

**Utility Maximization  
With Habit Formation:  
Dynamic Programming  
and  
Stochastic PDE's**

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# The Model

Probability space:  $(\Omega, \mathcal{F}, P)$ .

Brownian Motion:  $W(\cdot) = (W_1(\cdot), \dots, W_d(\cdot))^*$

Information filtration:  $\mathbb{F} = (\mathcal{F}_t^W)_{t \geq 0}$

Complete Financial Market:

- finite time horizon  $[0, T]$
  - a riskless asset  $B(\cdot)$
  - $d$  stocks  $S_i(\cdot)$ ,  $i = 1, 2, \dots, d$
- such that

$$dB(t) = r(t)B(t)dt, \quad B(0) = 1,$$

$$dS_i(t) = S_i(t) \left[ b_i(t)dt + \sum_{j=1}^d \sigma_{ij}(t)dW_j(t) \right], \quad i = 1, \dots, d.$$

Interest rate:  $r(\cdot)$  is bounded.

Vector of rates-of-return:  $b(\cdot) = (b_1(\cdot), \dots, b_d(\cdot))^*$   
is integrable almost surely.

Volatility matrix:  $\sigma(\cdot) = \{\sigma_{ij}(\cdot)\}_{1 \leq i, j \leq d}$   
is square-integrable almost surely and  
 $\sigma(t)$  has full rank for every  $t$ .

All processes are  $\mathbb{F}$ –progressively measurable.  
No anticipation of the future.

**Economic agent:** At each time  $t$  he can decide

- **proportion (portfolio)**  $\pi_i(t)$  of his **wealth**  $X(t)$  to be invested in the  $i$ th stock
- remaining amount is invested in the riskless asset
- **consumption** rate  $c(t) \geq 0$
- **initial endowment**  $x > 0$ .

For  $X(\cdot) \equiv X^{x,\pi,c}(\cdot)$  we have the equation

$$dX(t) = X(t) \left[ \sum_{i=1}^d \pi_i(t) \cdot \frac{dS_i(t)}{S_i(t)} + \left( 1 - \sum_{i=1}^d \pi_i(t) \right) r(t) dt \right] - c(t) dt$$

$$= [r(t)X(t) - c(t)]dt + X(t)\pi^*(t)\sigma(t) [dW(t) + \vartheta(t)dt],$$

subject to the initial condition  $X(0) = x > 0$ .

**Market price of risk:**  $\vartheta(t) \triangleq \sigma^{-1}(t)[b(t) - r(t)\mathbf{1}]$

Equivalently, we have

$$\frac{X(t)}{B(t)} + \int_0^t \frac{c(s)}{B(s)} ds = x + \int_0^t \frac{X(s)}{B(s)} \pi^*(s) \sigma(s) [dW(s) + \vartheta(s) ds]$$

**Utility function:**  $u : [0, T] \times \mathbb{R}^+ \rightarrow \mathbb{R}$ ,

such that  $u(t, \cdot)$  is strictly increasing, strictly concave, of class  $C^1(\mathbb{R}^+)$ , and  $u'(t, 0^+) = \infty$ ,  $u'(t, \infty) = 0$ ;

**Standard of living:** average of past consumption.

For  $\alpha(\cdot), \delta(\cdot) \geq 0$   $\mathbb{F}$ -adapted processes,

$$z(t) \equiv z(t; c) = z e^{-\int_0^t \alpha(v) dv} + \int_0^t \delta(s) e^{-\int_s^t \alpha(v) dv} c(s) ds$$

# The Habit-Forming Maximization Problem

Value function: For given  $(x, z) \in \mathcal{D}$

$$V(x, z) \triangleq \sup_{(\pi, c) \in \mathcal{A}(x, z)} E \left[ \int_0^T u(t, c(t) - z(t; c)) dt \right]$$

Admissible controls:  $(\pi, c) \in \mathcal{A}(x, z)$  such that

- $c(\cdot) \geq 0$
- $X^{x, \pi, c}(t) \geq 0$ , for all  $t \in [0, T]$
- $c(t) - z(t; c) > 0$ ,  $\rightsquigarrow$  **“addiction”**

