# Optimal Execution in a General One-Sided Limit-Order Book

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# Optimal Execution

Setup and main results

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- Agent - a large investor, purchases from a limit-order book with resilience.
- Purchases can be continuous (at rate) or in lumps.
- Objective: Minimize expected total cost of purchase.

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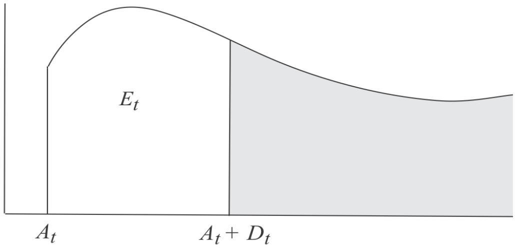

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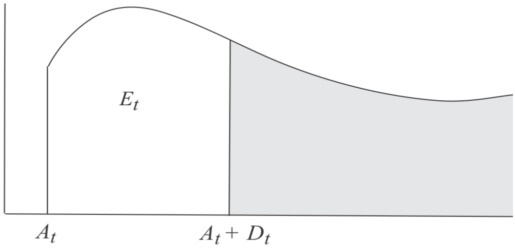


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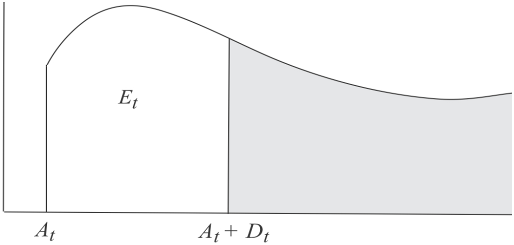


  - The shaded area shows the orders in the book. This is the actual limit-order book.
  - The white area $E_t$ shows orders missing from the shadow book.

# One-sided limit-order book, resilience

- $X_t, 0 \leq t \leq T$ — Cumulative purchases by our agent up to time $t$. $X_{0-} = 0$, $X_T = \overline{X}$, Nondecreasing and right-continuous.

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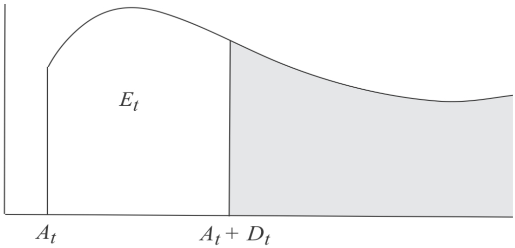
- $D_t, 0 \leq t \leq T$ — Price displacement due to the combined effect of agent's purchases and book's resilience.

# One-sided limit-order book, price displacement

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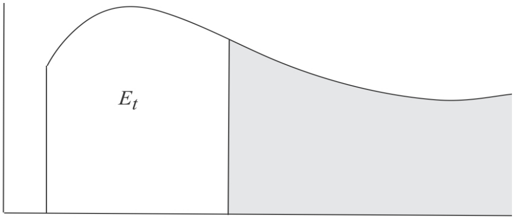


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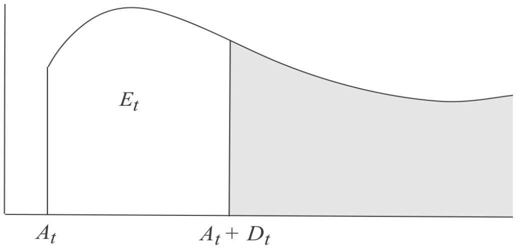


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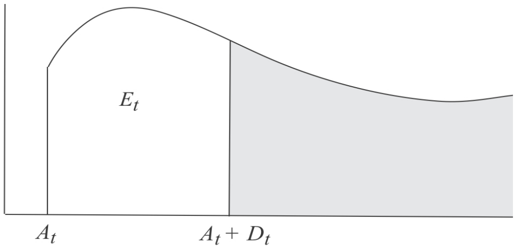


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- $D_t \triangleq \psi(E_t), 0 \leq t \leq T$.
- The current ask price is $A_t + D_t$.

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    !!! each purchase being exactly the amount by which LOB has recovered since the preceding purchase !!!

