Functional Itô Calculus and Applications

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Outline

1) Functional Itô Calculus

- Functional Itô formula
- Functional Feynman-Kac
- PDE for path dependent options

2) Applications

- Volatility expansion in LVM
- Vega decomposition
- Robust hedge with Vanillas
- Super-replication and claim decomposition

1) Functional Itô Calculus

Why?

Most often, the impact of uncertainty is cumulative. The link

$$\text{Cause} \rightarrow \text{Consequence}$$

$$X_t \xrightarrow{f} y_t = f(X_t)$$

depends on certain patterns:

temperature $\rightarrow$ water level

price history $\rightarrow$ path dependent payoff

exposure to antigen $\rightarrow$ viral reaction

interest rate path $\rightarrow$ MBS prepayment

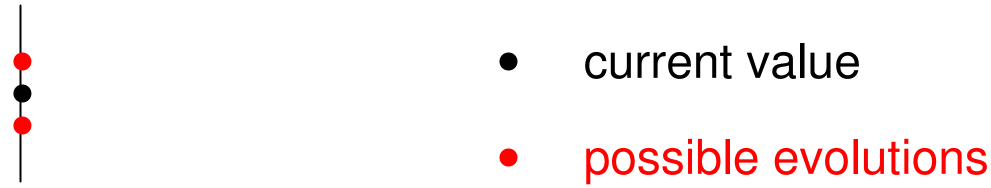
Itô calculus deals with functions of processes.

We extend it to functions of the current path,
or history, of the process.

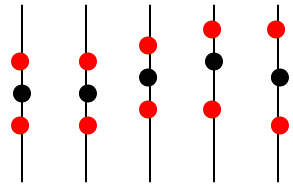
Review of Itô Calculus

D. hhttp://ssrn.com/abstract=1435551

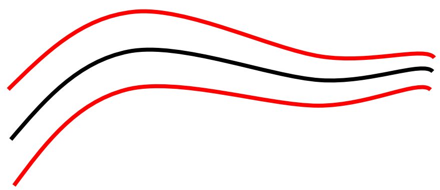
- 1D



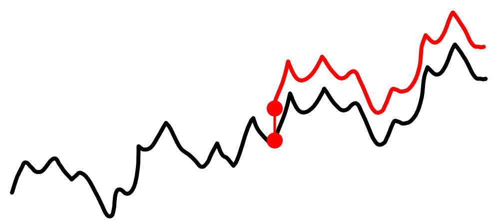
- nD



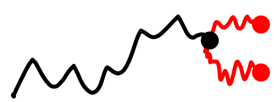
- infiniteD



- Malliavin Calculus



- Functional Itô Calculus



Functionals of running paths

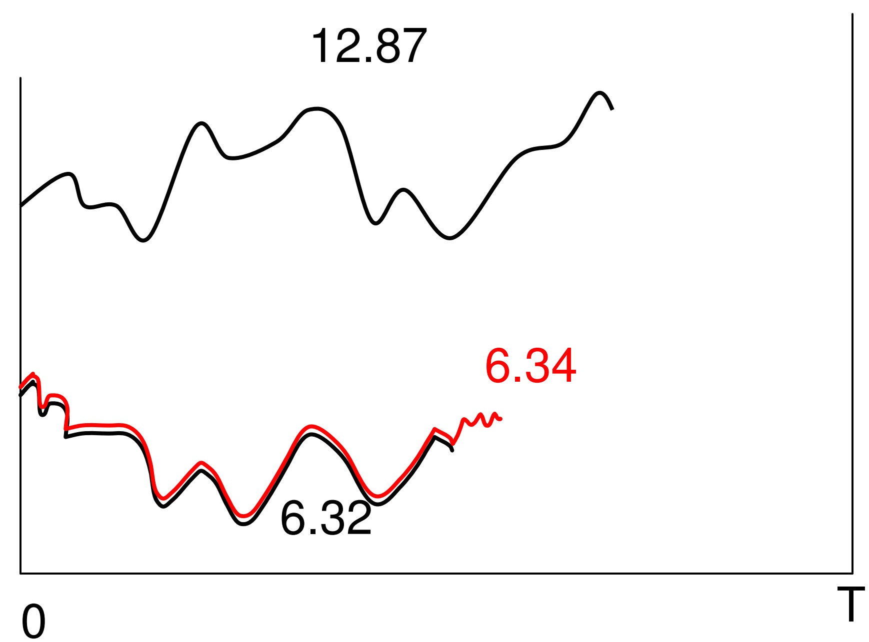
Functionals defined not only on the whole path [0, T] but on all starting sections [0, t]

$$\Lambda_t \equiv \{\text{RCLL functions } [0,t] \rightarrow \Re\}$$

$$\Lambda \equiv \bigcup_{t \in [0,\bar{T}]} \Lambda_t$$

f functional : $f : \Lambda \rightarrow \Re$, for $X_t \in \Lambda_t$, $f(X_t) \in \Re$

The value of X_t at $s \leq t$ is $X_t(s)$ and $x_t \equiv X_t(t)$



Examples of Functionals

- Current average
- Current drawdown
- Conditional expectation of final value
- Super - replicating price

The last one covers the important case of path dependent option price

Finite number of state variables (excluding time) :

- European (1)
- Asian (2)
- Option on range (3)

Infinite number of state variables

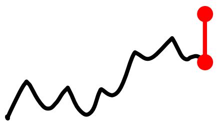
- Max of rolling average
- First time last value is hit

Derivatives

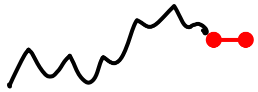
$$\Lambda_t \equiv \{\text{bounded RCLL functions } [0,\mathrm{t}] \to \Re\},\ \Lambda \equiv \bigcup_{t\in[0,\overline{T}]} \Lambda_t$$

For $X_t \in \Lambda_t,\ f : \Lambda \to \Re,$

$$X_t^h(s) \equiv X_t(s) \text{ for } s < t \quad X_t^h(t) = X_t(t) + h$$



$$X_{t,\delta t}(s) = X_t(s) \text{ for } s \le t \quad X_{t,\delta t}(s) = X_t(t) \text{ for } s \in [t, t+\delta t]$$



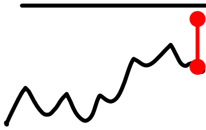
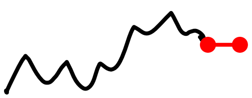
Space derivative

$$\Delta_x f(X_t) \equiv \lim_{h\to 0} \frac{f(X_t^h) - f(X_t)}{h}$$

Time derivative

$$\Delta_t f(X_t) \equiv \lim_{\delta t\to 0} \frac{f(X_{t,\delta t}) - f(X_t)}{\delta t}$$

Examples

	f	x_t	$\int_0^t x_u du$	QV_t
	$\Delta_x f$	1	0	$2(x_t - x_{t^-})$
	$\Delta_t f$	0	x_t	0

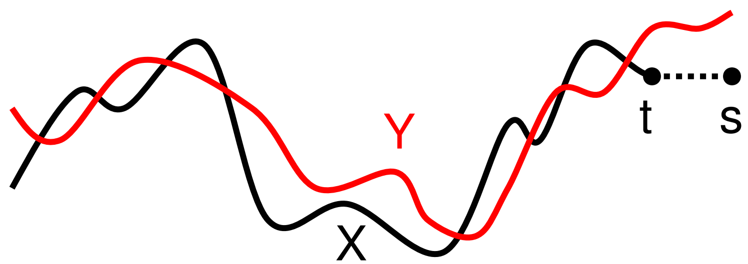
$$\text{If } f(X_t) = h(x_t, t), \text{ then } \begin{cases} \Delta_x f = \dfrac{\partial h}{\partial x} \\ \Delta_t f = \dfrac{\partial h}{\partial t} \end{cases}$$

Topology and Continuity

Λ - distance :

For all X_t, Y_s in Λ, (we can assume $t \leq s$)

$$d_\Lambda(X_t, Y_s) \equiv \left\| X_{t,s-t} - Y_s \right\|_\infty + s - t$$



Λ - continuity :

A functional $f : \Lambda \to \Re$ is $\Lambda -$continuous at $X_t \in \Lambda$ if :

$$\forall \varepsilon > 0, \ \exists \alpha > 0 : \forall Y_s \in \Lambda,$$

$$d_\Lambda(X_t, Y_s) < \alpha \Rightarrow \left| f(Y_s) - f(X_t) \right| < \varepsilon$$

Functional Itô Formula

Definition : a functional $f : \Lambda \to \Re$ is *smooth* if it is Λ - continuous, C^2 in x and C^1 in t, with these derivatives themselves Λ - continuous.

Theorem

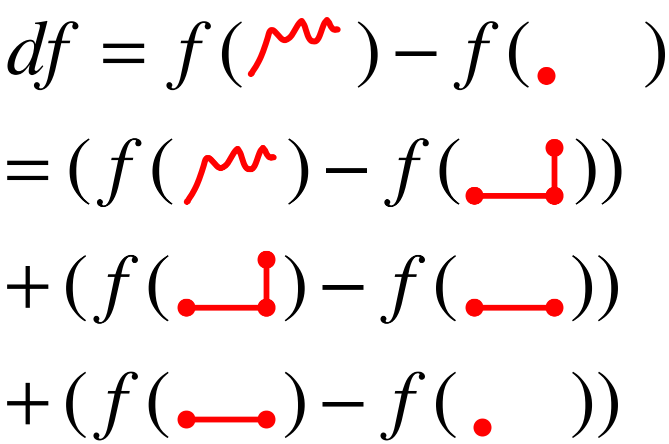
If x is a continuous semi - martingale and X_t denotes its path over [0, t], then, for all smooth functional f and for all $T \geq 0$,

$$f(X_T) = f(X_0) + \int_0^T \Delta_x f(X_t)\, dx_t + \int_0^T \Delta_t f(X_t)\, dt + \frac{1}{2}\int_0^T \Delta_{xx} f(X_t)\, d\langle x \rangle_t$$

or, in more concise notation,

$$df = \Delta_x f\, dx + \Delta_t f\, dt + \frac{1}{2}\Delta_{xx} f\, d\langle x \rangle$$

Fragment of proof



From Taylor expansion of $f(Y_i) - f(Z_i)$ we get,
with $\delta x_i \equiv X_T(\tau_i) - X_T(\tau_{i-1})$ and $\delta t_i \equiv \tau_i - \tau_{i-1}$,

$$B = \sum_{i=1}^{N} \left(f(Y_i) - f(Z_i)\right) = \sum_{i=1}^{N} \Delta_x f(Z_i)\,\delta x_i \qquad B_1$$

$$+ \frac{1}{2}\sum_{i=1}^{N} \Delta_{xx} f(Z_i^{\theta_i \delta x_i})(\delta x_i)^2 \qquad B_2$$

with $\theta_i \in (0,1)$

$$B_1 = \sum_{i=1}^{N} \Delta_x f(Z_i)\,\delta x_i = \sum_{i=1}^{N} \Delta_x f(X_{i-1})\,\delta x_i \qquad B_{11}$$

$$+ \sum_{i=1}^{N} \left(\Delta_x f(U_i) - \Delta_x f(X_{i-1})\right)\delta x_i \qquad B_{12}$$

$$+ \sum_{i=1}^{N} \left(\Delta_x f(Z_i) - \Delta_x f(U_i)\right)\delta x_i \qquad B_{13}$$

$$B_{11} = \sum_{i=1}^{N} \Delta_x f(X_{i-1})\,\delta x_i \xrightarrow{L_1} \int_0^T \Delta_x f(X_t)\,dx_t$$

As $\Delta_x f(U_i) - \Delta_x f(X_{i-1}) = \Delta_{xt} f(X_{i-1,\theta_i\delta t_i})\delta t_i \le \beta \max \Delta_{xt} f$, $B_{12} \xrightarrow{L_1} 0$

As $\Delta_x f(Z_i) - \Delta_x f(U_i) \le k_x \alpha$, $B_{13} \xrightarrow{L_1} 0$

$$2B_2 = \sum_{i=1}^{N} \Delta_{xx} f(Z_i^{\theta_i \delta x_i})(\delta x_i)^2 = \sum_{i=1}^{N} \Delta_{xx} f(X_{i-1})(\delta x_i)^2 \qquad B_{21}$$

$$+ \sum_{i=1}^{N} \left(\Delta_{xx} f(Z_i^{\theta_i \delta x_i}) - \Delta_{xx} f(Z_i)\right)(\delta x_i)^2 \qquad B_{22}$$

$$+ \sum_{i=1}^{N} \left(\Delta_{xx} f(Z_i) - \Delta_{xx} f(X_{i-1})\right)(\delta x_i)^2 \qquad B_{23}$$

$$B_{21} = \sum_{i=1}^{N} \Delta_{xx} f(X_{i-1})(\delta x_i)^2 \Rightarrow L_2 - \lim_{\alpha,\beta\to 0} B_{21} = \int_0^T \Delta_{xx} f(X_t)\,d\langle x\rangle_t$$

As $\left|\Delta_{xx} f(Z_i^{\theta_i \delta x_i}) - \Delta_{xx} f(Z_i)\right| \le k_{xx}\alpha$, $L_2 - \lim_{\alpha,\beta\to 0} B_{22} = 0$

As $\left|\Delta_{xx} f(Z_i) - \Delta_{xx} f(X_{i-1})\right| = \left|\Delta_{xxt} f(X_{i-1,\theta_i\delta t_i})\delta t_i\right| \le \beta \max \Delta_{xxt} f$, $L_2 - \lim_{\alpha,\beta\to 0} B_{23} = 0$

$$L_2 - \lim_{\alpha\to 0} B_1 = \int_0^T \Delta_x f(X_t)\,dx_t$$

$$L_2 - \lim_{\alpha\to 0} B_2 = \frac{1}{2}\int_0^T \Delta_{xx} f(X_t)\,d\langle x\rangle_t$$

Functional Feynman-Kac Formula

Generalisation of FK to non Markov dynamics and path dependent payoff.

