A factor contagion model for portfolio credit derivatives with interacting recovery rate

Geon Ho Choe[1] Hyun Jin Jang[2] Soon Won Kwon[1]

[1]KAIST
[2]Samsung FN

6th World Congress of the Bachelier Finance Society, 2010, Toronto

Introduction

Goal

- Valuation of portfolio credit derivatives when there is a contagion effect

Outline

- Marshall-Olkin copula & Contagion model
- Distribution of the kth default time
- Portfolio loss distribution
- Interacting recovery rate

Introduction

Copula Model

- Li(2000) : Gaussian copula function approaches
- Laurent & Gregory(2005), Andersen & Sidenius(2003), Bastide et al.(2007) : pricing method with various copula functions
- Andersen et al.(2003), Hull & White(2004) : recursive techniques to derive loss distributions

Contagion Model

- Davis & Lo(2001) : infection model
- Jarrow & Yu(2001) : primary-secondary framework
- Frey & Backhaus(2008) : pricing method using Markov process technique

Default Probability & Recovery Rate

- Altman et al.(2005) : relations between default probabilities and recovery rates

Bivariate Marshall-Olkin copula

Assumptions

- $Z_1 \sim \exp(\lambda_1)$, $Z_2 \sim \exp(\lambda_2)$, $Z \sim \exp(\lambda)$: independent
- $X_1 = \min(Z, Z_1)$ and $X_2 = \min(Z, Z_2)$
- $s_i(x_i) = e^{-(\lambda_i+\lambda)x_i}, \quad s(x_1, x_2) = e^{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda \max(x_1, x_2)}$
  : marginal and joint survival functions of X_1 and X_2

Bivariate Marshall-Olkin copula

- By Sklar's theorem, there exists a unique survival copula C satisfying

$$s(x_1, x_2) = C(s_1(x_1), s_2(x_2)).$$

- Marshall-Olkin copula is given by

$$C(u_1, u_2) = \min(u_2 u_1^{1-\theta_1}, u_1 u_2^{1-\theta_2}),$$

where $\theta_i = \frac{\lambda}{\lambda+\lambda_i}$.

One factor Marshall-Olkin copula model

Assumptions

- $V_0 \sim \exp(\alpha)$: systematic factor, $0 \leq \alpha \leq 1$
  $V_i \sim \exp(1-\alpha)$: idiosyncratic factor, $i = 1, \ldots, n$
- V_0 and V_i are independent
- $\overline{V}_i = \min(V_0, V_i) \sim \exp(1)$

Definition of default times

- The default time τ_i of name i is defined as
$$\tau_i = \inf\left\{ t > 0 : \int_0^t \lambda_i(s)ds \geq \overline{V}_i \right\}.$$
- The distribution function of the kth default time is
$$F^{(k)}(t) = \int_0^\infty \mathbb{P}(\tau^k \leq t | V_0 = v)\phi(v)dv,$$
where ϕ is the probability density function of V_0 and τ^k is the kth default time.

kth default time under one factor contagion model

Default intensity and default times

- $\lambda_i(t) = a(t) + c(t) \sum_{j=1, j\neq i}^{n} \mathbf{1}_{\{\tau_j \leq t\}}$: default intensity

Assuming that the jth default occurs, we define the following random variables conditional on $V_0 = v$,

- $\tau_i(v) = \inf\left\{t > 0 : \int_0^t (a(s) + jc(s))ds \geq \min(v, V_i)\right\}$: conditional default time after jth default
- $\tau_{i:n-j}(v)$: ith order statistic of $\tau_1(v), \ldots, \tau_{n-j}(v)$, i.e.,

$$\tau_{1:n-j}(v) < \cdots < \tau_{i:n-j}(v) < \cdots < \tau_{n-j:n-j}(v).$$

kth default time and its distribution

- The kth default time $\tau^k(v)$ conditional on $V_0 = v$ is defined by

$$\tau^k(v) = \sum_{j=1}^{k} \tau_{1:n-j+1}(v).$$

- The distribution function of the kth default time can be written as

$$F^{(k)}(t) = \int_0^\infty \mathbb{P}\left(\sum_{j=1}^{k} \tau_{1:n-j+1}(v) \le t \right) \phi(v) dv,$$

where ϕ is the probability density function of V_0.