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# Optimal policy (Three-jump strategy)

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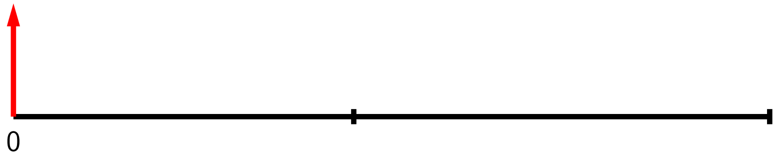


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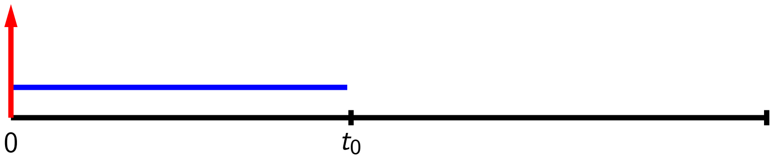


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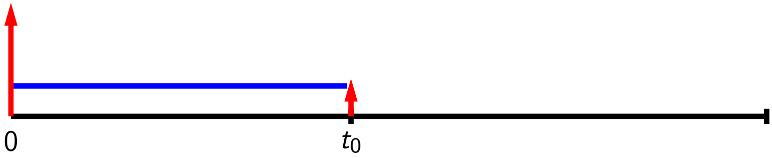


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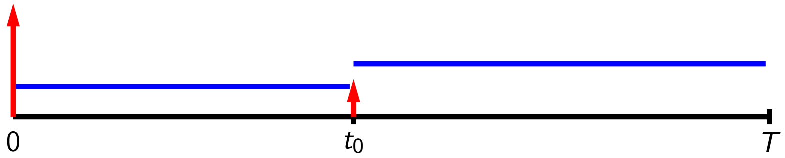


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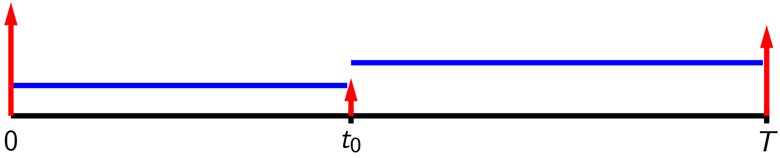
- At time $T$, make a final purchase.




Purchase lumps and rate

# Optimal Execution

Solution. The main ideas.

# Cost of execution

Suppose for the moment that $A_t \equiv 0$ and no purchases have been made prior to the present time.

- The cost of purchasing all the shares available at prices in $[0, x)$ is

$$\varphi(x) \triangleq \int_{[0,x)} \xi \, dF(\xi).$$

- The cost of purchasing $y$ shares is

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Suppose only that $A_t \equiv 0$. Recall that $\Delta X_t = \Delta E_t$.

- Then the cost of the purchasing strategy $X_t, 0 \leq t \leq T$, is

$$C(X) = \int_0^T D_t \, dX_t^c + \sum_{0 \leq t \leq T} \big[\Phi(E_t) - \Phi(E_{t-})\big].$$

# Cost simplification

- $A_t \equiv 0$ may be assumed without loss of generality.

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- $A_t \equiv 0$ may be assumed without loss of generality.
- The search for an optimal trading strategy can be restricted to deterministic strategies.

# Cost simplification (cont.)

**Theorem**

*The cost of using trading strategy* $X_t, 0 \leq t \leq T$,

$$C(X) = \int_0^T D_t \, dX_t^c + \sum_{0 \leq t \leq T} \left[ \Phi(E_t) - \Phi(E_{t-}) \right]$$

*is equal to*

$$\begin{aligned} C(X) &= \Phi(E_T) + \int_0^T D_t h(E_t) \, dt \\ &= \Phi(E_T) + \int_0^T g\big(h(E_t)\big) \, dt \quad , \end{aligned}$$

*where* $g(y) \triangleq y\psi\big(h^{-1}(y)\big)$.

# When "three-jump" strategy becomes "two-jump"

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**Theorem**

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**Observation**

Our function $g(y) \triangleq y\psi\big(h^{-1}(y)\big)$ is convex for the models in Obizhaeva & Wang (2005), Alfonsi, Fruth and Schied (2010).