$dx_t = a(X_t)\,dt + b(X_t)\,dW_t$

For g suitably integrable, $g : \Lambda_T \to \Re, r : \Lambda \to \Re$. We define $f : \Lambda \to \Re$ by

$$f(Y_t) \equiv E[e^{-\int_t^T r(Z_u)\,du} g(Z_T) \mid Z_t = Y_t]$$

where $\begin{cases} \text{for } u \in [0,t], Z_T(u) = Y_t(u) \\ \text{for } u \in [t,T], dz_u = a(Z_u)\,du + b(Z_u)\,dW_u \end{cases}$

Then, if f is smooth, it satisfies

$$\Delta_t f(X_t) + a(X_t)\Delta_x f(X_t) - r(X_t) f(X_t) + \frac{b^2(X_t)}{2}\Delta_{xx} f(X_t) = 0$$

(apply functional Itô formula to the martingale $e^{-\int_0^t r(X_u)\,du} f(X_t)$)

Delta Hedge/Clark-Ocone

If f defined by $f(Y_t) \equiv E[g(Z_T) \mid Z_t = Y_t]$ is smooth,
then we have the explicit Martingale Representation :

$$g(X_T) = E[g(X_T) \mid X_0] + \int_0^T b(X_t)\Delta_x f(X_t)\, dW_t$$

From functional Itô and Feynman - Kac with $r = 0$,

$$df(X_t) = b(X_t)\Delta_x f(X_t)\, dW_t$$

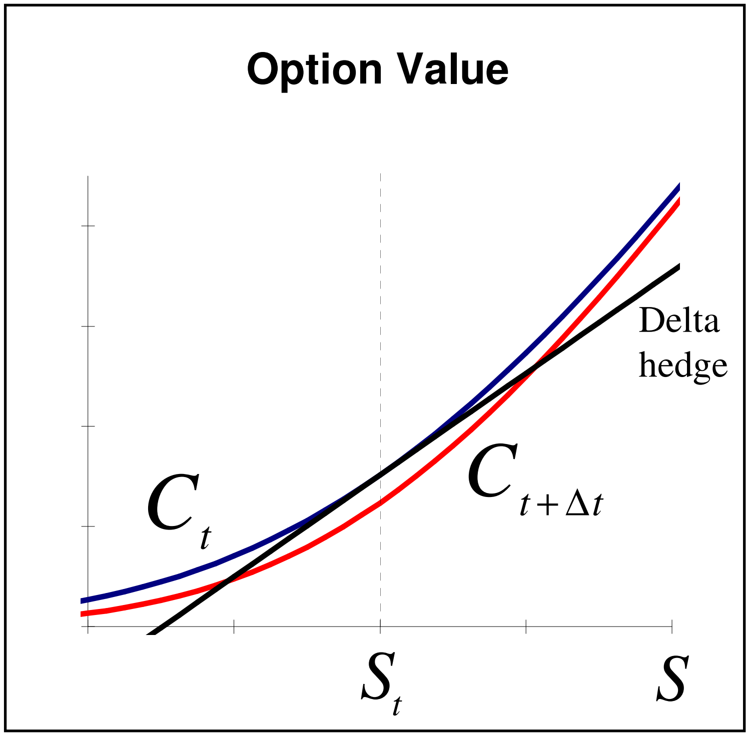
and

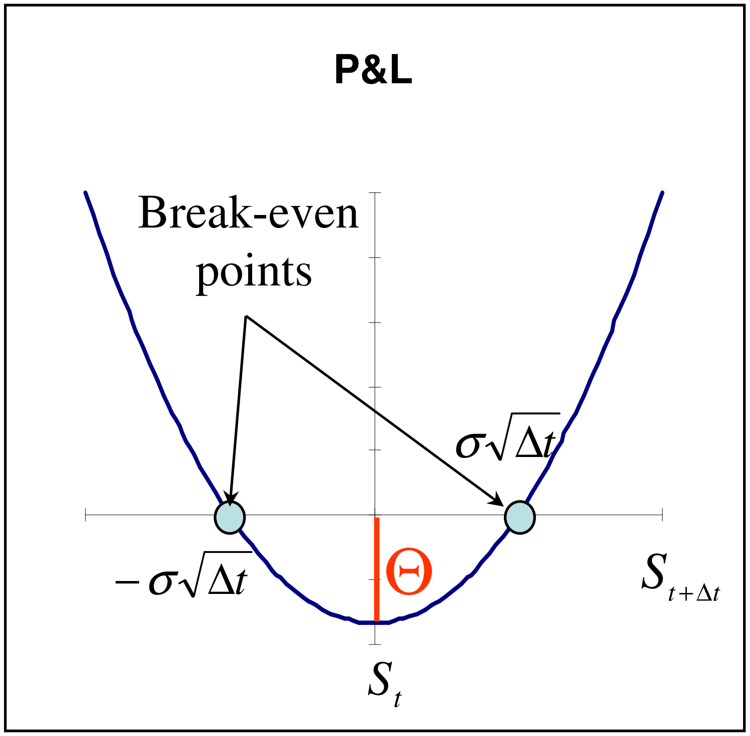
$$g(X_T) = f(X_T) = f(X_0) + \int_0^T b(X_t)\Delta_x f(X_t)\, dW_t$$

It can be compared to the Clark - Ocone formula
from Malliavin Calculus :

$$g(X_T) = E[g(X_T) \mid X_0] + \int_0^T b(X_t)\, E[D_t g(X_T) \mid X_t]\, dW_t$$

P&L of a delta hedged Vanilla





Functional PDE for Exotics

If f given by $f(Y_t) \equiv E[e^{-\int_t^T r(Z_u)\,du} g(Z_T) \mid Z_t = Y_t]$ is smooth, then it satisfies

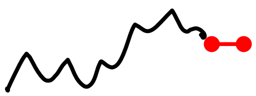
$$\Delta_t f(X_t) + \frac{1}{2} b^2 \Delta_{xx} f(X_t) + r(X_t)(\Delta_x f(X_t)\, x_t - f(X_t)) = 0$$

The Γ/Θ trade-off for European options also holds for path dependent options, even with an infinite number of state variables. However, in general Γ and Θ will be path dependent.

Classical PDE for Asian

Payoff of Asian Call: $g(X_T) = (\int_0^T x_u \, du - K)^+$

Assume $dx_t = b(x_t, t)\, dW_t$ Define $I_t \equiv \int_0^t x_u \, du$, $f(X_t) \equiv l(x_t, I_t, t)$



$$\Delta_t I = x_t \Rightarrow \Delta_t f = x \frac{\partial l}{\partial I} + \frac{\partial l}{\partial t}$$

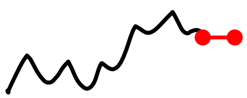
$$\Delta_x I = 0 \Rightarrow \Delta_x f = \frac{\partial l}{\partial x}, \ \Delta_{xx} f = \frac{\partial^2 l}{\partial x^2}$$

$$\Delta_t f + \frac{1}{2} b^2 \Delta_{xx} f = 0 \Rightarrow \frac{\partial l}{\partial t} + x \frac{\partial l}{\partial I} + \frac{1}{2} b^2 \frac{\partial^2 l}{\partial x^2} = 0$$

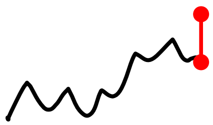
$x \dfrac{\partial l}{\partial I}$ is a bothering convection term.

Better Asian PDE

Define $J_t \equiv E_t[\int_0^T x_u \, du] = \int_0^t x_u \, du + (T-t)\, x_t, \ f(X_t) \equiv h(x_t, J_t, t)$



$$\Delta_t J = 0 \Rightarrow \Delta_t f = \frac{\partial h}{\partial t}$$



$$\Delta_x J = (T-t) \Rightarrow \begin{cases} \Delta_x f = \dfrac{\partial h}{\partial x} + (T-t)\dfrac{\partial h}{\partial J} \\ \Delta_{xx} f = \dfrac{\partial^2 h}{\partial x^2} + 2(T-t)\dfrac{\partial^2 h}{\partial x \partial J} + (T-t)^2 \dfrac{\partial^2 h}{\partial J^2} \end{cases}$$

$$\Delta_t f + \frac{1}{2} b^2 \Delta_{xx} f = 0 \Rightarrow \frac{\partial h}{\partial t} + \frac{1}{2} b^2 \left(\frac{\partial^2 h}{\partial x^2} + 2(T-t)\frac{\partial^2 h}{\partial x \partial J} + (T-t)^2 \frac{\partial^2 h}{\partial J^2}\right) = 0$$

2) Robust Volatility Hedge

Local Volatility Model

- Simplest model to fit a full surface
- Forward volatilities that can be locked

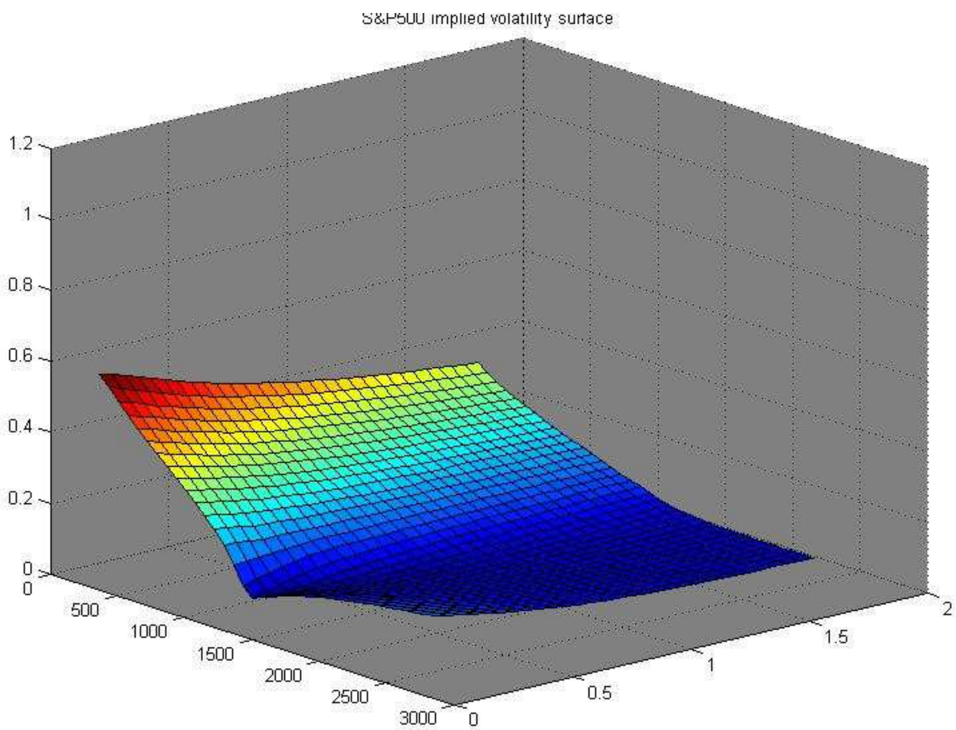
$$\frac{dS}{S} = (r-q)\,dt + \sigma(S,t)\,dW$$

