Implementation and Calibration of the Extended Affine Heston Model for Basket Options and Volatility Derivatives[*]

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Introduction

As observed in the market, the empirical distributions of equity log-returns are skewed and heavy-tailed. In addition, equity prices exhibit jumps, stochastic volatility clustering, and autocorrelation in the squared returns. All these properties of the stock dynamics considered in the risk neutral measure result in the “smiles” and “smirks” of the corresponding implied volatility surfaces.

Presented paper elaborates a special case of the Multi-Factor Affine Extended Heston model with displaced stochastic volatility and stochastic interest rates correlated with the underlyings developed in Levin (2008, 2009). This diffusion model belongs to a broad affine jump-diffusion class of models within a general framework of Duffie, Pan and Singleton (2000). A system of SDE’s considered in the presentation has one common stochastic variance described by the CIR process. Multiple stocks have different average volatilities and correlations with this stochastic variance providing different levels and “smirks” of the individual implied volatility surfaces. The Quasi-Elliptical Heston model is extended in the affine way by different Gaussian displacements in the stock stochastic variance. They allow for different levels of “smiles” in the implied volatilities and for correlations between stock log-returns and stochastic Hull-White interest rates and equity continuous dividend yields.

Similar “quasi-elliptical” construction for multi-factor models have been considered in Levin and Tchernitser (2003), Leoni and Schoutens (2008) for jump-stochastic volatility, and in many articles on stochastic time change models (e.g., Carr and Wu (2004)). A time-dependent mean reversion level for the Heston stochastic variance is considered for better fit into the term structure of the ATM implied volatilities and variance swap prices. Time-dependent parameters in Heston model were considered, for example, in Mikhailov and Nogel (2003) and Zhu and Zhang (2007) (for VIX). This paper assumes only mean reversion level is time-dependent (piece-wise constant) and other parameters are constant in order to preserve analytical tractability for the European option prices and multivariate characteristic function.

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General Affine Diffusion models

A diffusion model considered in this presentation belongs to a broad affine jump-diffusion class of models within a general framework of Duffie, Pan and Singleton (2000).

Suppose the risk neutral dynamics of the state variables X(t) under the equivalent martingale measure Q is defined by the following Markovian process

$$dX(t) = \mu(t, X(t))dt + \sigma(X(t))dW$$

Here the drift and covariance matrix are affine in state variables:

$$\mu(t,x) = K_0(t) + K_1 x\,,\ K_0(t) \in R^N\,,\ K_1 \in R^{NxN}\,;$$

$$\sigma(x)\sigma(x)^T = H_0 + H_1 x\,,\ H_0(t) \in R^{NxN}\,,\ H_1 \in R^{NxNxN}$$

Vector $W(t) \in R^N$ is a standard Q-Brownian motion with independent components. Coefficient $K_0(t)$ is time-dependent (including equations for the stochastic variances) to provide consistency with the interest rate dynamics and allow for the exact fit into initial equity forward price curves and variance swap price term structures. Coefficients K_1, H_0, and H_1 are constant to ensure analytical tractability of the model.

According to Dai and Singleton (2000), it is sufficient for the affinity of the diffusions with affine drifts that the volatility matrix $\sigma(X)$ is of the following canonical form:

$$\sigma(X) = \Sigma \begin{pmatrix} \sqrt{v_1(X)} & 0 & \cdots & 0 \\ 0 & \sqrt{v_2(X)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{v_N(X)} \end{pmatrix}$$

Here Σ is a constant matrix in R^{NxN} and $v_j(X)$ are affine functions with constant coefficients,

$v_j(X) = \chi_j + \lambda_j \cdot X$, $\chi_j \in R$, $\lambda_j \in R^N$.

Cheridito, Filipovic and Kimmel (2007) and Collin-Dufresne, Goldstein and Jones (2008) suggest more general canonical form with the number of Wiener processes possibly greater than the number of state variables, constant matrix $\Sigma \in R^{NxM}$ $(N \le M)$, and $k \ge 0$ Gaussian and $M - k$ square root components:

$$\sigma(X) = \Sigma \begin{pmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \sqrt{X_{k+1}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & 0 & 0 & \cdots & \sqrt{X_M} \end{pmatrix}$$

We use the latter canonical form with square root processes only for the stochastic variances.

<u>Example of non-affine multi-factor extension of the Heston model with Hull-White interest rate:</u>

$$dr = \kappa_r(\theta_r - r)dt + \sigma_r dW_r$$

$$dV = \kappa_V(\theta_V - V)dt + \eta\sqrt{V}dW_V$$

$$dX = (r - 0.5\,V)dt + \sqrt{V}dW_S$$

Main results for Affine Diffusion models

The “extended transform” from the Duffie, Pan and Singleton (2000) paper can be presented in a more natural for the Heston model form of a “discounted characteristic function”

$$\varphi(u, X_t, T, t) = E\left(\exp\left(-\delta \int_t^T r(X_S)\, ds \right) e^{iu X_T} \middle| \mathcal{F}_t \right)$$

that combines together a definition of the “discounted characteristic function” and regular multivariate characteristic function using a flag $\delta = 1$ and $\delta = 0$ correspondingly.