Optimal Policies by Detemple & Zapatero (1992):

- $\exists$  optimal pair  $(\pi_0, c_0)$
- For  $z_0(\cdot) \equiv z(\cdot, c_0)$  and  $I \triangleq (u')^{-1} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$

$$c_0(t) - z_0(t) = I(t, y_0 \Gamma(t)) > 0 :$$

optimal scalar:  $y_0 \in \mathbb{R}^+$

“adjusted” state-price density: for  $E_t[\cdot] \triangleq E[\cdot | \mathcal{F}(t)]$

$$\Gamma(t) \triangleq H(t) + \delta(t) \cdot E_t \left( \int_t^T e^{\int_t^s (\delta(v) - \alpha(v)) dv} H(s) ds \right)$$

state-price density:  $H(t) \triangleq e^{-\int_0^t r(s) ds} Z(t)$

density:  $Z(t) \triangleq e^{-\int_0^t \vartheta^*(s) dW(s) - \frac{1}{2} \int_0^t \|\vartheta(s)\|^2 ds}$

**Theorem 1.**  $V(\cdot, z)$  satisfies all the conditions of a utility function as defined previously, for any given  $z \geq 0$ .

**Optimal wealth:**  $X_0(\cdot) \equiv X^{x, \pi_0, c_0}(\cdot)$  is given by

$$X_0(t) = \frac{1}{H(t)} E_t \left[ \int_t^T H(s) c_0(s) ds \right],$$

and substituting the optimal  $c_0(\cdot)$  we can show that

$$X_0(t) - \mathcal{W}(t) z_0(t) = \frac{1}{H(t)} E_t \left[ \int_t^T \Gamma(s) I(s, y_0 \Gamma(s)) ds \right],$$

where

$$\mathcal{W}(t) \triangleq E_t \left[ \int_t^T e^{\int_t^s (\delta(v) - \alpha(v)) dv} \frac{H(s)}{H(t)} ds \right] \text{ and } \mathcal{W}(0) = w.$$

**Theorem 2.** The **effective state space** of  $(X_0(t), z_0(t))$  is identified as the **random wedge**

$$\mathcal{D}_t \triangleq \{(x', z') \in \mathbb{R}^+ \times [0, \infty); x' > \mathcal{W}(t) z'\},$$

$$\mathcal{D}_T \triangleq \{(0, z'); z' \in [0, \infty)\}, \text{ and } \mathcal{D}_0 = \mathcal{D}.$$

•  $\mathcal{W}(\cdot)$  is the “**marginal**” **cost of subsistence consumption per unit of standard of living at t.**

**Next Goal:** Dependence of the optimal pair  $(\pi_0, c_0)$  on the wealth  $X_0(\cdot)$  and standard of living  $z_0(\cdot)$ .

Assumption:  $\vartheta(\cdot)$  bounded away from 0 and  $\infty$

**Change of measure:**  $P^0(A) \triangleq E[Z(T) \mathbf{1}_A]$ ,  $A \in \mathcal{F}(T)$

**New  $d$ -dim BM:**  $W_0(t) \triangleq W(t) + \int_0^t \vartheta(s) ds$  under  $P^0$

For  $(t, y) \in [0, T] \times \mathbb{R}^+$  and  $t \leq s \leq T$ , consider

- $Z^t(s) \triangleq Z(s)/Z(t)$ ,
- $H^t(s) \triangleq \exp \left\{ - \int_t^s r(v) dv \right\} Z^t(s)$ ,
- $\Gamma^t(s) \triangleq H^t(s) + \delta(s) \cdot E_s \left( \int_s^T e^{\int_s^\theta (\delta(v) - \alpha(v)) dv} H^t(\theta) d\theta \right)$

Assumption:  $r(\cdot) - \delta(\cdot) + \alpha(\cdot)$  is deterministic

- $\Gamma^t(s) = H^t(s)\mu(s)$ , with

$$\mu(t) \triangleq 1 + \delta(t)w(t)$$

$$w(t) \triangleq \int_t^T e^{\int_t^s (-r(v) + \delta(v) - \alpha(v)) dv} ds = \mathcal{W}(t)$$