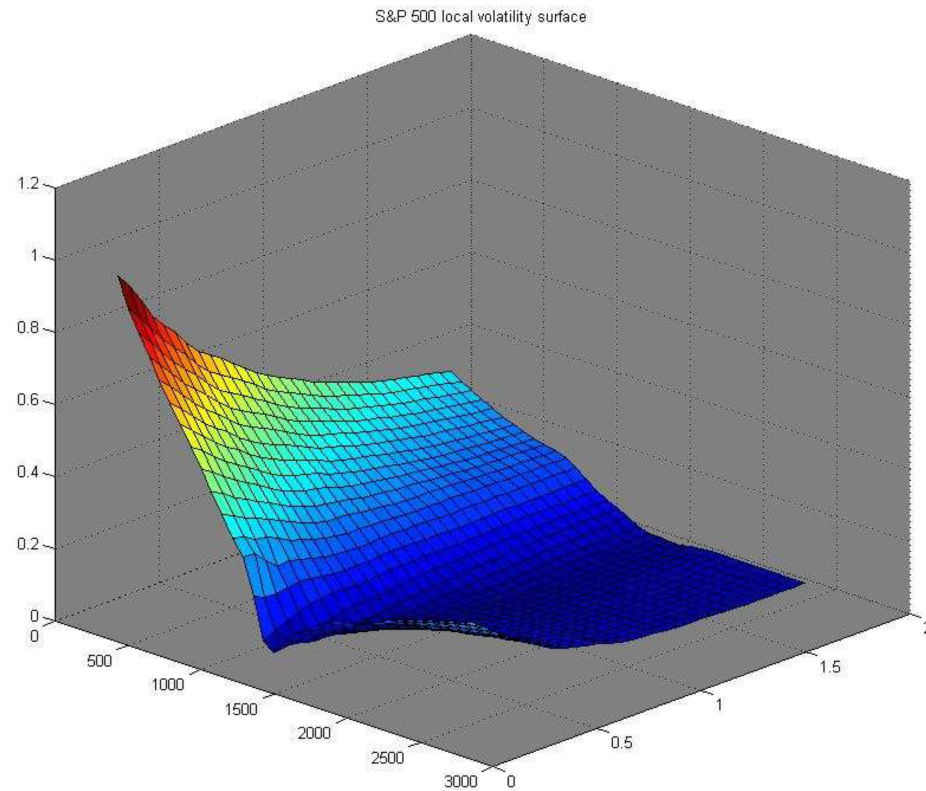
$$\frac{\partial C}{\partial T} = \frac{\sigma^2(K,T)}{2} K^2 \frac{\partial^2 C}{\partial K^2} - (r-q)K\frac{\partial C}{\partial K} - q \cdot C$$

Summary of LVM

- Simplest model that fits vanillas
- In Europe, second most used model (after Black-Scholes) in Equity Derivatives
- Local volatilities: fwd vols that can be locked by a vanilla PF
- Stoch vol model calibrated $E[\sigma_t^2 \mid S_t = S] = \sigma_{loc}^2(S,t)$
- If no jumps, deterministic implied vols => LVM

S&P500 implied and local vols





Hedge within/outside LVM

- 1 Brownian driver => complete model
- Within the model, perfect replication by Delta hedge
- Hedge outside of (or against) the model: hedge against volatility perturbations
- Leads to a decomposition of Vega across strikes and maturities

P&L from Delta hedging

Assume $r = q = 0, dx_t = \sqrt{v_0(x_t, t)}\, dW_t$.

For $g \in \Lambda_T$, define $f \in \Lambda$ by $f(X_t) \equiv E^{Q_{v0}}[g(X_T) \mid X_t]$

By functional PDE, $\Delta_t f(X_t) + \frac{1}{2} v_0(x_t, t)\, \Delta_{xx} f(X_t) = 0$

If y follows $dy_t = \sqrt{v_t}\, dW_t$ with $y_0 = x_0$, by the functional Itô formula,

$$g(Y_T) = f(Y_T) = f(Y_0) + \int_0^T \Delta_x f(Y_t)\, dy_t + \int_0^T \Delta_t f(Y_t)\, dt + \frac{1}{2}\int_0^T v_t \Delta_{xx} f(Y_t)\, dt$$

$$= f(X_0) + \int_0^T \Delta_x f(Y_t)\, dy_t + \frac{1}{2}\int_0^T (v_t - v_0(y_t, t))\, \Delta_{xx} f(Y_t)\, dt$$

Model Impact

Recall

$$g(Y_T) = f(Y_T) = f(X_0) + \int_0^T \Delta_x f(Y_t)\, dy_t + \frac{1}{2}\int_0^T (v_t - v_0(y_t, t))\, \Delta_{xx} f(Y_t)\, dt$$

Hence, with $\Pi_g(v) \equiv E^{Q_v}[g(Y_T) \mid X_0]$ and $\phi^v(x,t)$ the transition density for v,

$$\Pi_g(v) = \Pi_g(v_0) + \frac{1}{2} E^{Q_v}[\int_0^T (v_t - v_0(x_t, t))\, \Delta_{xx} f(X_t)\, dt]$$

$$= \Pi_g(v_0) + \frac{1}{2}\int_0^T \int \phi^v(x,t)\, E^{Q_v}[(v_t - v_0(x,t))\, \Delta_{xx} f(X_t) \mid x_t = x]\, dx\, dt$$

Comparing calibrated models

Recall

$$\Pi_g(v) = \Pi_g(v_0) + \frac{1}{2}\int_0^T \int \phi^v(x,t)\, E^{Q_v}[(v_t - v_0(x,t))\,\Delta_{xx} f(X_t) \mid x_t = x]\, dx\, dt$$

For a knock-out Barrier option,

$$\Pi_g(v) - \Pi_g(v_0) = \frac{1}{2}\int_0^T \int \phi^v(x,t)\, E^{Q_v}[(v_t - v_0(x,t))\,\Delta_{xx} f(X_t) \mid x_t = x]\, dx\, dt$$

$$= \frac{1}{2}\int_0^T \int \phi^v(x,t)\, P_{alive}(x,t) E^{Q_v}[(v_t - v_0(x,t))\,\Delta_{xx} f(X_t) \mid x_t = x, alive]\, dx\, dt$$

$$= \frac{1}{2}\int_0^T \int \phi^v_{alive}(x,t)\, \frac{\partial^2 f_{alive}(x,t)}{\partial x^2} E^{Q_v}[(v_t - v_0(x,t)) \mid x_t = x, alive]\, dx\, dt$$

If v and v_0 are calibrated on the same vanilla options, $E^{Q_v}[v_t \mid x_t = x] = v_0(x,t)$

It amounts to evaluate the impact of conditioning by "alive":

$$E^{Q_v}[v_t \mid x_t = x, alive] - E^{Q_v}[v_t \mid x_t = x]$$

Volatility Expansion in LVM

In general,

$$\Pi_g(v) = \Pi_g(v_0) + \frac{1}{2}\int_0^T \int \phi^v(x,t)\, E^{Q_v}[(v_t - v_0(x,t))\,\Delta_{xx} f(X_t) \mid x_t = x]\, dx\, dt$$

In the case where v is a LVM of the form $v_0 + u : dx_t = \sqrt{v_0(x_t,t) + u(x_t,t)}\, dW_t$

$$\Pi_g(v_0 + u) = \Pi_g(v_0) + \frac{1}{2}\int_0^T \int \phi^{v_0+u}(x,t)\, u(x,t)\, E^{Q_{v_0+u}}[\Delta_{xx} f(X_t) \mid x_t = x]\, dx\, dt$$

$$= \Pi_g(v_0) + \int_0^T \int m(x,t)\, u(x,t)\, dx\, dt$$

where $m(x,t) \equiv \frac{1}{2}\phi^{v_0+u}(x,t) E^{Q_{v_0+u}}[\Delta_{xx} f(X_t) \mid x_t = x]$

Fréchet Derivative in LVM

In particular,

$$\Pi_g(v_0+\varepsilon u)=\Pi_g(v_0)+\frac{\varepsilon}{2}\int_0^T\int \phi^{v_0+\varepsilon u}(x,t)\,u(x,t)\,E^{Q_{v_0+\varepsilon u}}[\Delta_{xx}f(X_t)\,|\,x_t=x]\,dx\,dt$$

The Fréchet derivative in the direction of u satisfies :

$$<\nabla_v\Pi_g,u>\equiv\lim_{\varepsilon\to 0}\frac{\Pi_g(v_0+\varepsilon u)-\Pi_g(v_0)}{\varepsilon}$$

$$=\frac{1}{2}\int_0^T\int \phi^{v_0}(x,t)\,u(x,t)\,E^{Q_{v_0}}[\Delta_{xx}f(X_t)\,|\,x_t=x]\,dx\,dt$$

$$=\int_0^T\int m(x,t)\,u(x,t)\,dx\,dt$$

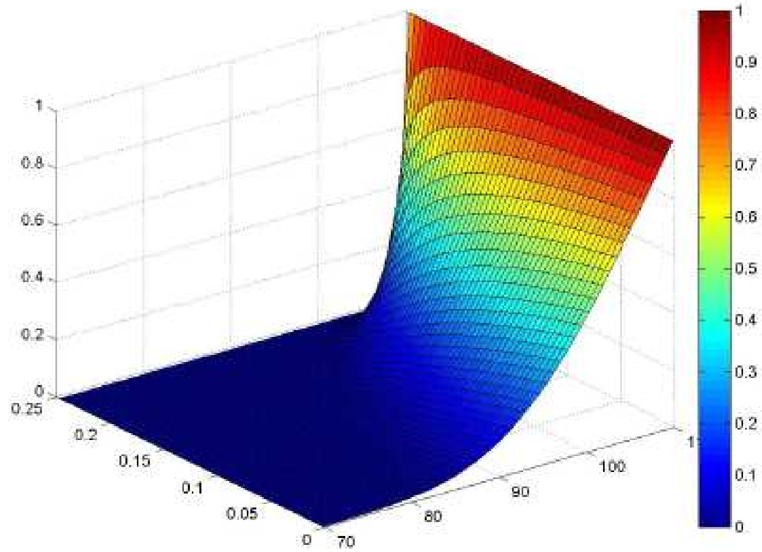
where

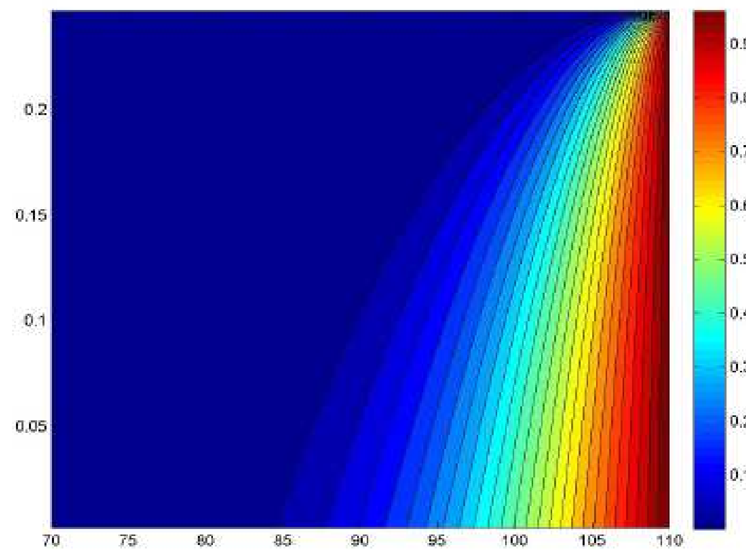
$$m(x,t)\equiv\frac{1}{2}\phi^{v_0}(x,t)E^{Q_{v_0}}[\Delta_{xx}f(X_t)\,|\,x_t=x]$$

is the sensitivity of g to the local variance at (x,t) (Fréchet derivative)

