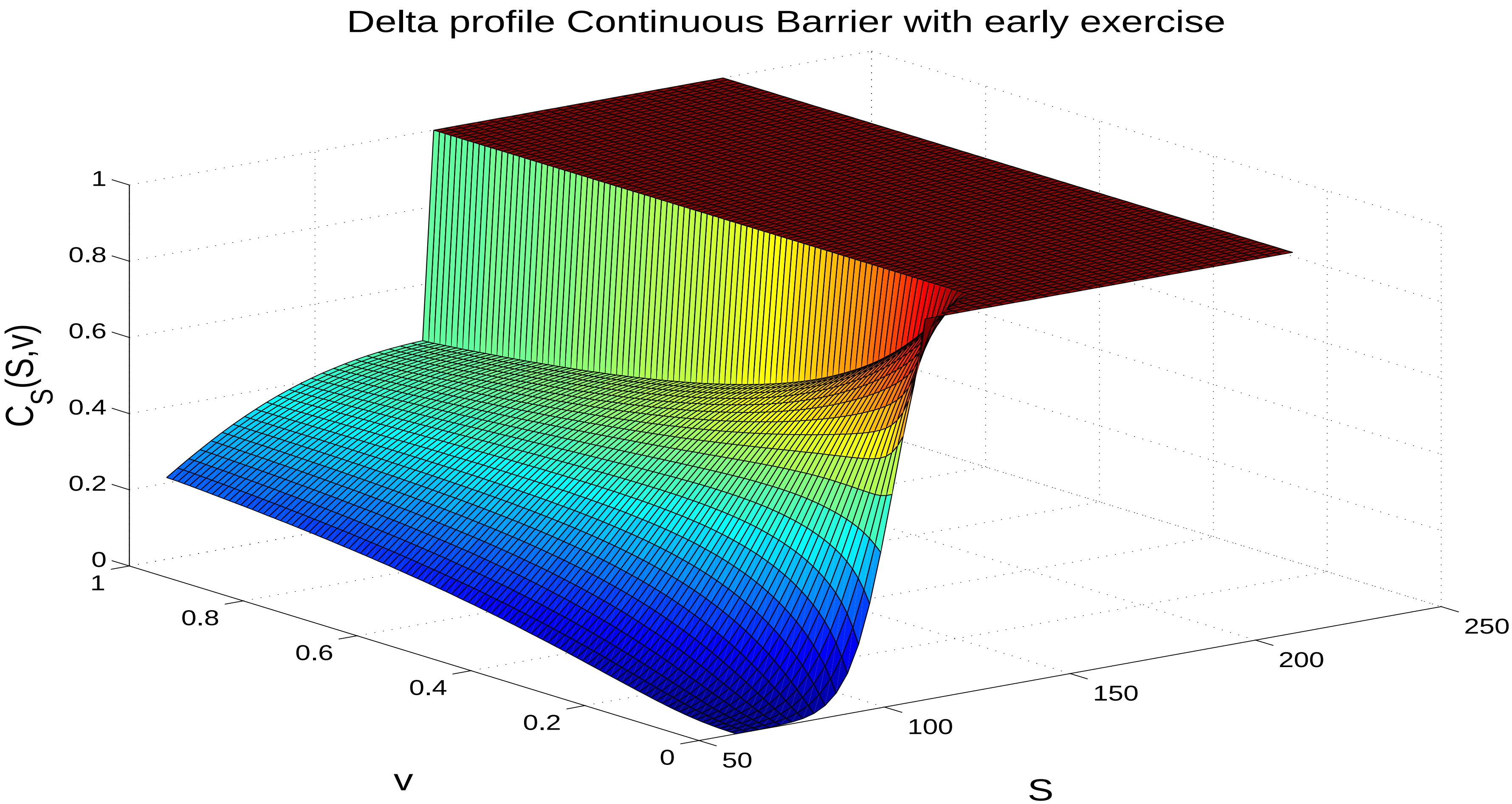


Figure 10: Delta profile of a continuously monitored up-and-out call option with early exercise opportunities.