**Key Process:**  $Y^{(t,y)}(s) \triangleq y\Gamma^t(s)$ ,  $t \leq s \leq T$

Rewrite the relationship derived for  $X_0(\cdot)$  and  $z_0(\cdot)$  as

$$X_0(t) - w(t)z_0(t) = \mathfrak{X} \left( t, \frac{Y^{(0,y_0)}(t)}{\mu(t)} \right),$$

in terms of

**Random field:**  $\mathfrak{X} : [0, T] \times \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}^+$  given by

$$\mathfrak{X}(t, y) \triangleq E_t^0 \left[ \int_t^T e^{-\int_t^s r(v) dv} \mu(s) I(s, yY^{(t,1)}(s)) ds \right]$$

In the following slide, we recall a generalized **Itô-Kunita-Wentzell formula** for *random fields*.

**Generalized Itô-Kunita-Wentzell (GIKW):** Let the random field  $\mathbf{F}$  be of class  $C^{0,2}([0, T] \times \mathbb{R}^n)$  and satisfy

$$\mathbf{F}(t, \mathbf{x}) = \mathbf{F}(0, \mathbf{x}) + \int_0^t \mathbf{f}(s, \mathbf{x}) ds + \int_0^t \mathbf{g}^*(s, \mathbf{x}) dW(s),$$

where  $\mathbf{g} = (\mathbf{g}^{(1)}, \dots, \mathbf{g}^{(d)})$  are  $C^{0,2}$ ,  $\mathbb{F}$ -adapted random fields, and  $\mathbf{f}$  is a  $C^{0,1}$  random field. Furthermore, let  $\mathbf{X} = (\mathbf{X}^{(1)}, \dots, \mathbf{X}^{(n)})$  be a vector of continuous semimartingales with decompositions

$$\mathbf{X}^{(i)}(t) = \mathbf{X}^{(i)}(0) + \int_0^t \mathbf{b}^{(i)}(s) ds + \int_0^t (\mathbf{h}^{(i)}(s))^* dW(s),$$

where  $\mathbf{h}^{(i)} = (\mathbf{h}^{(i,1)}, \dots, \mathbf{h}^{(i,d)})$  is an  $\mathbb{F}$ -progressively measurable, almost surely square integrable vector process, and  $\mathbf{b}^{(i)}(\cdot)$  is an almost surely integrable process. Then  $\mathbf{F}(\cdot, \mathbf{X}(\cdot))$  is also a continuous semimartingale, with decomposition

$$\begin{aligned} \mathbf{F}(t, \mathbf{X}(t)) &= \mathbf{F}(0, \mathbf{X}(0)) + \sum_{i=1}^n \int_0^t \mathbf{F}_{\mathbf{x}_i}(s, \mathbf{X}(s)) d\mathbf{X}^{(i)}(t) \\ &\quad + \int_0^t \mathbf{f}(s, \mathbf{X}(s)) ds + \int_0^t \mathbf{g}^*(s, \mathbf{X}(s)) dW(s) \\ &\quad + \sum_{j=1}^d \sum_{i=1}^n \int_0^t \mathbf{g}_{\mathbf{x}_i}^{(j)}(s, \mathbf{X}(s)) \mathbf{h}^{(i,j)}(s) ds \\ &\quad + \frac{1}{2} \sum_{i,k=1}^n \int_0^t \mathbf{F}_{\mathbf{x}_i \mathbf{x}_k}(s, \mathbf{X}(s)) d\langle \mathbf{X}^{(i)}, \mathbf{X}^{(k)} \rangle(s). \end{aligned}$$

**Notation:** Consider the class of *pair* random fields

$$\begin{aligned} \mathbf{G}_{\mathbb{F}} &\triangleq C_{\mathbb{F}}([0, T]; \mathbb{L}^2(\Omega; C^3(\mathbb{R}^+))) \\ &\quad \times \mathbb{L}_{\mathbb{F}}^2(0, T; \mathbb{L}^2(\Omega; C^2(\mathbb{R}^+; \mathbb{R}^d))) \end{aligned}$$



## The Role of Stochastic PDE's

Under reasonable assumptions on the utility preferences  $(u, I, \dots)$ , and the study of parabolic BSPDE's by Ma and Yong (1997), we reach the following result.