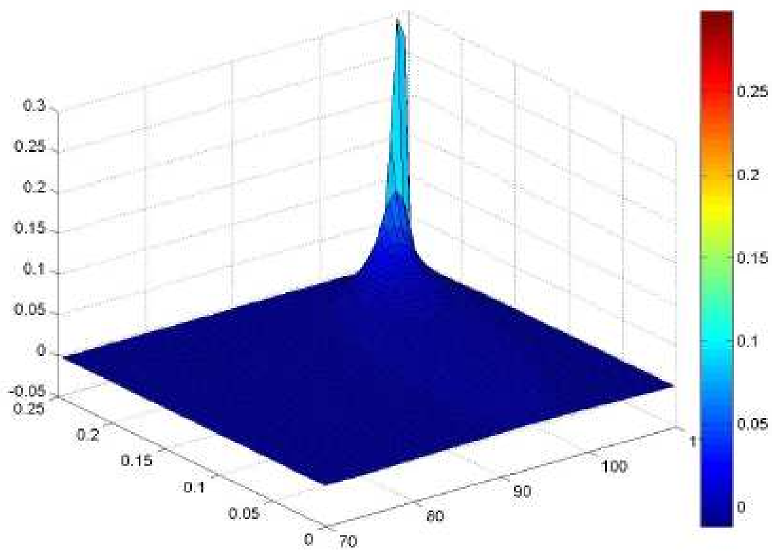
One Touch Option - Price

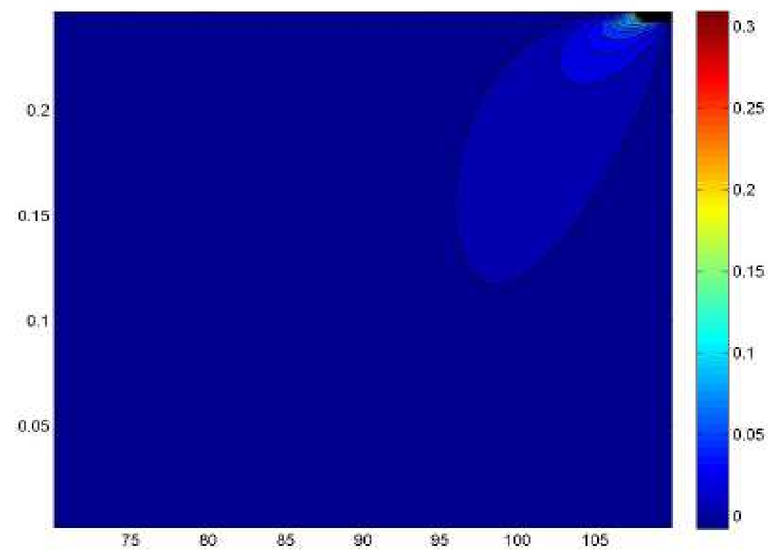
Black-Scholes model S_0=100, H=110, σ=0.25, T=0.25



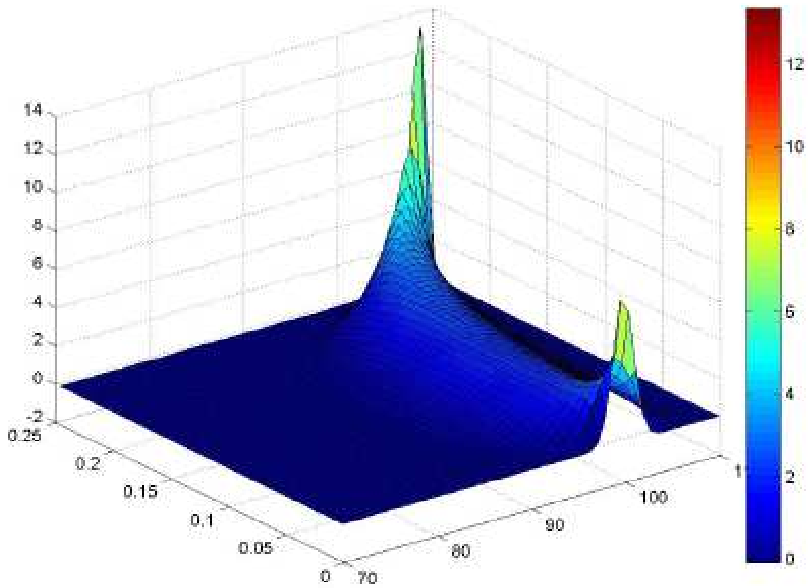


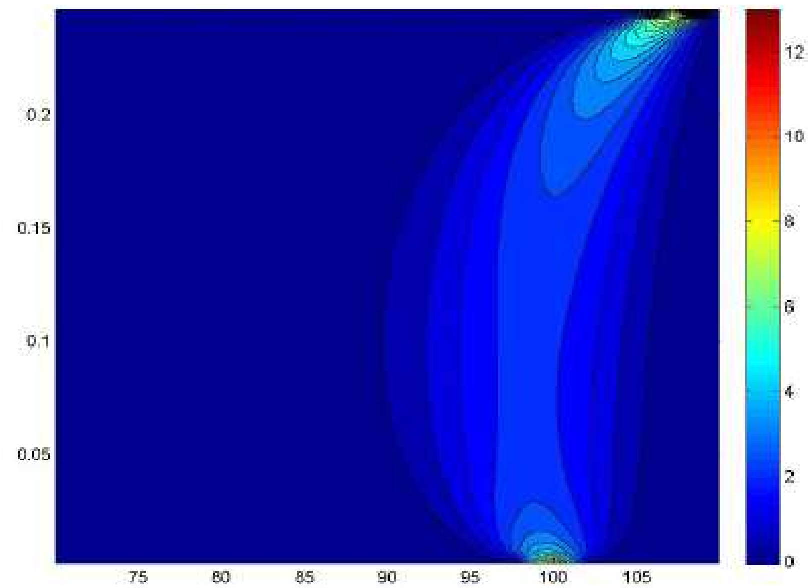
One Touch Option - Γ





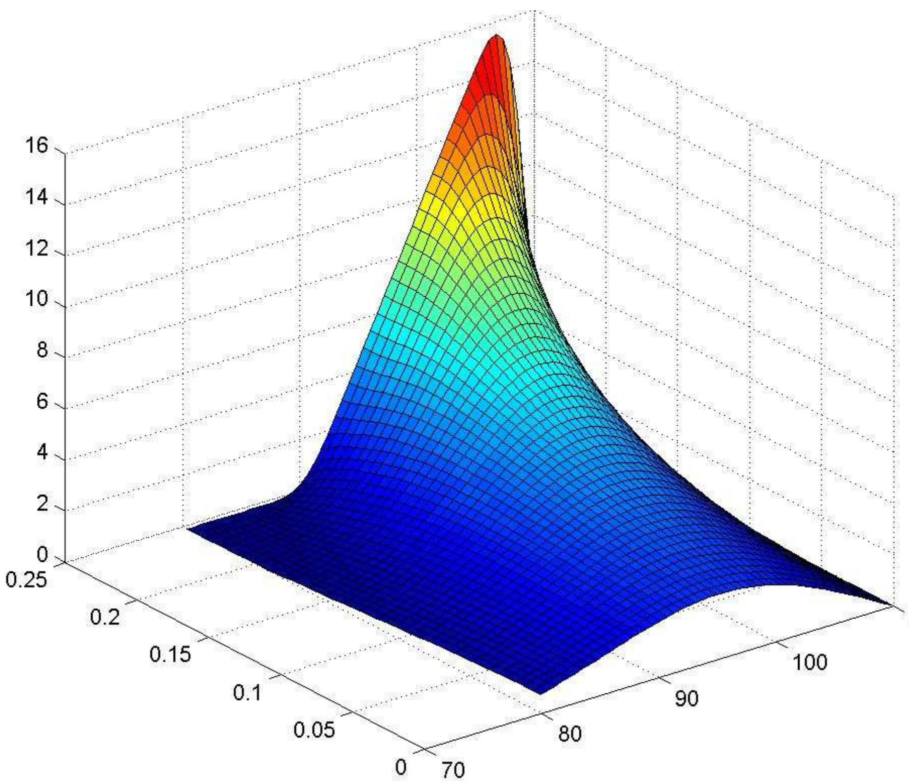
$$O.T.: m(S,t) = \frac{1}{2}\Gamma.\varphi.P$$