Theorem (Distribution function of kth default time - General case)

Let the default time τ_i and its default intensity $\lambda_i(t)$ be defined as above. Let

$$Q_i(t) = \int_0^t (a(x) + ic(x))dx$$

and

$$f_{\ell,n}(t) = (1-\alpha)(n-\ell+1)Q'_{\ell-1}(t)e^{-(1-\alpha)(n-\ell+1)Q_{\ell-1}(t)}.$$

Let ζ_i be the function whose Laplace transform is given by

$$\widehat{\zeta}_i(s) = \frac{1}{s^{i+1}} \prod_{\ell=1}^{i} \mathbb{E}\left[f_{\ell,n}\left(\frac{X}{s}\right)\right],$$

where X be a unit exponential random variable.

Then the distribution function of τ^k is

$$F^{(k)}(t) = \zeta_k(t)e^{-\alpha Q_{k-1}(t)} + \sum_{i=1}^{k-1} \zeta_i(t)(e^{-\alpha Q_{i-1}(t)} - e^{-\alpha Q_i(t)}) + 1 - e^{-\alpha Q_0(t)}.$$

Theorem (Distribution function of kth default time - Special case)

Let $a(t) \equiv a > 0$ and $c(t) \equiv c \geq 0$ be constants.
Let $Q_i(t) = (a + ic)t$ and let ζ_i be the function which is given by

$$\zeta_i(t) = 1 - \sum_{\ell=1}^{i} A_{\ell,i} \exp\left(-\frac{1-\alpha}{p_{\ell,n}} t\right)$$

where

$$A_{\ell,i} = \frac{p_{\ell,n}^{i-1}}{\prod_{j=1,j\neq\ell}^{i}(p_{\ell,n} - p_{j,n})} \quad \text{and} \quad p_{\ell,n} = \frac{1}{(n-\ell+1)q_{\ell-1}}.$$

Then the distribution function of τ^k is

$$F^{(k)}(t) = \zeta_k(t)e^{-\alpha Q_{k-1}(t)} + \sum_{i=1}^{k-1} \zeta_i(t)(e^{-\alpha Q_{i-1}(t)} - e^{-\alpha Q_i(t)}) + 1 - e^{-\alpha Q_0(t)}.$$

Corollary (The number of defaults up to time t)

Let $N(t)$ be the number of defaults up to time t. Then

$$\mathbb{P}(N(t)=k)=\begin{cases}(1-\zeta_1(t))e^{-\alpha Q_0(t)} & \text{if } k=0\\ (\zeta_k(t)-\zeta_{k+1}(t))e^{-\alpha Q_k(t)} & \text{if } k=1,\dots,n-1\\ F^{(n)}(t) & \text{if } k=n\end{cases}$$

where $\zeta_k(t)$, $Q_k(t)$ and $F^{(n)}(t)$ are given in the previous theorems.

kth-to-default swaps

Now we compute premiums of portfolio credit derivatives.

We use the following notations:

- R : recovery rate
- T : maturity of the contracts
- $B(0,t)$: price of risk-free zero coupon bond with maturity t
- t_i : premium payment dates, $0 = t_0 < \cdots < t_N = T$ and $\Delta_{i-1,i} = t_i - t_{i-1}$

Proposition (Premium of kth-to-default swap)

The annualized premium s_k of a kth-to-default swap is equal to

$$s_k = \frac{(1-R)\int_0^T B(0,t)dF^{(k)}(t)}{\sum_{i=1}^N \left\{ \Delta_{i-1,i} B(0,t_i)(1-F^{(k)}(t_i)) + \Delta_{i-1,i}^{-1} \int_{t_{i-1}}^{t_i} (t-t_{i-1}) B(0,t) dF^{(k)}(t) \right\}}.$$

Single tranche CDOs

Define a cumulative percentage loss $L(t)$ on the homogeneous reference portfolio up to time t

$$L(t) = \frac{1-R}{n}\sum_{i=1}^{n}\mathbf{1}_{\{\tau_i \le t\}} = \frac{1-R}{n}N(t).$$

Consider a CDO tranche with an attachment point A and a detachment point B where $0 \le A < B \le 1$. The percentage loss $L(t,A,B)$ on the tranche $[A,B]$ up to time t is defined by

$$L(t,A,B) = \frac{\max(L(t)-A,0) - \max(L(t)-B,0)}{B-A}.$$

Proposition (Premium of single tranche CDO)

The premium $s_{A,B}$ of the tranche with an attachment point A and a detachment point B is equal to

$$s_{A,B} = \frac{B(0,T)\mathbb{E}[L(T,A,B)] + \int_0^T f(0,s)B(0,s)\mathbb{E}[L(s,A,B)]ds}{\sum_{i=1}^{N} \Delta_{i-1,i}B(0,t_i)(1-\mathbb{E}[L(t_i,A,B)])}$$

where

$$\mathbb{E}[L(t,A,B)] = 1 - \frac{1}{B-A}\int_A^B \sum_{\ell=0}^{\lfloor \frac{n}{1-R}x \rfloor} \mathbb{P}(N(t)=\ell)dx,$$

$\lfloor y \rfloor$ is the largest integer not greater than y and $f(0,t)$ is the spot forward rate.