### Proposition 3.

There exists a random field  $\Psi^{\mathfrak{X}}$  such that the pair  $(\mathfrak{X}, \Psi^{\mathfrak{X}}) \in \mathbf{G}_{\mathbb{F}}$  is the **unique  $\mathbb{F}$ -adapted solution** of the **linear** parabolic BSPDE

$$\begin{aligned}
 -d\mathfrak{X}(t, y) = & \left[ \frac{1}{2} \|\vartheta(t)\|^2 y^2 \mathfrak{X}_{yy}(t, y) \right. \\
 & + \left( \|\vartheta(t)\|^2 - r(t) \right) y \mathfrak{X}_y(t, y) - r(t) \mathfrak{X}(t, y) \\
 & \left. - \vartheta^*(t) y \Psi_y^{\mathfrak{X}}(t, y) + \mu(t) I(t, y \mu(t)) \right] dt \\
 & - \left( \Psi^{\mathfrak{X}}(t, y) \right)^* dW_0(t) \quad \text{on } [0, T) \times \mathbb{R}^+, \\
 \mathfrak{X}(T, y) = & 0 \quad \text{on } \mathbb{R}^+.
 \end{aligned}$$

### Remark 4.

Integrating over  $[t, T]$ , the above BSPDE yields the semimartingale decomposition of the process  $\mathfrak{X}(\cdot, y)$ , for all  $y \in \mathbb{R}^+$ .

For  $t \in [0, T)$

- $\mathfrak{X}(t, \cdot)$  is strictly decreasing,
- $\mathfrak{X}(t, 0^+) = \infty$ ,  $\mathfrak{X}(t, \infty) = 0$

**Inverse random field:**

of class  $C_{\mathbb{F}}([0, T); C^3(\mathbb{R}^+))$ .  $\mathfrak{Y}(t, \cdot) \triangleq \mathfrak{X}^{-1}(t, \cdot)$

## Feedback Formulae

Inverting  $\mathfrak{X}$  in the equation of  $X_0(\cdot)$  and  $z_0(\cdot)$ , we have

$$Y^{(0,y_0)}(t) = \mu(t)\mathfrak{J}(t),$$

where

$$\mathfrak{J}(t) \triangleq \mathfrak{Y}\left(t, X_0(t) - w(t)z_0(t)\right).$$

Then the optimal consumption process  $c_0(\cdot)$  is expressed by

$$c_0(t) = z_0(t) + I\left(t, \mu(t)\mathfrak{J}(t)\right)$$

**Theorem 5a.** The optimal consumption policy  $c_0(\cdot)$  admits the **stochastic feedback form** of

$$c_0(t) = \mathfrak{C}(t, X_0(t), z_0(t)), \quad 0 \leq t < T,$$

determined by the random field

$$\mathfrak{C}(t, x, z) \triangleq z + I\left(t, \mu(t)\mathfrak{Y}(t, x - w(t)z)\right), \quad (x, z) \in \mathcal{D}_t.$$

Employing the GIKW's rule to the equation of  $X_0(\cdot)$  and  $z_0(\cdot)$ , and using the semimartingale decomposition of Proposition 3, we obtain the integral equation

$$\begin{aligned} \frac{X_0(t)}{B(t)} + \int_0^t \frac{c_0(s)}{B(s)} ds \\ = x - \int_0^t \frac{1}{B(s)} \left[ \vartheta(s) \frac{\tilde{\mathfrak{J}}(s)}{\tilde{\mathfrak{J}}_x(s)} - \Psi^{\mathfrak{X}}(s, \tilde{\mathfrak{J}}(s)) \right]^* dW_0(s), \end{aligned}$$

where  $\tilde{\mathfrak{J}}_x(t) \triangleq \mathfrak{Y}_x(t, X_0(t) - w(t)z_0(t))$ .