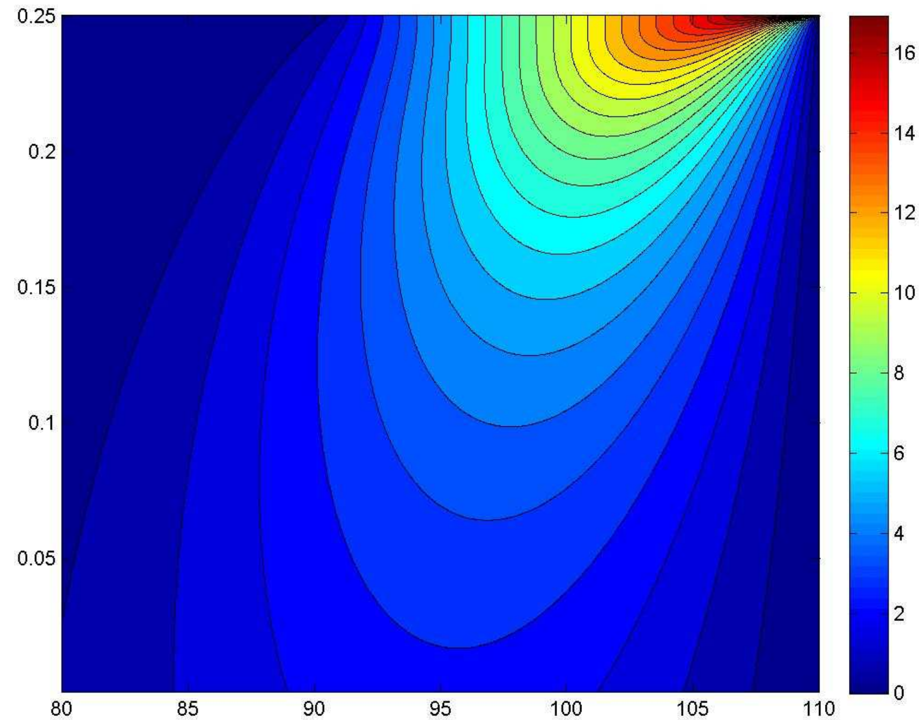




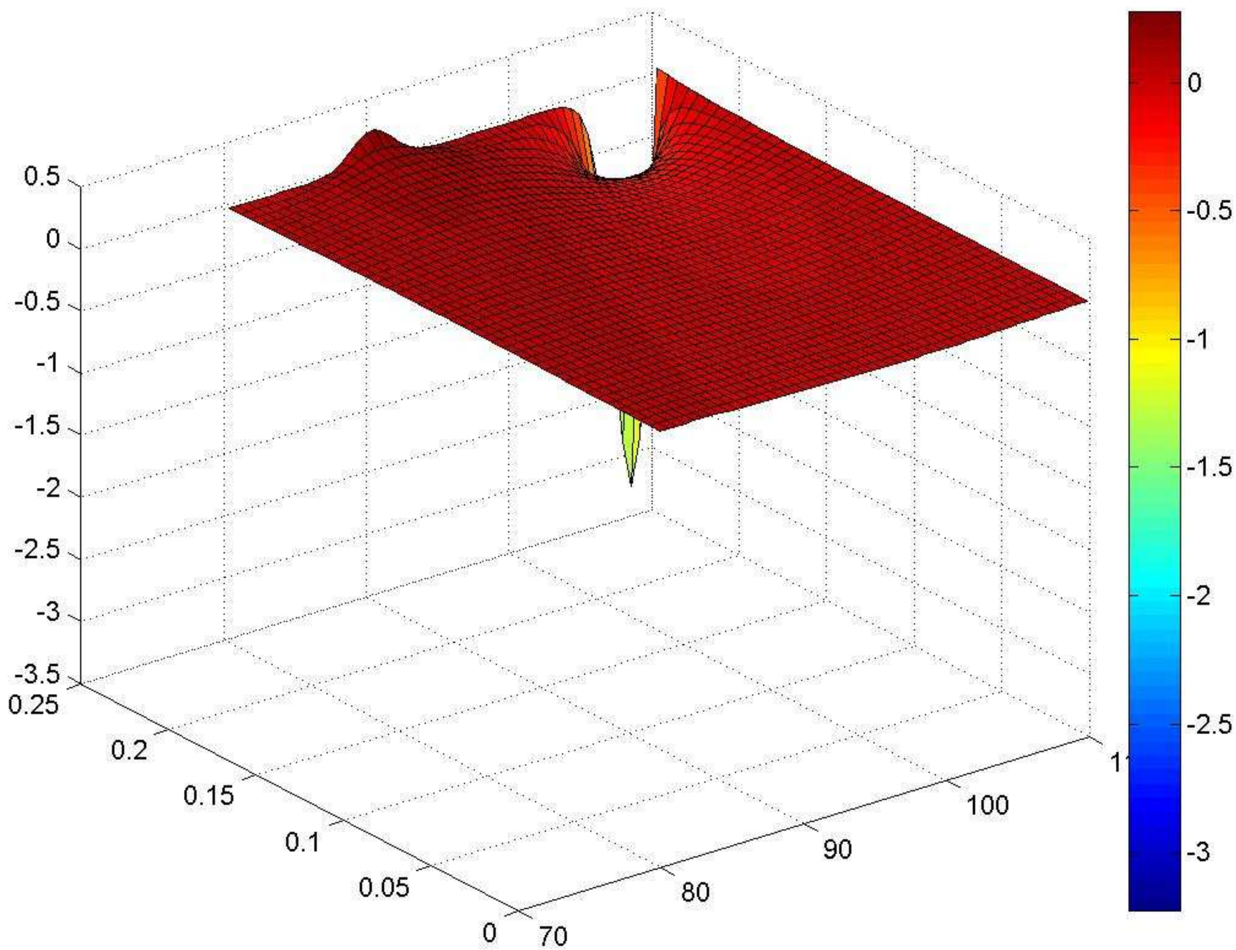
Up-Out Call - Price

Black-Scholes model S_0=100, H=110, K=90, σ=0.25, T=0.25

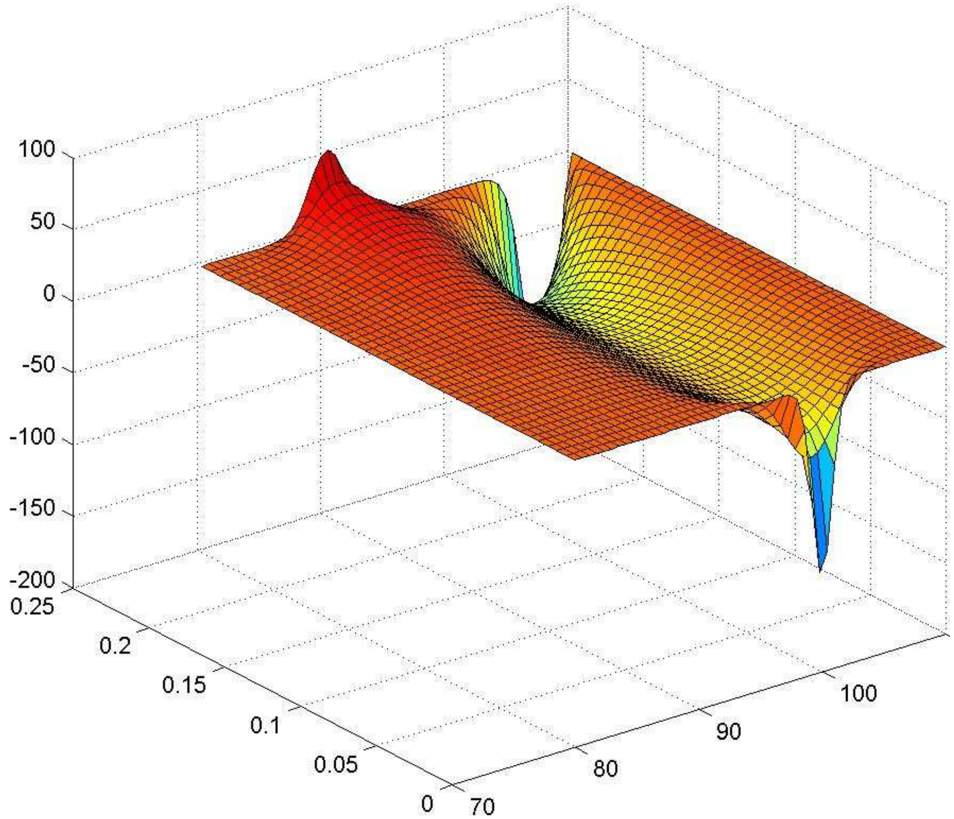


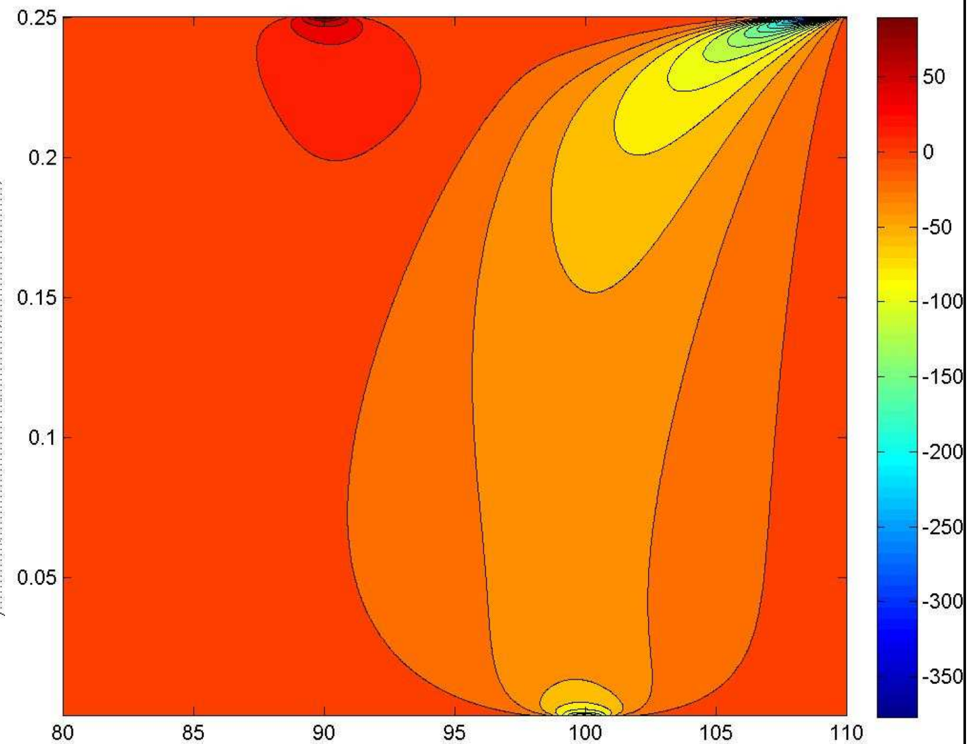


Up-Out Call - Γ

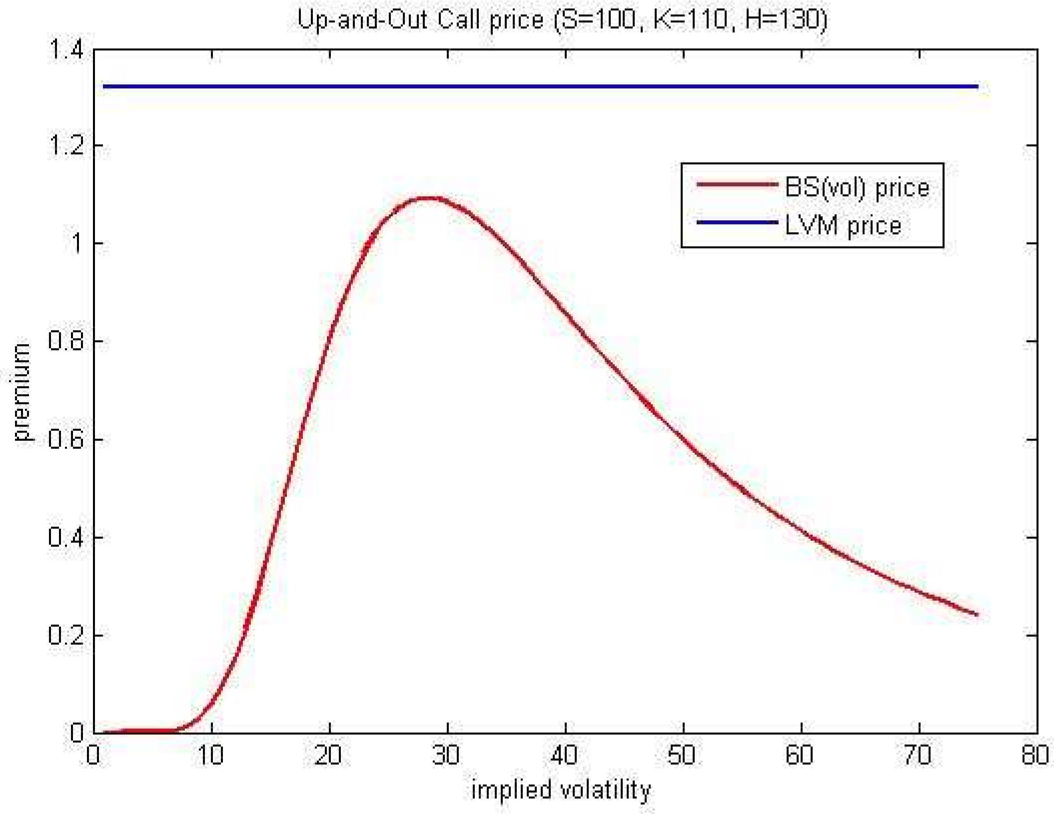


$$UOC: m(S,t) = \frac{1}{2}\Gamma.\varphi.P$$





Black-Scholes/LVM comparison



In this case, no volatility input of the Black - Scholes enables to reach the LVM price.

Vanilla hedging portfolio I

Recall $m(x,t) \equiv \frac{1}{2}\phi^{v_0}(x,t)E^{Q_{v_0}}[\Delta_{xx}f(X_t)\,|\,x_t = x]$ is the sensitivity of Π_g to the local variance at *(x,t)*.

Portfolio $PF = \iint \alpha(K,T)\,C_{K,T}\,dK\,dT$ of vanillas hedges all small volatility moves if and only if for all (x, t)

$$\frac{\partial^2 PF(x,t)}{\partial x^2} = E^{Q_{v_0}}[\Delta_{xx}\Pi_{PF}(X_t)\,|\,x_t = x] = E^{Q_{v_0}}[\Delta_{xx}f(X_t)\,|\,x_t = x] \equiv h(x,t)$$

How can we get the function $\alpha(K,T)$?

Vanilla hedging portfolios II

a) For $k(x,t)$, we define $L(k) \equiv \frac{\partial k}{\partial t} + \frac{1}{2}\frac{\partial^2 (v_0 k)}{\partial x^2}$

For a call $C_{K,T}$, $L(\frac{\partial^2 C_{K,T}}{\partial x^2}) = 0$ with boundary condition $\frac{\partial^2 C_{K,T}}{\partial x^2}(x,T) = \delta_K(x)$

b) $PF \equiv \iint \alpha(K,T)\, C_{K,T}\, dK\, dT \Rightarrow L(\frac{\partial^2 PF}{\partial x^2})(x,t) = -\alpha(x,t)$

$k(x,t) \equiv E^{Q_{v0}}[\Delta_{xx} f(X_t) - \frac{\partial^2 PF}{\partial x^2}(x,t) \big| x_t = x] = h(x,t) - \frac{\partial^2 PF}{\partial x^2}(x,t)$ with $h(x,t) \equiv E^{Q_{v0}}[\Delta_{xx} f(X_t) \big| x_t = x]$

Thus, $L(k) = L(h) + \alpha$.