Interacting recovery rates

Define an interacting recovery rate $R(t)$ by

$$R(t) = R_0 - \gamma \sum_{k=1}^{n} \mathbf{1}_{\{\tau^k \leq t\}}.$$

In this case, the cumulative percentage loss is given by

$$\widetilde{L}(t) = \frac{1}{n} \sum_{k=1}^{n} (1 - (R_0 - k\gamma))\mathbf{1}_{\{\tau^k \leq t\}}$$

and the loss on a tranche $[A, B]$ is

$$\widetilde{L}(t, A, B) = \frac{\max(\widetilde{L}(t) - A, 0) - \max(\widetilde{L}(t) - B, 0)}{B - A}.$$

Proposition (Interacting recovery rate)

The premium $\widetilde{s}_{A,B}$ of the tranche with an attachment point A and a detachment point B is equal to

$$\widetilde{s}_{A,B} = \frac{B(0,T)\mathbb{E}[\widetilde{L}(T,A,B)] + \int_0^T f(0,s)B(0,s)\mathbb{E}[\widetilde{L}(s,A,B)]ds}{\sum_{i=1}^N \Delta_{i-1,i}B(0,t_i)(1-\mathbb{E}[\widetilde{L}(t_i,A,B)])}$$

where

$$\mathbb{E}[\widetilde{L}(t,A,B)] = 1 - \frac{1}{B-A}\int_A^B \sum_{\ell=0}^{\lfloor \beta(x) \rfloor} \mathbb{P}(N(t)=\ell)dx$$

and

$$\beta(x) = \frac{1}{2\gamma}\Bigg(-(2(1-R_0)+\gamma) + \sqrt{(2(1-R_0)+\gamma)^2 + 8\gamma n x}\Bigg).$$

Numerical results

Default intensity

Assume that the default intensity is given by

$$\lambda_i(t) = ae^{-\delta t}\left(1 + \theta \sum_{j=1, j\neq i}^{n} \mathbf{1}_{\{\tau_j \leq t\}}\right).$$

- a : the base default intensity
- δ : the rate of exponential decay
- θ : the level of contagion.

Distributions of kth default time

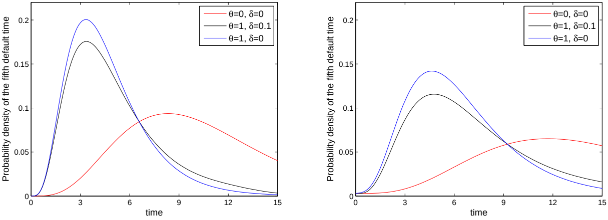

Figure: Probability densities of the 5th default times with $\alpha = 0$ (left) and $\alpha = 0.3$ (right)

Probability distribution the number of defaults

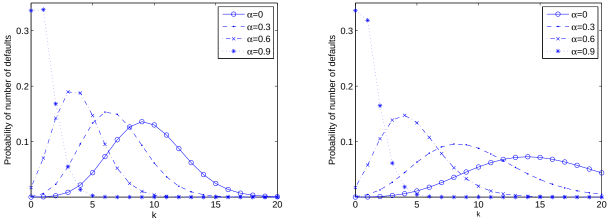

Figure: Probabilities of the number of defaults up to time $t = 5$ with $\alpha = 0, 0.3, 0.6, 0.9$ for the contagion level $\theta = 0$ (left) and $\theta = 0.1$ (right).

BDS-Contagion level

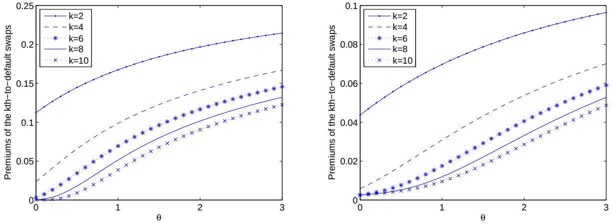

Figure: Premiums of kth-to-default swaps against contagion levels $0 \leq \theta \leq 3$ with $\alpha = 0$ (left) and $\alpha = 0.5$ (right) for $k = 2, 4, 6, 8, 10$

