Averaging principle for an order book model

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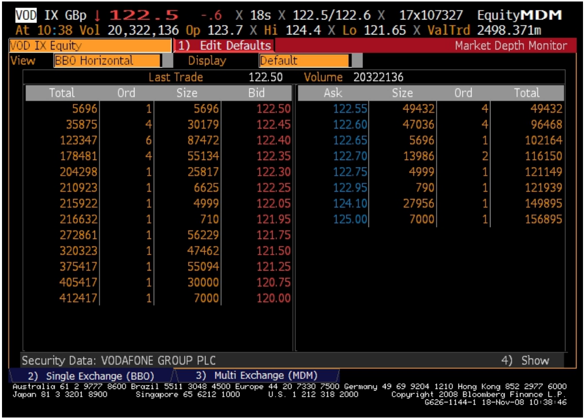
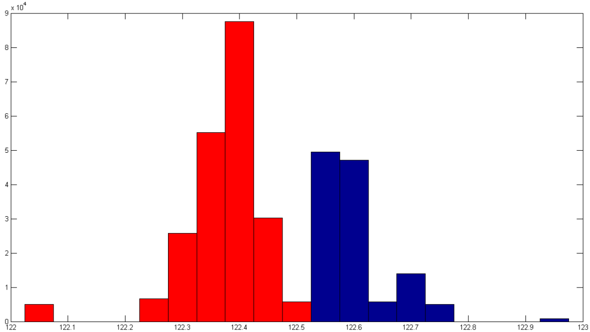
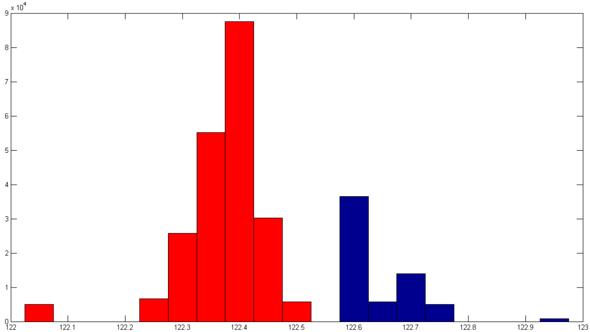


Total	Ord	Size	Bid	Ask	Size	Ord	Total
5696	1	5696	122.50	122.55	49432	4	49432
35875	4	30179	122.45	122.60	47036	4	96468
123347	6	87472	122.40	122.65	5696	1	102164
178481	4	55134	122.35	122.70	13986	2	116150
204298	1	25817	122.30	122.75	4999	1	121149
210923	1	6625	122.25	122.95	790	1	121939
215922	1	4999	122.05	124.10	27956	1	149895
216632	1	710	121.95	125.00	7000	1	156895
272861	1	56229	121.75				
320323	1	47462	121.50				
375417	1	55094	121.25				
405417	1	30000	120.75				
412417	1	7000	120.00				

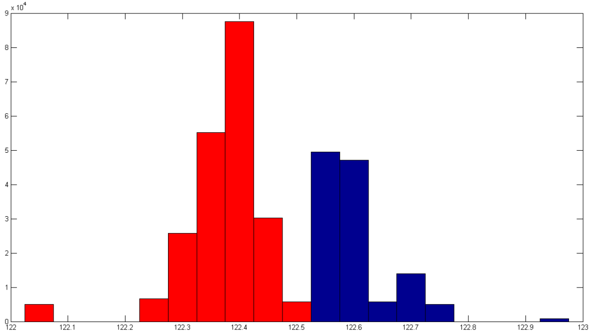
A screen shot of an electronic limit order book.



Initial state of the order book.



After **market buy order** placement.



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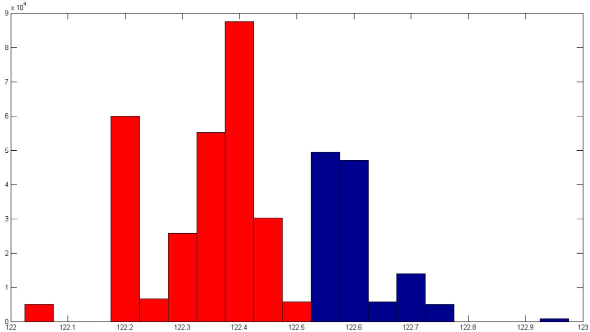
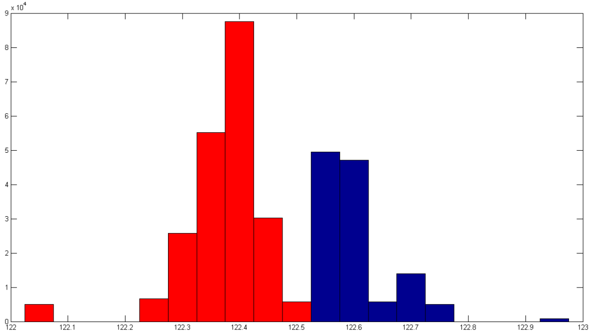


After **buy limit order** placement at 122.20.