# Two-jump strategies: Idea of the Proof

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Minimize the last expression over $E_T$ to determine the constant:

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If $e^*$ minimizes (1), then $h(E_t) = \frac{\overline{X} - e^*}{T}$ on $[0, T)$.

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Strategy: $\Delta X_0 = h^{-1}\left(\frac{\overline{X}-e^*}{T}\right)$, purchase at rate $\frac{\overline{X}-e^*}{T}$ on $[0, T)$, at the end $\Delta X_T = e^* - h^{-1}\left(\frac{\overline{X}-e^*}{T}\right)$.

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$$C(X) \;\; \geq \;\; \Phi(E_T) + T\,\widehat{g}\left(\frac{1}{T}\int_0^T h(E_t)\,dt\right) \quad \text{(Jensen) \& } \; (g \geq \widehat{g})$$

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C(X) &\geq \Phi(E_T) + T\,\widehat{g}\left(\frac{1}{T}\int_0^T h(E_t)\,dt\right) \qquad \text{(Jensen) \& } (g \geq \widehat{g}) \\
&= \Phi(E_T) + T\,\widehat{g}\left(\frac{\overline{X} - E_T}{T}\right), \qquad (2)
\end{aligned}
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# Three-jump strategies: Idea of the Proof

If $g$ is not convex, replace $g$ by its convex hull $\widehat{g}$. We have

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Can we still achieve the lower bound if $g(y^*) > \widehat{g}(y^*)$ ?

# Three-jump strategies: Idea of the Proof

There exist $\rho \in (0,1)$, and $\alpha < \beta$, s.t. $y^* = \rho\alpha + (1-\rho)\beta$ and

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Set $t_0 = \rho T$ and achieve the lower bound $T\,\widehat{g}(y^*)$ by

$$T\,\widehat{g}(y^*) = \int_0^{t_0} g(h(E_s))ds + \int_{t_0}^{T} g(h(E_s))ds \, ,$$

where $h(E_t) \equiv \alpha$ on $[0, t_0)$ and $h(E_t) \equiv \beta$ on $[t_0, T)$.

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The corresponding THREE-JUMP strategy:

$\Delta X_0 = h^{-1}(\alpha)$, then purchase at rate $\alpha$ on $(0, t_0)$;

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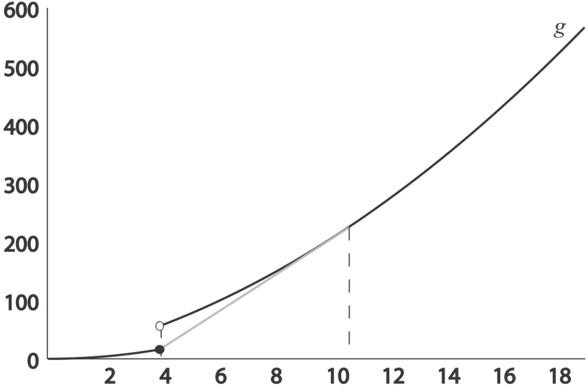
$\Delta X_T = e^* - h^{-1}(\beta)$.

# Three-jump strategies

If $g$ is not convex, replace $g$ by its convex hull.




To achieve a constant consumption on the graph of the convex hull that is not on the graph of $g$, say at 6, consume a while at 4 and a while at 10.324. The switch from 4 to 10.324 creates an intermediate jump.

## Example (Block order book)

Let $q$ and $\rho$ be a positive constants. Set

$$F(x) = qx, \quad h(x) = \rho x.$$

Then

$$\psi(y) = \frac{y}{q}, \quad \Phi(y) = \frac{y^2}{2q}, \quad g(y) = \frac{y^2}{\rho q}.$$

Optimal strategy:

- Initial lump purchase of size $\frac{\overline{X}}{2+\rho T}$,
- Intermediate purchases at rate $\frac{\rho \overline{X}}{2+\rho T}$,
- Terminal lump purchase of size $\frac{\overline{X}}{2+\rho T}$.