BDS-Correlation

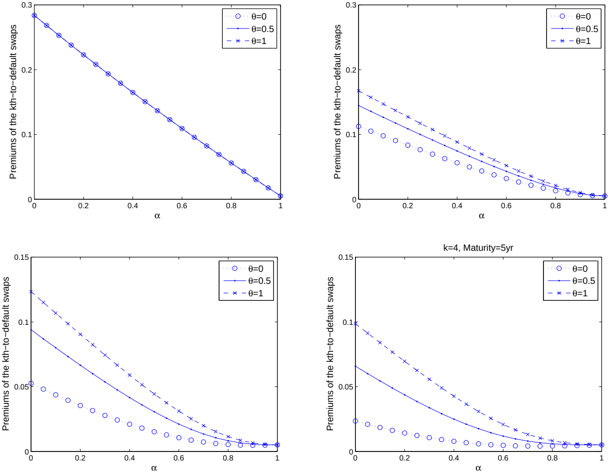

Figure: Premiums of kth-to-default swaps against correlation $0 \leq \alpha \leq 1$ with $T = 5$ for $k = 1$ (upper left), $k = 2$ (upper right), $k = 3$ (lower left) and $k = 4$ (lower right)

STCDO-Contagion level

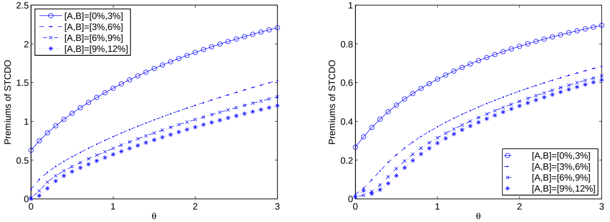

Figure: Premiums of STCDOs against contagion levels $0 \leq \theta \leq 3$ with $\alpha = 0$ (left) and $\alpha = 0.5$ (right) for $[A, B]$=[0%, 3%], [3%, 6%], [6%, 9%], [9%, 12%]

STCDO-Correlation

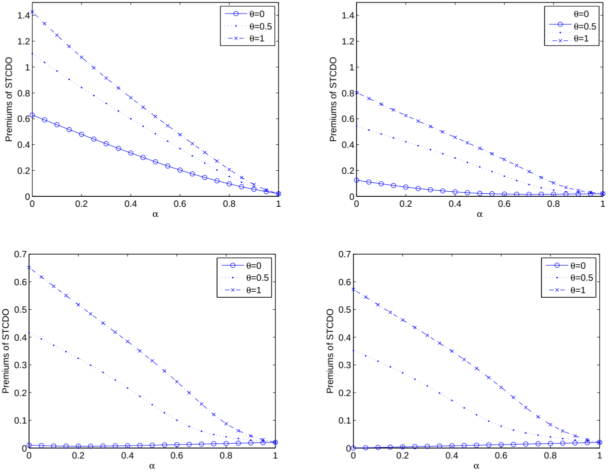

Figure: Premiums of STCDOs against correlations $0 \leq \alpha \leq 1$ with $[A, B]$=[0%, 3%] (upper left), [3%, 6%] (upper right), [6%, 9%] (lower left), [9%, 12%] (lower right)

STCDO-Interacting recovery rate

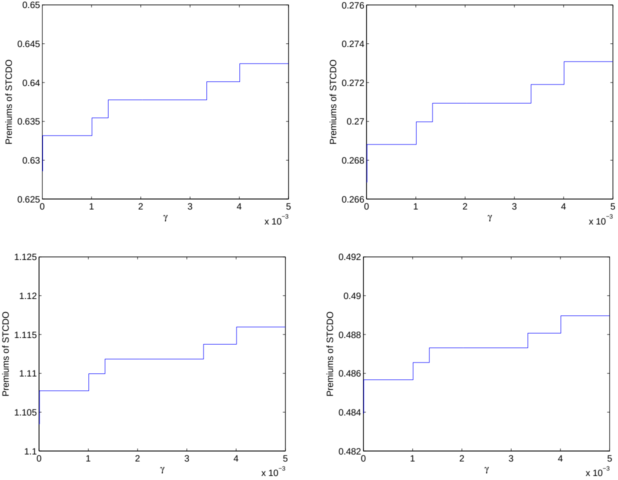

Figure: Premiums of STCDOs against $0 \leq \gamma \leq 0.005$ with $\theta = 0$, $\alpha = 0$ (upper left), $\theta = 0$, $\alpha = 0.5$ (upper right), $\theta = 0.5$, $\alpha = 0$ (lower left) and $\theta = 0.5$, $\alpha = 0.5$ (lower right)

Summary

- A homogeneous reference portfolio with correlation risks as well as contagion effect
- One factor contagion model with Marshall-Olkin copula
- A simple and efficient method for finding the distribution of the kth default time and pricing portfolio credit derivatives
- The reference portfolio with interacting recovery rates
- The relationship between premiums and parameters such as default correlation and the level of contagion

Thank you!