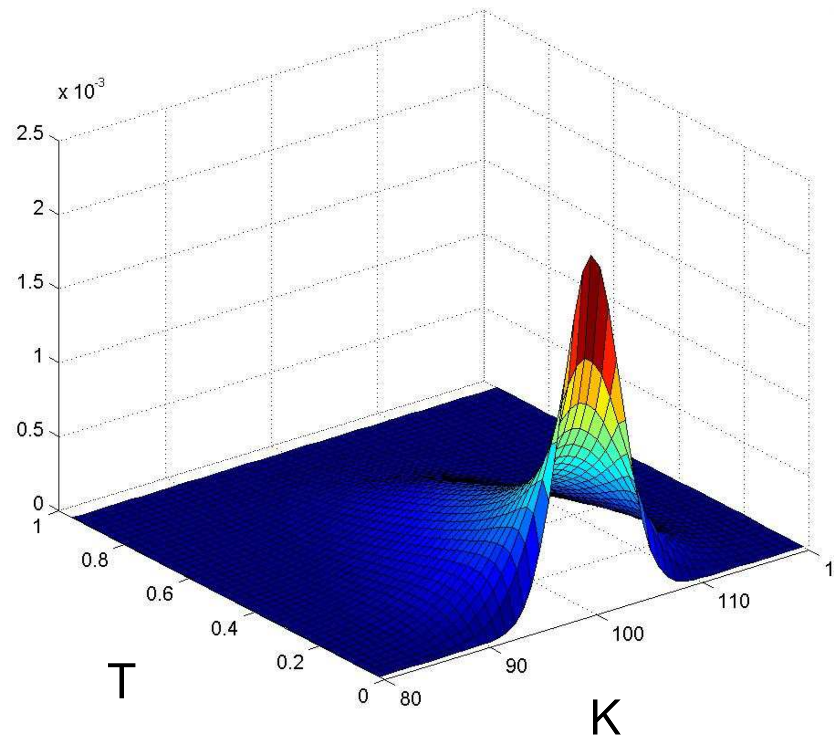
c) If we take $\alpha = -L(h) = -\frac{\partial h}{\partial t} - \frac{1}{2}\frac{\partial^2 (v_0 h)}{\partial x^2}$ then $L(k) = 0$ with $k(x,T) = 0 \Rightarrow k \equiv 0$

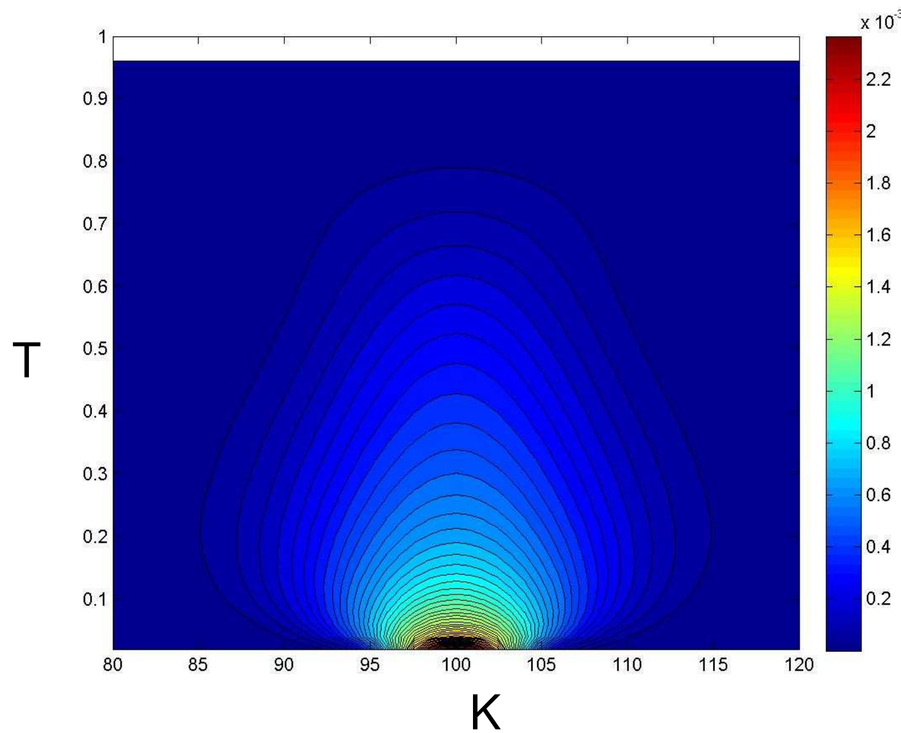
$\forall x,t,\ E^{Q_{v0}}[\Delta_{xx} f(X_t) - \frac{\partial^2 PF}{\partial x^2}(x,t) \big| x_t = x] = 0$ and f - PF has no sensitivity to local vol bumps hence no sensitivity to implied vol bumps.

Example : Asian option

$$dx_t = \sqrt{v_0}\, dW_t \quad \text{Pay-off}: g(X_t) = \left(\int_0^T x_t dt - K \right)^+, S_0 = K = 100 \text{ volatility } \sqrt{v_0} = 20 \text{ maturity } T = 1$$

$m(x,t) \equiv \frac{1}{2} \phi^{v_0}(x,t) E^{Q_{v_0}}[\Delta_{xx} f(X_t) | x_t = x]$ is the sensitivity of Π_g to the local variance at (x,t).

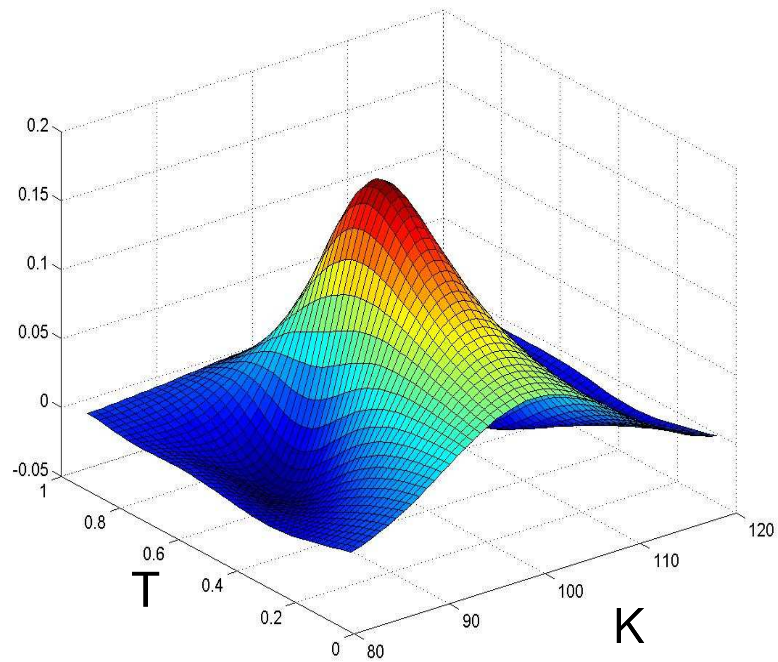


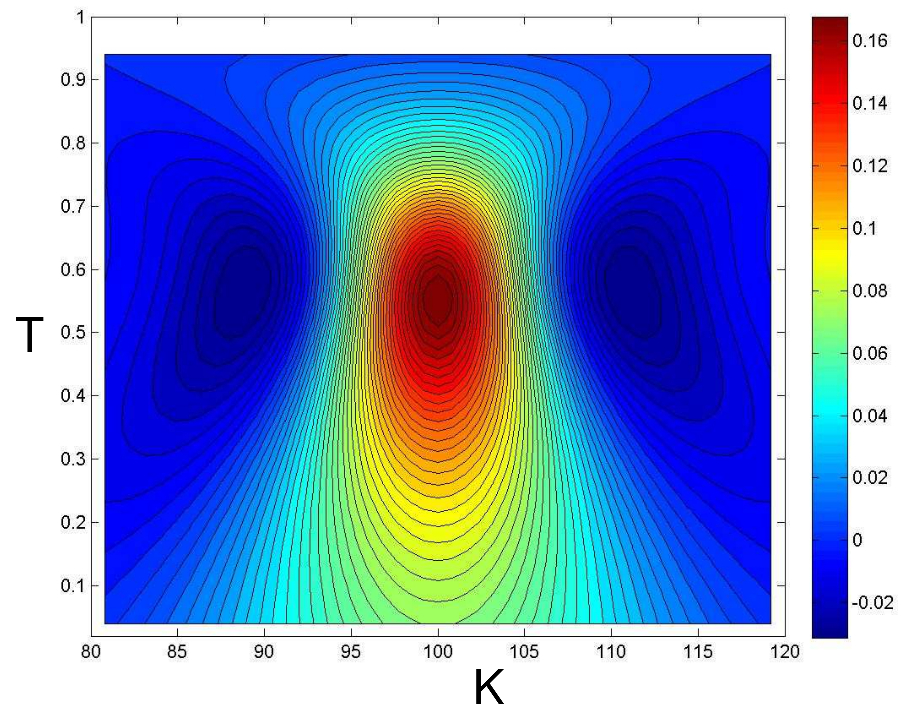


Asian Option Hedge

Robust volatility hedge with $PF \equiv \iint \alpha(K,T)\, C_{K,T}\, dK\, dT$

$$\alpha(K,T) = -\left(\frac{\partial h(K,T)}{\partial t} + \frac{1}{2} \frac{\partial^2 \left(v_0(K,T) h(K,T) \right)}{\partial x^2} \right)$$

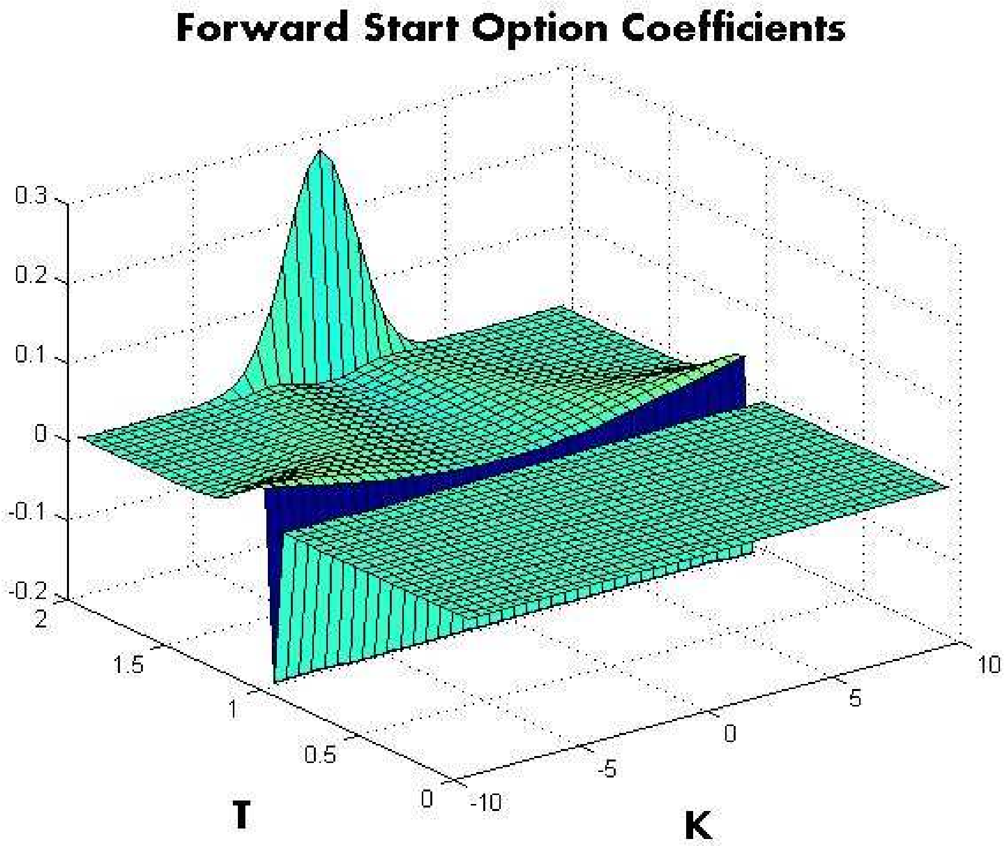




Fwd Start Option Hedge

$g(X_{T_2}) = (x_{T_2} - x_{T_1})^+$

Robust volatility hedge with $PF \equiv \iint \alpha(K,T)\, C_{K,T}\, dK\, dT$



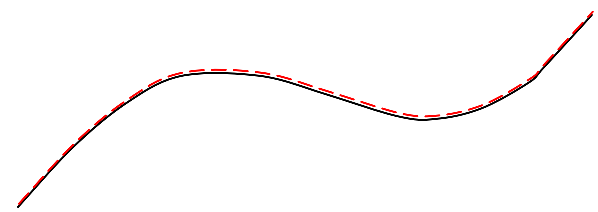
Link *Γ/Vega*

Cancel		Insensitive to
Γ	$\Gamma = 0$	temporary shocks in σ
$Vega$	$\mathbb{E}\left[\int_0^T S_t^2 \Gamma_t dt\right] = 0$	persistent shocks in σ
Superbucket	$\forall S, t \mathbb{E}\left[\Gamma_t \middle| S_t = S\right] = 0$	•any temporary shocks in σ in the future •any small shocks in implied volatility today