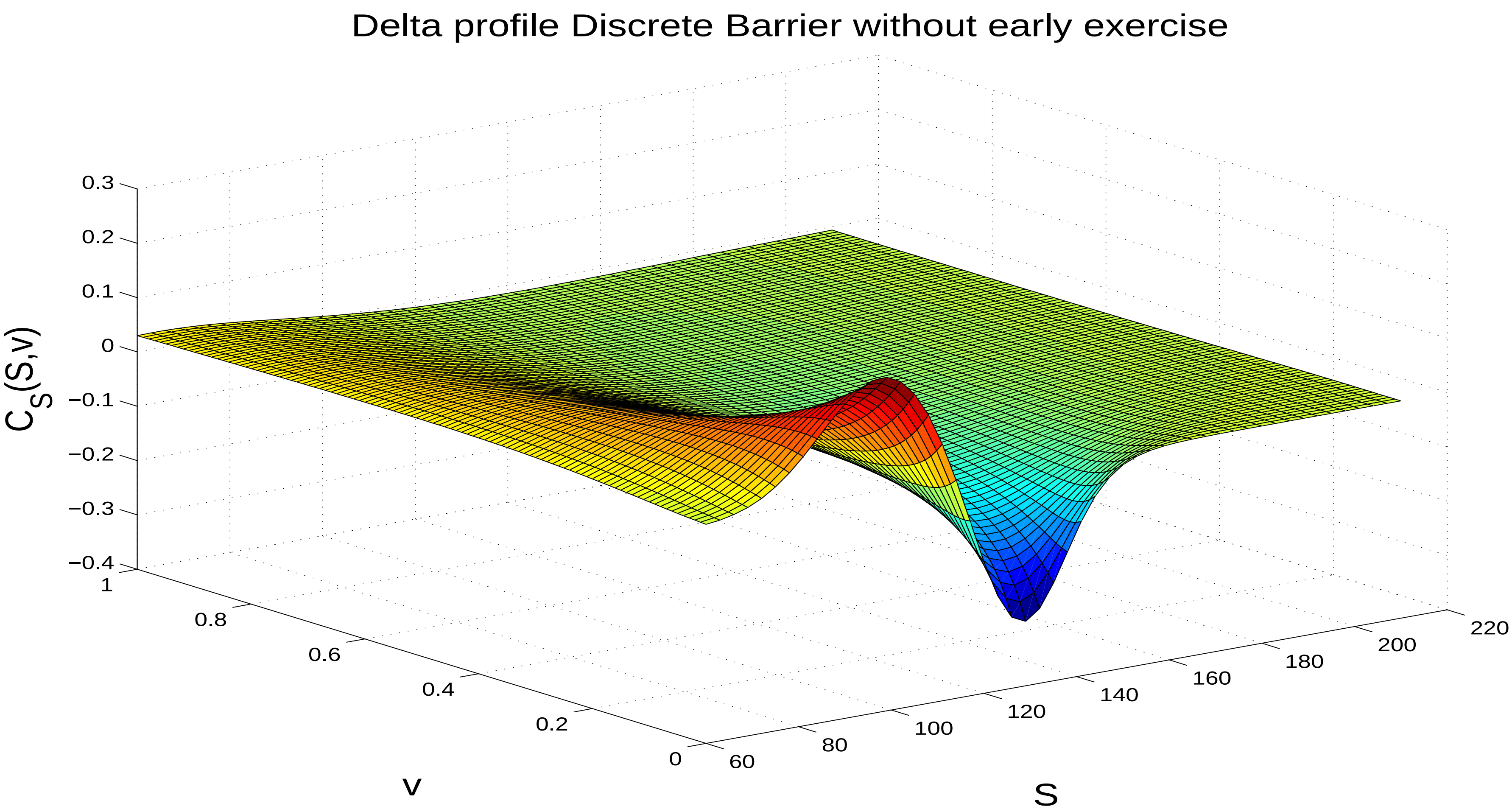


Figure 11: Delta profile of a discretely monitored up-and-out call option without early exercise opportunities.