# Example (Modified block order book)

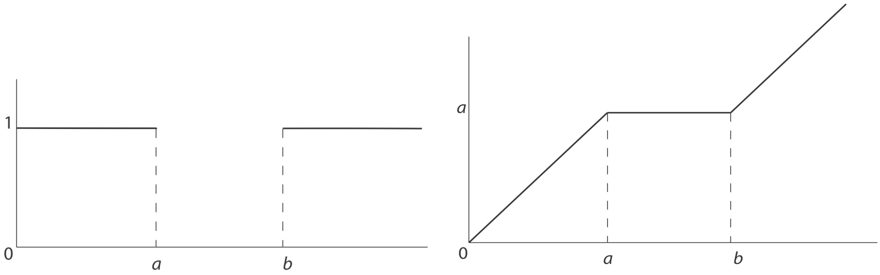



Figure: Density and cumulative distribution of the modified block order book

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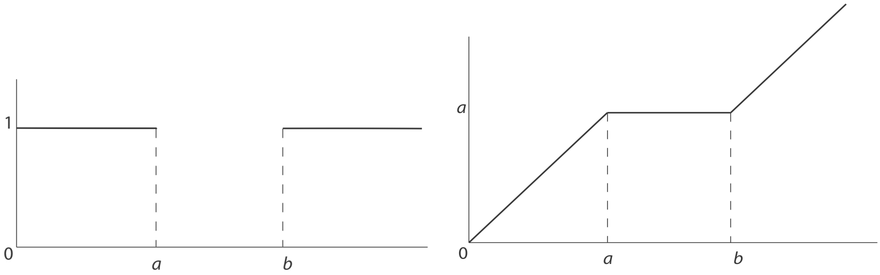


Figure: Density and cumulative distribution of the modified block order book

$$\psi(y) = \begin{cases} y, & 0 \leq y \leq a, \\ y + b - a, & a < y < \infty, \end{cases}$$

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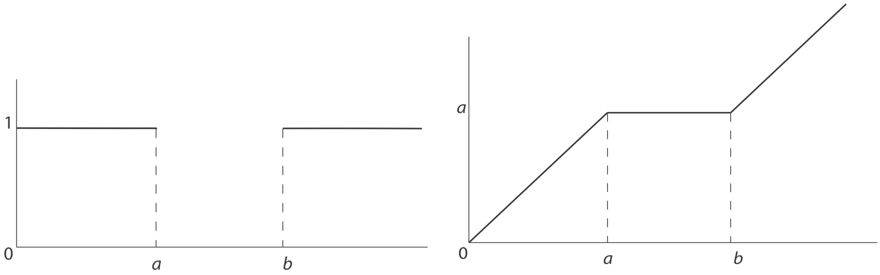


Figure: Density and cumulative distribution of the modified block order book

$$\psi(y) = \begin{cases} y, & 0 \le y \le a, \\ y + b - a, & a < y < \infty, \end{cases}$$

$$\Phi(y) = \begin{cases} \frac{1}{2}y^2, & 0 \le y \le a, \\ \frac{1}{2}\left((y+b-a)^2 + a^2 - b^2\right), & a \le y < \infty. \end{cases}$$

## Example (Modified block order book, continued)

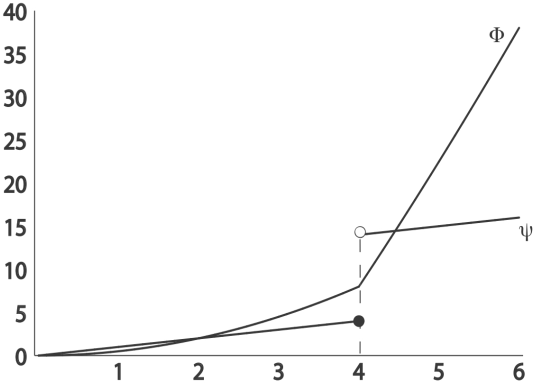


Figure: Functions $\Phi$ and $\psi$ for the modified block order book with parameters $a = 4$ and $b = 14$

## Example (Modified block order book, continued)

$$g(y) = \begin{cases} y^2, & 0 \le y \le a, \\ y^2 + (b-a)y, & a < y < \infty. \end{cases}$$