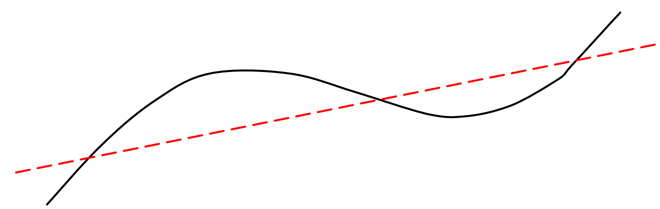
SUPER-REPLICATION/ CLAIM DECOMPOSITION

Claim Pricing

- Complete Markets
    - perfect replication
    - unique price



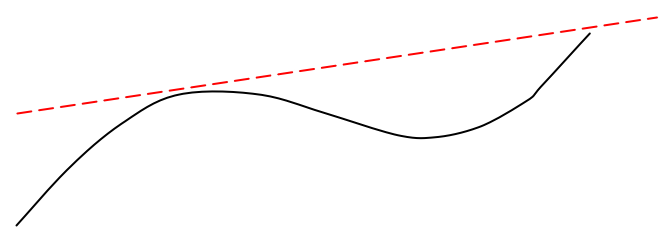
- Incomplete Markets
    - imperfect replication
    - range of prices



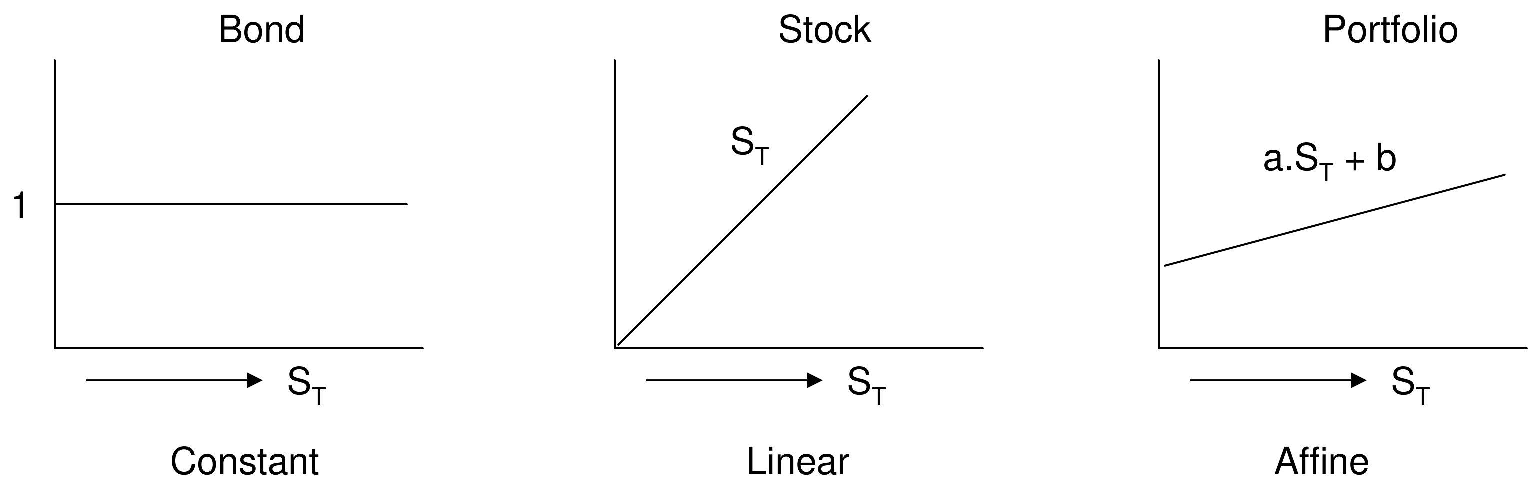
  – Minimum variance hedge

  – Indifference pricing (utility based)

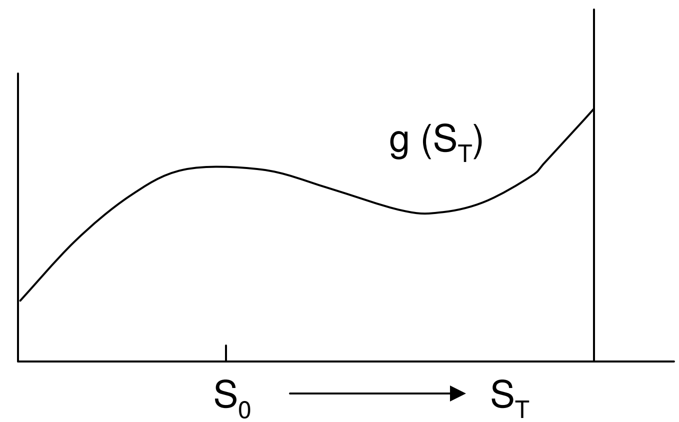
  – Super-replication



Static hedge in Stocks and Bonds



Claim gives $g(S_T)$ at T

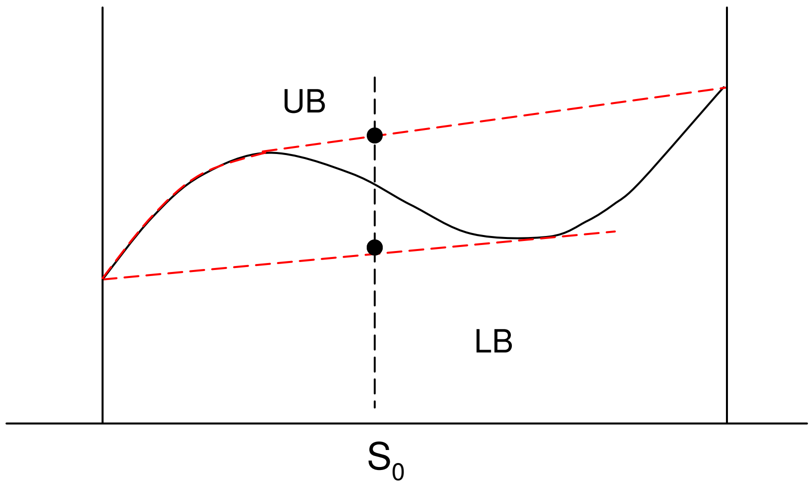


Arbitrage free prices for the claim at 0 ?

Arbitrage free prices for g

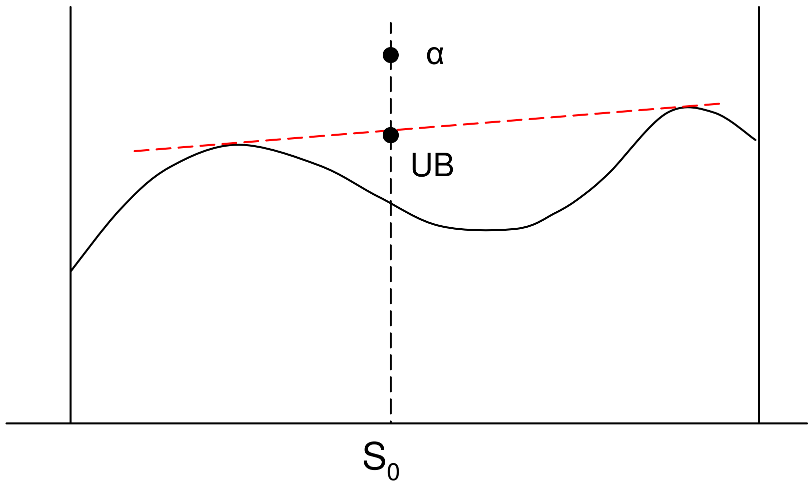
1. Take convex hull of (S, g(S)) in the plane

2. Intersect with the vertical line at S_0

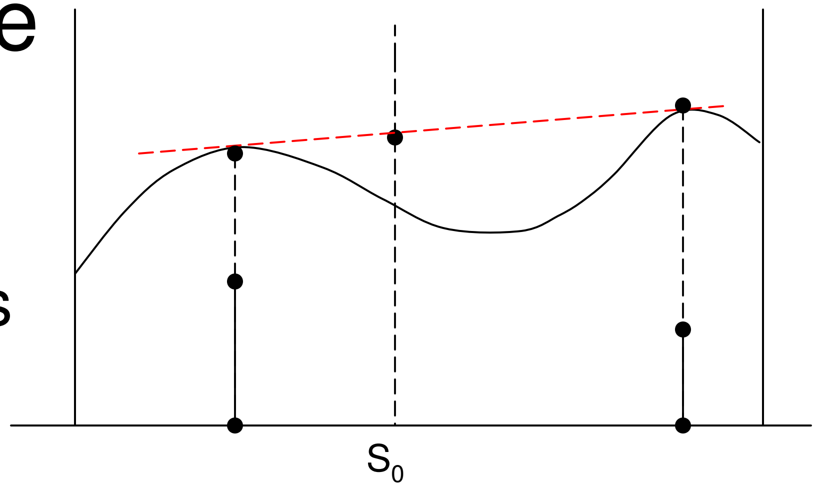


Price Range

1. If α > UB
   - Sell Claim for α
   - Buy super-replication for UB



2. Prices in (LB, UB) are possible
   - Binomial model with states = contact points



Theory of static case

- Delbaen-Schachermayer (1994): Duality Result

- Price decomposition

  - With static position in options and dynamic position in stocks

  where C_i^0 is the price of option C_i at 0

Continuous time case

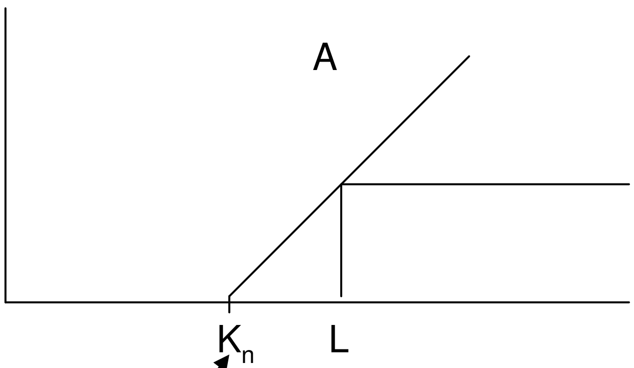
- El Karoui - Quenez (1995): LB is a submartingale
- Dynamic version of Delbaen - Schachermayer (1994)
- Kramkov (1996): Optimal decomposition theorem

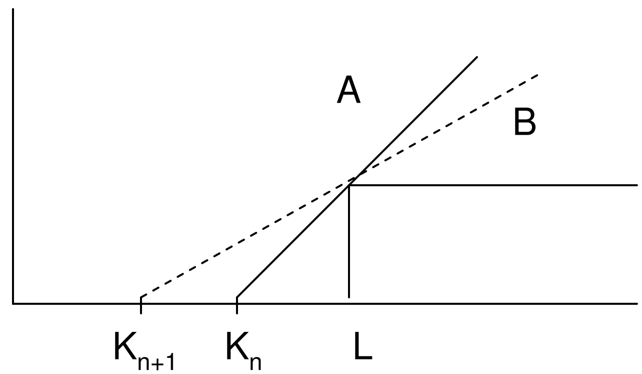
where k is a non decreasing process

- Interpretation: The increasing process *k* comes from improving LB whenever you can
- In other words, if you prepare for the worst, you can only have good surprises

One Touch example

- A super-replication A put in place today will still work tomorrow. If a cheaper one, B is available tomorrow, roll from A to B to collect the improvement





Optimal strike at t_n: minimize $\frac{1}{L-K}C_K \geq OT_L$

Optimal strike at t_{n+1}

- If at t_{n+1} $B = \frac{1}{L-K_{n+1}}C_{K_{n+1}}$ is cheaper than $A = \frac{1}{L-K_n}C_{K_n}$

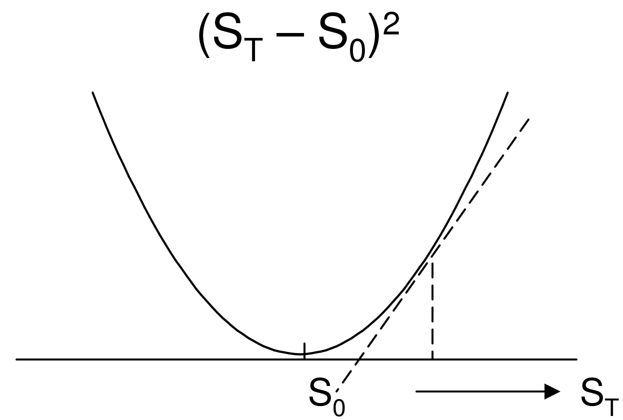
- Sell $A(t_n)$ and buy $B(t_{n+1})$: you collect the difference and still super-replicate