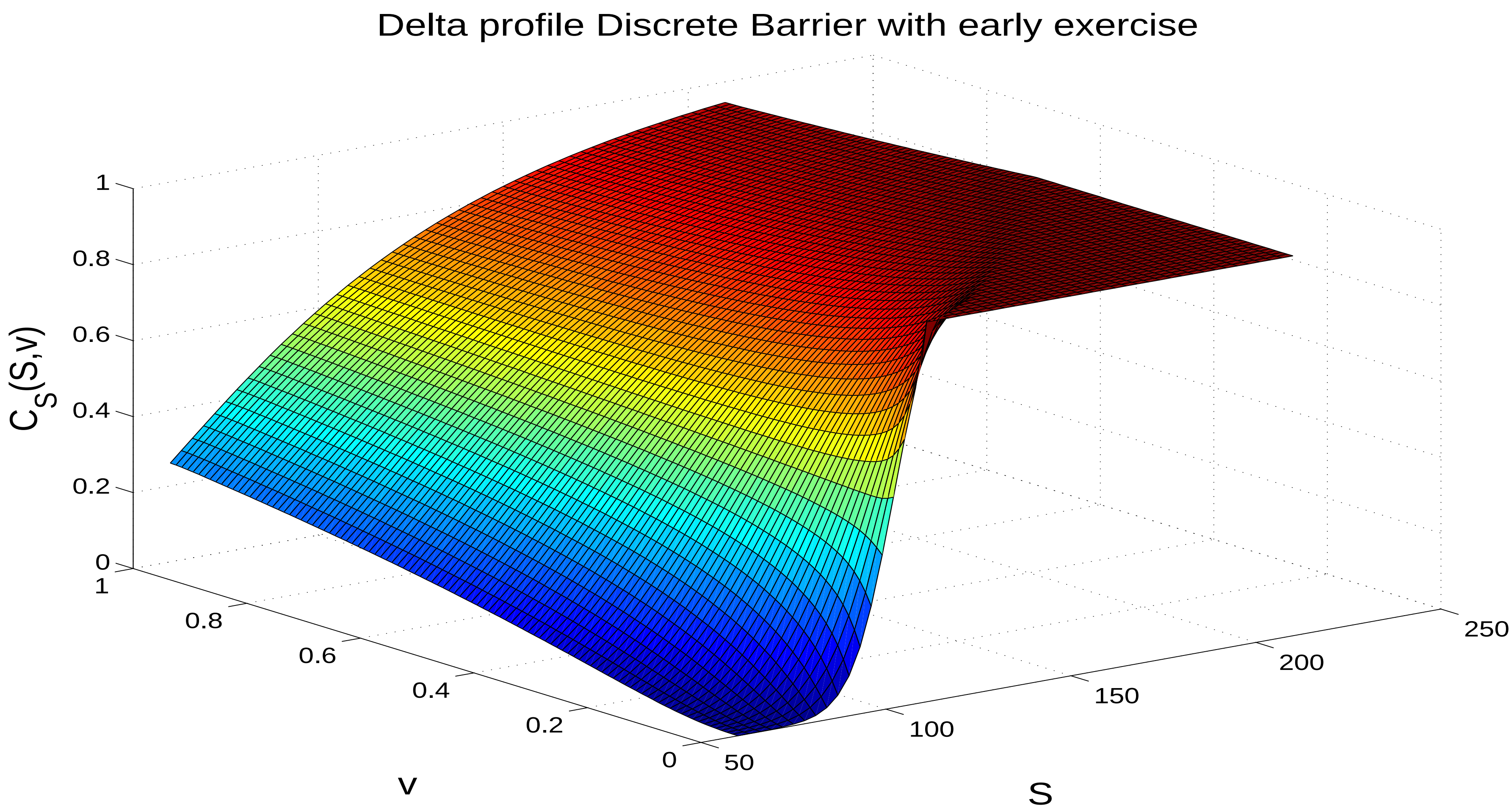


Figure 12: Delta profile of a discretely monitored up-and-out call option with early exercise opportunities.

# 6 Conclusions

- Set up a framework for pricing Barrier options under SV.
- Allow for early exercise features.
- Unify both continuously and discretely monitored options.
- Implement the method of lines approach.
- Some numerical examples.
- Future work:
  - Incorporating jump diffusion as well,
  - Pricing knock-in options under SV with early exercise features.

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