A comparison of the later with the wealth equation implies that

$$X_0(t)\pi_0^*(t)\sigma(t) = - \left[ \vartheta(t) \frac{\tilde{\mathfrak{J}}(t)}{\tilde{\mathfrak{J}}_x(t)} - \Psi^{\mathfrak{X}}(t, \tilde{\mathfrak{J}}(t)) \right]^*.$$

**Theorem 5b.** The optimal portfolio strategy  $\pi_0(\cdot)$  admits the **stochastic feedback form** of

$$\pi_0(t) = \mathfrak{P}(t, X_0(t), z_0(t)), \quad 0 \leq t < T$$

determined by

$$\begin{aligned} \mathfrak{P}(t, x, z) \triangleq - \frac{1}{x} (\sigma^*(t))^{-1} \left[ \vartheta(t) \frac{\mathfrak{Y}(t, x - w(t)z)}{\mathfrak{Y}_x(t, x - w(t)z)} \right. \\ \left. - \Psi^{\mathfrak{X}}(t, \mathfrak{Y}(t, x - w(t)z)) \right], \quad (x, z) \in \mathcal{D}_t. \end{aligned}$$

# Dynamic Programming

Generalized time horizon:  $[t, T]$ .

Value random field:

$$\mathbf{V}(t, x, z) \triangleq \operatorname{ess\,sup}_{(\pi, c)} E_t \left[ \int_t^T u(s, c(s) - z(s)) ds \right],$$

Extension:  $\mathbf{V}(0, \cdot, \cdot) = V(\cdot, \cdot)$

By analogy with the previous analysis, we get

$$\mathbf{V}(t, x, z) = \mathfrak{G}(t, \mathfrak{Y}(t, x - w(t)z)), \quad (x, z) \in \mathcal{D}_t, \quad t \in [0, T),$$

where

$$\mathfrak{G}(t, y) \triangleq E_t \left[ \int_t^T u(s, I(s, yY^{(t,1)}(s))) ds \right]$$

Boundary conditions:

$$\begin{aligned} \mathbf{V}(T, x, z) &= 0, \\ \lim_{(x,z) \rightarrow (\chi, \zeta)} \mathbf{V}(t, x, z) &= \int_t^T u(s, 0^+) ds, \quad \forall (\chi, \zeta) \in \partial \mathcal{D}_t. \end{aligned}$$

Likewise with  $\mathfrak{X}$ ,

- $\exists$  random field  $\Phi^{\mathfrak{G}}$  such that
- $(\mathfrak{G}, \Phi^{\mathfrak{G}}) \in \mathbf{G}_{\mathbb{F}}$  is the  $\mathbb{F}$ -**adapted unique solution** of another **linear** parabolic BSPDE

# Stochastic Hamilton-Jacobi-Bellman(HJB) Equation

## Theorem 6.

The pair of random fields  $(\mathbf{V}, \Xi)$ , where

$$\begin{aligned} \Xi(t, x, z) \triangleq & \Phi^{\mathfrak{G}}\left(t, \mathfrak{Y}(t, x - w(t)z)\right) \\ & - \mathfrak{Y}(t, x - w(t)z) \Psi^{\mathfrak{X}}\left(t, \mathfrak{Y}(t, x - w(t)z)\right), \end{aligned}$$

belongs to  $\mathbf{G}_{\mathbb{F}}$  and is an  $\mathbb{F}$ -adapted **classical solution** of the **stochastic HJB dynamic programming PDE** (cf. Peng (1992))

$$\begin{aligned} -d\mathbf{V}(t, x, z) = & \operatorname{ess\,sup}_{\substack{0 \leq c < \infty \\ \pi \in \mathbb{R}^d}} \left\{ \frac{1}{2} \|\sigma^*(t)\pi\|^2 x^2 \mathbf{V}_{xx}(t, x, z) \right. \\ & + \left[ r(t)x - c + \pi^* \sigma(t) \vartheta(t)x \right] \mathbf{V}_x(t, x, z) \\ & + \left[ \delta(t)c - \alpha(t)z \right] \mathbf{V}_z(t, x, z) \\ & \left. + \pi^* \sigma(t)x \Xi_x(t, x, z) + u(t, c - z) \right\} dt \\ & - \Xi(t, x, z) dW(t), \quad t \in [0, T), \quad (x, z) \in \mathcal{D}_t \end{aligned}$$

with the boundary conditions of the previous slide.