Parabola Example

- If you delta hedge at stopping times τ_i, you collect the discrete quadratic variation

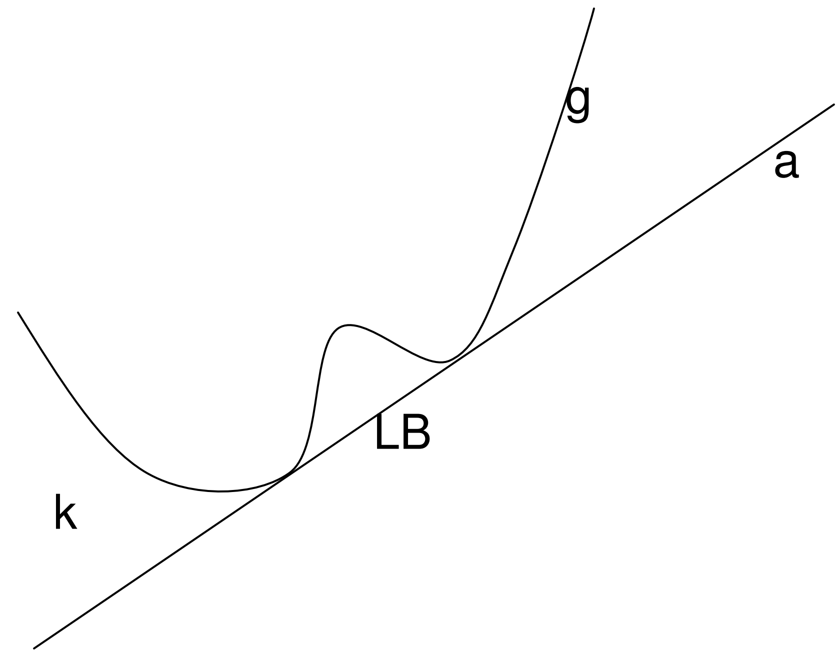
$$\sum (S_{\tau_i} - S_{\tau_{i-1}})^2$$

(mechanics of Variance Swaps)



- No hedge $(S_T - S_0)^2$
- Hedge at τ_i $\sum (S_{\tau_i} - S_{\tau_{i-1}})^2$
- Continuous hedge $\langle S \rangle_T$ (quadratic variation of S)

European Case 1 period



$g = LB + a + k$

g : claim

LB : lower bound

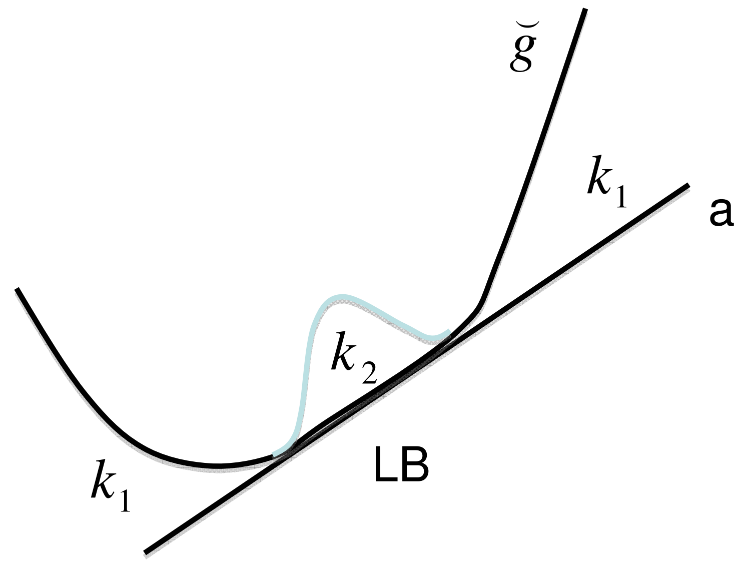
a : costless attainable claim

k : non negative claim

$LB + a$ is the sub - hedge, a static position in ZC and asset

k is a possible payment to be collected at maturity

European Case Continuous Time



$g = LB + a + k_1 + k_2$

k_1, k_2 : non negative claims

$LB + a + k_1$: convex hull of X

k_1 : convex claim that can be hedged in many ways

k_2 is a possible payment to be collected at maturity

$\text{ConvexHull}(k_2) = 0$

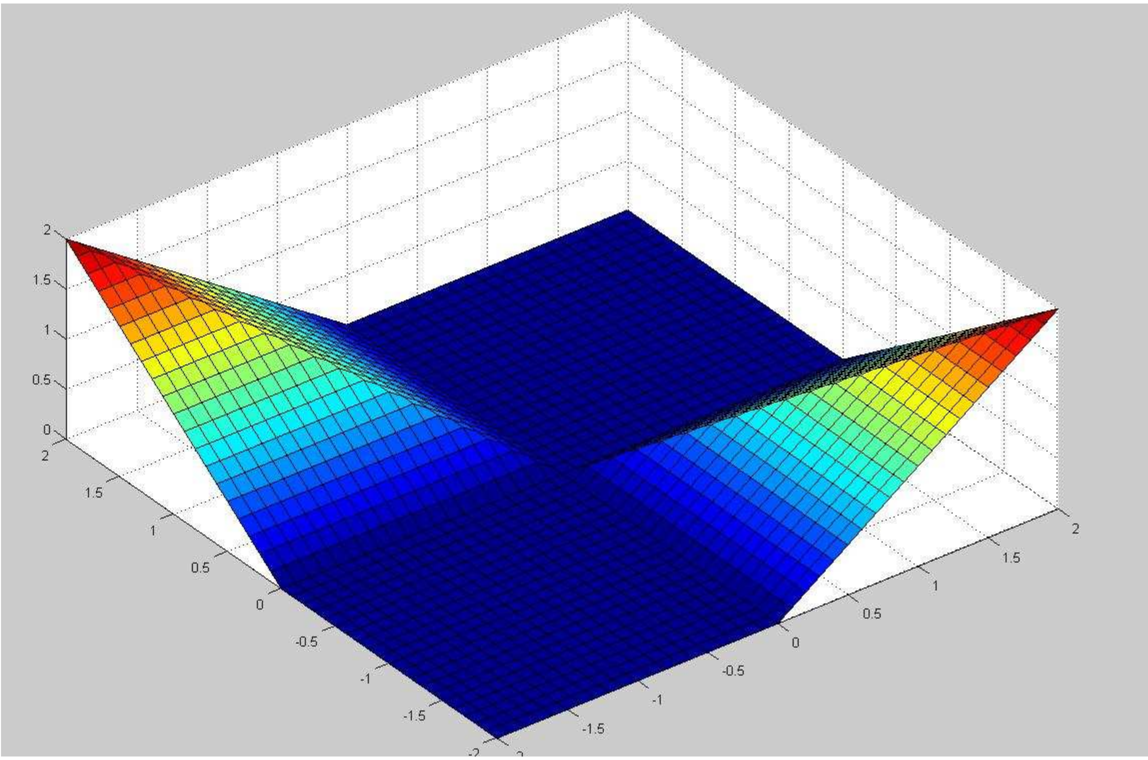
Basket Option: k2

Let the underlying of the Basket option be X and Y

1. Payoff = $(X + Y - K)^+$

   For simplicity, K = 0: $(X+Y)^+ \leq X^+ + Y^+$

   So, $X^+ + Y^+ - (X+Y)^+$



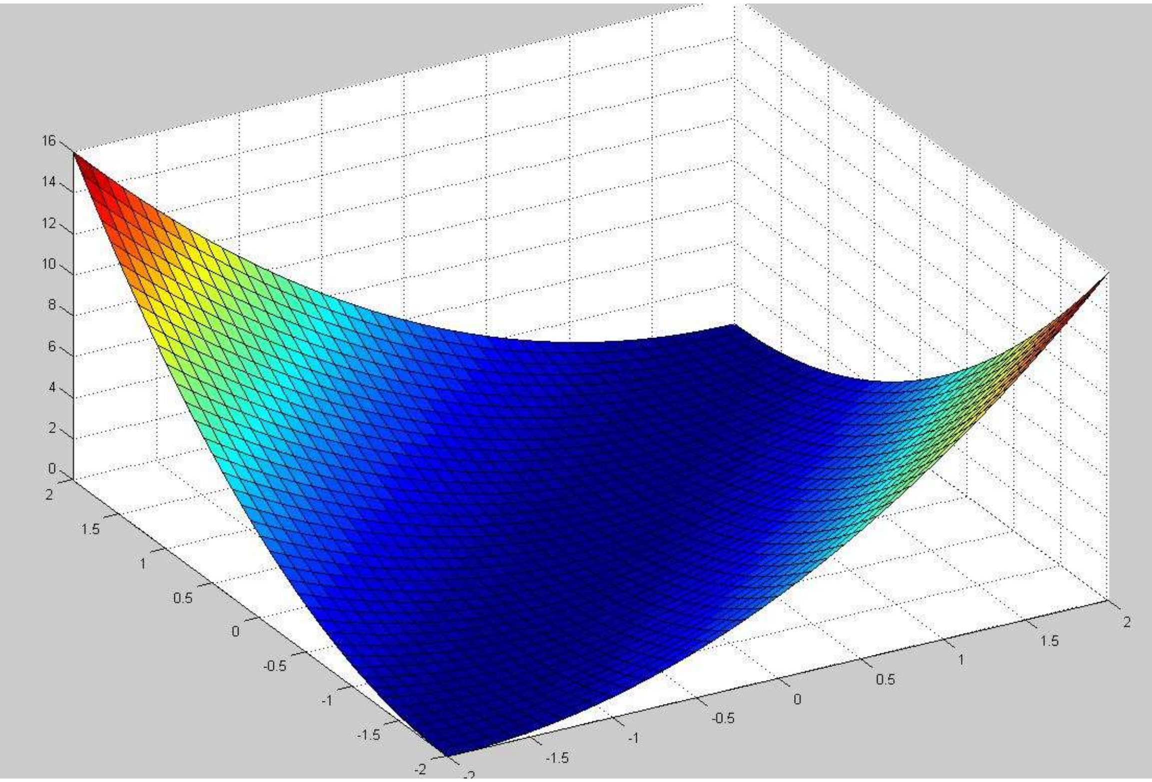
Basket Parabola: k1

Let the underlying of the Basket option be X and Y

2. Payoff = $(X + Y - K)^2$

   For simplicity, K = 0: $(X+Y)^2 \le 2(X^2 + Y^2)$

   So, $2(X^2 + Y^2) - (X+Y)^2 = (X-Y)^2$



Path Dependent Case

The same decomposition

$$g = LB + a + k_1 + k_2$$

still holds.

We make use of Functional Ito Formula

Subfunctionals

LB and Functional Itô Formula

From Functional Ito Formula, if LB is smooth,

$$g(X_T) = LB(X_T)$$

$$= LB(X_0) + \int_0^T \Delta_x LB(X_u)\, dx_u + \frac{1}{2}\int_0^T \Delta_{xx} LB(X_u)\, d\langle x\rangle_u + \int_0^T \Delta_t LB(X_u)\, du$$

which gives the same decomposition $g = LB + a + k_1 + k_2$

$$k_1(X_t) \equiv \frac{1}{2}\int_0^t \Delta_{xx} LB(X_u)\, d\langle x\rangle_u$$

$$k_2(X_t) \equiv \int_0^t \Delta_t LB(X_u)\, du$$

are non decreasing with disjoint supports

$$g = LB + a + k_1 + k_2$$

The claim k_1 is $k_1(X_T)$

The claim k_2 is $k_2(X_T)$

$\Delta_{xx} LB(X_t) > 0 \Leftrightarrow$ minimizing measure : freeze

If prices does not freeze, capture convexity (Parabola)

$\Delta_t LB(X_t) > 0 \Leftrightarrow$ minimizing measure : jump

If prices does not jump, capture time value (Range accrual)

CONCLUSION

Conclusion

- Itô calculus can be extended to functionals of price paths
- Price difference between 2 models can be computed
- We get a variational calculus on volatility surfaces
- It leads to a strike/maturity decomposition of the volatility risk of the full portfolio
- (Path dependent) Claims can be decomposed in a canonical way. It refines the Kramkov result and splits the increasing process in 2