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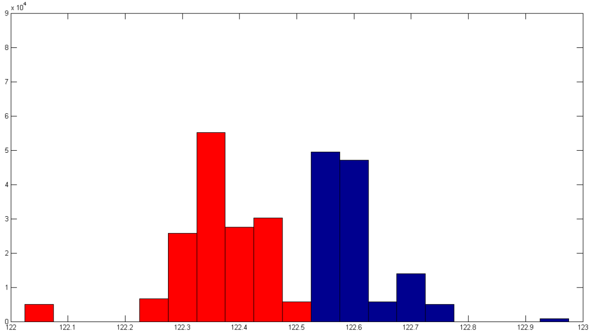

After **buy limit order cancelation** at 122.40.

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Inherent complexity in order book modeling: **high dimensionality** and **state-dependent random dynamics**.

Outline

- Introduce **discrete order book model** (Microscopic setting, order flow).

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- Prove **averaging principle** and consider examples (Macroscopic aggregates, ODE).

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Deriving the model dynamics

trading mechanisms + random order flow $\Rightarrow$ model dynamics

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- when orders arrive - **inter arrival times**.
- what happens - **limit/market order/cancelation volumes**.
- where it happens - **limit order prices**, **cancelation prices**.
- randomness is dependent on the **current state of the order book** (e.g. state-dependent intensities, densities).

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Summarizing

Given input parameters and trading rules, we can derive the random incremental state change, denoted

$$\Delta S_k(S_k) := (\Delta B_k(S_k), \Delta A_k(S_k), \Delta h_{b,k}(S_k), \Delta h_{s,k}(S_k)) .$$

I: Mathematical framework

Define **sequence of models**, $S_k^{(n)} := (B_k^{(n)}, A_k^{(n)}, h_{b,k}^{(n)}, h_{s,k}^{(n)})$, on $(\Omega^{(n)}, \mathcal{F}^{(n)}, \mathbb{P}^{(n)})$, $\mathcal{F}_0^{(n)} \subset \mathcal{F}_1^{(n)} \subset \ldots \subset \mathcal{F}^{(n)}$, where

$$\mathcal{F}_k^{(n)} := \left\{ \left(\Delta S_k^{(n)}(\alpha), \Delta t_k^{(n)}(\alpha)\right), \alpha \in E \right\}, \quad k \geq 0, \qquad (1)$$

are jointly **independent** families of random variables with values in $E \times [0, \infty)$, **i.d.** $\forall k \geq 0$ **given** $\alpha \in E$.

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Recurrent sequences

$$t_0^{(n)} := 0, \qquad t_{k+1}^{(n)} := t_k^{(n)} + \Delta t_k^{(n)}(S_k^{(n)}), \qquad [time]. \tag{2}$$

$$S_0^{(n)} := s_0^{(n)}, \quad S_{k+1}^{(n)} := S_k^{(n)} + \Delta S_k^{(n)}(S_k^{(n)}) \qquad [state]. \tag{3}$$

II: LLN-scaling of process

Order book process in physical time:

$$S^{(n)}(t) := S_k^{(n)} \quad \text{as} \quad t \in [t_k^{(n)}, t_{k+1}^{(n)}), \quad t \geq 0. \tag{4}$$

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$$\lim_{n\to\infty} \frac{S^{(n)}(nt)}{n} \tag{5}$$

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(4) and (5):

$$\frac{S^{(n)}(nt)}{n} := \frac{S_k^{(n)}}{n} \quad \text{as} \quad t \in [\frac{t_k^{(n)}}{n}, \frac{t_{k+1}^{(n)}}{n}), \quad t \geq 0. \tag{6}$$

III: Scaling of input parameters and tick size

Idea

- We have $\frac{S_k^{(n)}}{n} \sim S_k$ and we scale input parameters s.t. **random incremental change** $\frac{1}{n}\Delta S_k^{(n)} \sim \frac{1}{n}\Delta S_k$ over **small time intervals**.

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Thus, we **gain regularity** as $n \to \infty$ and we have **infinitesimally small changes** over **infinitesimal time intervals**.

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- Identify **limit in probability** as **ODE on the state space** $\mathbb{R}_+ \times \mathbb{R}_+ \times L^2([0, K_b], \mathbb{R}_-) \times L^2([0, K_s], \mathbb{R}_+)$.

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- Use **time change theorem** for the composition i.e. the original scaled process.

I: Main result

Theorem (Horst, P.)

Given boundedness, convergence and Lipschitz continuity of the input parameters (as a function of the current order book state) it holds that

$$\frac{S^{(n)}(nt)}{n} \xrightarrow{\mathbb{P}} s^*(t), \text{ as } n \to \infty \tag{7}$$

where $s^*(t) \in E$ is the unique solution to the ODE

$$\begin{cases} \frac{\mathrm{d}s^*(t)}{\mathrm{d}t} = \frac{b^*(s^*(t))}{m^*(s^*(t))}, & t \in (0,T]. \\ s^*(0) = s_0 \end{cases} \tag{8}$$

$$b^*(x) := \mathbb{E}[\Delta S_1^*(x,\omega)] \quad \text{and} \quad m^*(x) := \mathbb{E}[\Delta t_1^*(x,\omega)]. \tag{9}$$

II: Corollary

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If the volume densities have a **stationary shape** *(e.g. block shaped), the ODE on the state space E (8)* **reduces** *to a coupled ODE on* $\mathbb{R}^2$ *for the* **best bid** *and* **ask prices***.*

Current work

- Diffusion approximation ("noise"), i.e. study

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- Alternative scaling$\Rightarrow$ PDE.

Thank you!