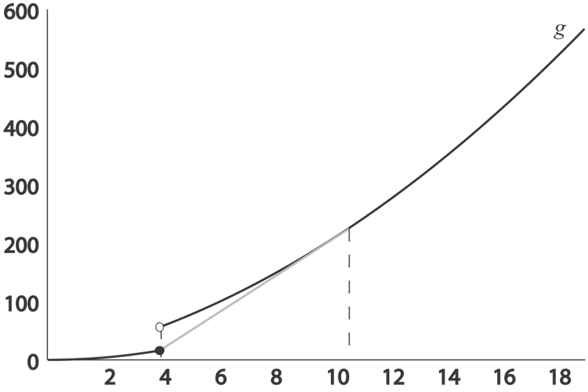


Figure: Function $g$ for the modified block order book with parameters $a = 4$ and $b = 14$. The convex hull $\widehat{g}$ is constructed by replacing a part $\{g(y), y \in (a, \beta)\}$ by a straight line connecting $g(a)$ and $g(\beta)$. Here $\beta = 10.324$

# Example (Discrete order book)

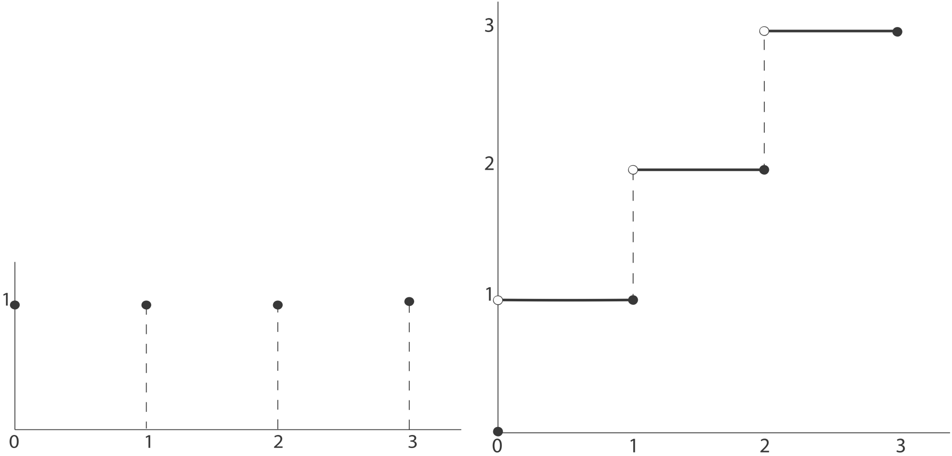


Figure: Measure and cumulative distribution function of the discrete order book

# Example (Discrete order book, continued)

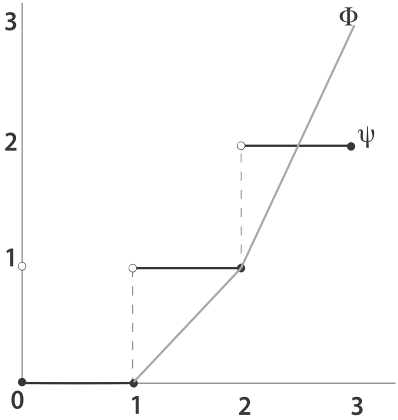


Figure: Functions $\Phi$ and $\psi$ for the discrete order book

## Example (Discrete order book, continued)

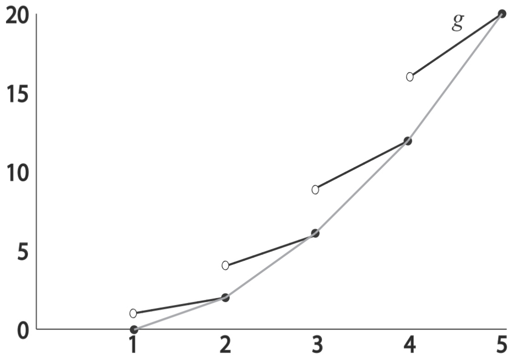


Figure: Function $g$ for the discrete order book. The convex hull $\widehat{g}$ interpolates linearly between the points $(k, (k-1)k)$ and $(k+1, k(k+1))$.