## Remark 7.

The **optimal values** for the above maximization are provided by  $(\mathfrak{P}(t, x, z), \mathfrak{C}(t, x, z))$  .

Carrying out the maximization, the stochastic HJB takes the

Conventional form:

$$-d\mathbf{V}_t(t, x, z) = \mathcal{H}\left(\mathbf{V}_{xx}, \mathbf{V}_x, \mathbf{V}_z, \Xi_x, t, x, z\right) dt - \Xi(t, x, z) dW(t),$$

where

$$\begin{aligned} \mathcal{H}(A, p, q, \mathbf{B}, t, x, z) \triangleq & -\frac{1}{2A} \|\vartheta(t)p + \mathbf{B}\|^2 \\ & + \left[ r(t)x - z - I(t, p - \delta(t)q) \right] p \\ & + \left[ (\delta(t) - \alpha(t))z + \delta(t)I(t, p - \delta(t)q) \right] q \\ & + u\left(t, I(t, p - \delta(t)q)\right) \end{aligned}$$

for  $A < 0$ ,  $p > 0$ ,  $q < 0$  and  $\mathbf{B} \in \mathbb{R}$ .

Remark 8.

We have achieved a **concrete solution** of the **strongly nonlinear** stochastic HJB equation by solving instead the two **linear** equations of  $\mathfrak{X}$  and  $\mathfrak{G}$ .

**Uniqueness:** Necessary and sufficient condition for

Convex dual of  $\mathbf{V}$  :

$$\widetilde{\mathbf{V}}(t, y) \triangleq \operatorname{ess\,sup}_{(x,z) \in \mathcal{D}_t} \left\{ \mathbf{V}(t, x, z) - (x - w(t)z)y \right\}, \quad y \in \mathbb{R}^+.$$

**Identity:**  $\widetilde{\mathbf{V}}(t, y) = \mathfrak{G}(t, y) - y\mathfrak{X}(t, y),$

**Theorem 9.**

The pair of random fields  $(\widetilde{\mathbf{V}}, \Lambda)$ , where

$$\Lambda(t, y) \triangleq \Phi^{\mathfrak{G}}(t, y) - y\Psi^{\mathfrak{X}}(t, y),$$

belongs to  $\mathbf{G}_{\mathbb{F}}$  and is the **unique  $\mathbb{F}$ -adapted solution** of the **linear BSPDE**

$$\begin{aligned} -d\widetilde{\mathbf{V}}(t, y) = & \left[ \frac{1}{2} \|\vartheta(t)\|^2 y^2 \widetilde{\mathbf{V}}_{yy}(t, y) - r(t)y\widetilde{\mathbf{V}}_y(t, y) \right. \\ & \left. - \vartheta^*(t)y\Lambda_y(t, y) + \tilde{u}(t, y\mu(t)) \right] dt \\ & - \Lambda^*(t, y)dW(t) \quad \text{on } [0, T) \times \mathbb{R}^+, \\ \widetilde{\mathbf{V}}(T, y) = & 0 \quad \text{on } \mathbb{R}^+. \end{aligned}$$

**Remark 10.**

To compute  $\mathbf{V}$ :

- resolve the above equation for  $(\widetilde{\mathbf{V}}, \Lambda)$
- invert the dual transformation to recover  $\mathbf{V}$  as

$$\mathbf{V}(t, x, z) = \operatorname{ess\,inf}_{y \in \mathbb{R}} \left\{ \widetilde{\mathbf{V}}(t, y) + (x - w(t)z)y \right\}, \quad (x, z) \in \mathcal{D}_t.$$

## Deterministic Coefficients

Case of deterministic model coefficients:

- asset prices become Markov processes
- the **feedback formulae** for  $(\pi_0, c_0)$  reduce to the **deterministic functions**

$$C(t, x, z) \triangleq z + I\left(t, \mu(t)\mathcal{Y}(t, x - w(t)z)\right),$$

$$\Pi(t, x, z) \triangleq -(\sigma^*(t))^{-1}\vartheta(t) \cdot \frac{\mathcal{Y}\left(t, x - w(t)z\right)}{x\mathcal{Y}_x\left(t, x - w(t)z\right)},$$

where

$\mathcal{Y}(t, \cdot)$  is the **inverse** of the function

$$\mathcal{X}(t, y) \triangleq E^0 \left[ \int_t^T e^{-\int_t^s r(v)dv} \mu(s) I(s, yY^{(t,1)}(s)) ds \right].$$

### Corollary 11.

An investor

- needs only his *current* level of  $X_0(t)$  and  $z_0(t)$ ,
  - doesn't need entire history of the market up to  $t$ .
- Therefore  $(X_0(t), z_0(t))$  is a **sufficient statistic** for the optimization problem.



## An Example: Logarithmic Utility

Let  $u(t, x) = \log x$ .

We have  $I(t, y) = 1/y$  and  $\tilde{u}(t, y) = -\log y - 1$ .

### Case 1: Deterministic Coefficients.

Unique solution of the Cauchy problem for  $\tilde{V}$ :

$$\tilde{v}(t, y) \triangleq -\nu(t) \log(y\mu(t)) - m(t), \quad \text{where}$$

$$\nu(t) = T - t$$

$$m(t) = \int_t^T \left[ 1 - (T - s) \left( \frac{1}{2} \|\vartheta(s)\|^2 + r(s) - \frac{\mu'(s)}{\mu(s)} \right) \right] ds.$$

Therefore inverting the dual transformation:

$$\mathcal{X}(t, y) = \frac{\nu(t)}{y}, \quad \mathcal{Y}(t, x) = \frac{\nu(t)}{x},$$

$$V(t, x) = \nu(t) \log \left( \frac{x - w(t)z}{\nu(t)\mu(t)} \right) + \nu(t) - m(t),$$

and the feedback formulae are

$$C(t, x) = z + \frac{x - w(t)z}{\nu(t)\mu(t)},$$

$$\Pi(t, x) = (\sigma^*(t))^{-1} \vartheta(t) \frac{x - w(t)z}{x}.$$

## Case 2: Stochastic Coefficients.

By analogy with the deterministic case, we introduce in this case the  $\mathbb{F}$ -adapted random field

$$\tilde{\mathfrak{v}}(t, y) \triangleq -\nu(t) \log(y\mu(t)) - \mathfrak{m}(t)$$

with

$$\nu(t) = T - t, \quad \mathfrak{m}(t) = E_t \left[ \int_t^T m(s) ds \right].$$

Moreover, the *completeness* of the market stipulates the existence of an  $\mathbb{R}^d$ -valued,  $\mathbb{F}$ -progressively measurable, square-integrable process  $\ell(\cdot)$ , such that the Brownian martingale (cf. Karatzas & Shreve (1991))

$$\mathfrak{M}(t) = E_t \left[ \int_0^T m(s) ds \right]$$

has the representation  $\mathfrak{M}(t) = \mathfrak{M}(0) + \int_0^t \ell^*(s) dW(s)$ .

It is verified directly that the pair  $(\tilde{\mathfrak{v}}, \ell)$  solves the BSPDE for  $(\tilde{\mathbf{V}}, \Lambda)$ . Consequently,

$$\mathfrak{X}(t, y) = \frac{\nu(t)}{y}, \quad \mathfrak{Y}(t, x) = \frac{\nu(t)}{x}$$

and

$$\mathfrak{V}(t, x) = \nu(t) \log \left( \frac{x - w(t)z}{\nu(t)\mu(t)} \right) + \nu(t) - \mathfrak{m}(t).$$

For this particular choice of utility preference,  $\mathfrak{X}$  (and so  $\mathfrak{Y}$ ) is deterministic, and the feedback formulas are the same as those of the previous case.

## Open Problems

### (1) Incomplete Markets:

- Number of stocks smaller than the dimension of  $W(\cdot)$
- Compute the **optimal policies**

### (2) Exponential utility function $u : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$

- $c(t) > z(t; c)$  is removed (non-addictive habits)
- Optimal policies exist by Detemple & Karatzas (2003)
- Develop the **Dynamic Programming Theory**

### (3) Generalized utility function $u(t, c(t), z(t))$

- Compute the **optimal policies**
- Develop the **Dynamic Programming Theory**

## Basic References

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