Under the same technical regularity conditions as in Duffie, Pan and Singleton (2000):

$$\varphi(u, X_t, T, t) = e^{A(\tau,u) + B(\tau,u)\cdot X_t}$$

Here $\tau = T - t$, and for a fixed $u \in C^N$ the vector-function $B(\tau) = B(\tau, u)$ and the function $A(\tau) = A(\tau, u)$ satisfy the following complex-valued ODEs:

$$\dot{B}(\tau) = -\delta\rho_1 + K_1^T B(\tau) + \frac{1}{2} B(\tau)^T H_1 B(\tau)$$

$$B(0) = iu$$

$$\dot{A}(\tau) = -\delta\rho_0(T - \tau) + K_0(T - \tau)\, B(\tau) + \frac{1}{2} B(\tau)^T H_0 B(\tau)$$

$$A(0) = 0$$

Where ρ_0 and ρ_1 describe the affine function of the domestic short interest rate $r(t) = \rho_0(t) + \rho_1 X_t$

Pricing of European options

Let $G_{a,b}(y)$ denote the price of a security that pays $e^{a\cdot X_T}$ at time T in the event that $b\cdot X_T \le y$ for any real number y and any a and b in R^n.

$G_{a,b}(y)$ has the following representation via the discounted characteristic function $\varphi(u, X_t, T, t)$:

$$G_{a,b}(y; x, T, t) = \frac{\varphi(-ia, x, T, t)}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{ivy}\varphi(-ia - vb, x, T, t)}{iv}\right]dv$$

Then a plain-vanilla European call option $(S_T - K)^+ = (e^{bX_T} - K)^+$ with expiration time T and strike K has a price at time t defined by the following formula:

$$C = G_{b,-b}(-\ln K; X, T, t) - K\, G_{0,-b}(-\ln K; X, T, t)$$

As the call option is in the money when $-bX_T \le -\ln K$ and in that case pays $e^{bX_T} - Ke^{0\cdot X_T}$ where b is a vector with j-th element equal to one and all other elements equal to zero.

The choice of a particular model in our paper is based on the following requirements we want to satisfy:

1. The model should be affine, i.e. with multiplication of Σ in the diffusion part by $\sqrt{V_i}$ by columns. An affine model results in closed-form European option prices and effective parameter calibration.
2. We restrict the model to one-factor stochastic variance for each stock for simple and stable calibration (otherwise the pairwise correlations between different stochastic variances and stochastic variances and stock log-returns need to be calibrated as well).
3. We require stochastic interest rates and dividend yields correlated with the equity prices.
4. We use Hull-White model for the interest rates and continuous dividend yields. Stochastic dividend yields ensure more realistic dynamics for the equity forward price curves.
5. We need to capture different "smirks" and "smiles" of the implied volatility surfaces.
6. The model should allow for accurate fit into the ATM implied volatility and variance swap price term structures.

There are two ways to satisfy conditions 1-2:

- One can take different independent Heston stochastic variances for different stocks. Then, to have one variance in each row (see point 2 above) the correlations between stock prices should be zero, which is unrealistic (Bergomi (2008) considered a two-factor stochastic variance with many more parameters for calibration).
- One can select <u>one</u> common stochastic variance corresponding to general market activity and preserve the correlations between stock prices.

We consider the latter approach called "quasi-elliptical model". Finally, we utilize Gaussian displacements in the SV to correlate stock prices with Gaussian interest rates and dividend yields.

Multi-Factor Affine Extended Heston Model with Displaced Stochastic Variance and Stochastic Interest Rates and Dividend Yields

A globally affine system of stochastic differential equations for one common "normalized" Heston stochastic variance $V(t),\, d \geq 1,$ stock log-prices $X_j(t) = \ln S_j(t), \quad j = 1,\ldots,d$, interest rate $r(t)$, dividend yields $q_j(t)$, $j = 1,\ldots,d$, and integrated stochastic variance $I_V(t) = \int_0^t V(s)ds$ (for variance swaps) is defined as follows:

$$dr = \kappa_r(\theta_r(t) - r)dt + \sigma_r \sum_{l=1}^{2d+1} a_l^r dW_l^{rqS}$$

$$dq_j = \kappa_{q_j}(\theta_j^q(t) - q_j)dt + \sigma_j^q \sum_{l=1}^{2d+1} a_{jl}^q dW_l^{rqS}, \quad j = 1,\ldots,d$$

$$dI_V = Vdt$$

$$dV = \kappa(\theta(t) - V)dt + \eta\sqrt{V}\, dW_0^H$$

$$dX_j = \left(r(t) - q_j(t) - \frac{\sigma_j^2}{2}\left(\theta(t)\sum_{l=1}^{2d+1} (a_{jl}^S)^2 + \tilde{\omega}_j^2\theta(t) + (1-\tilde{\omega}_j^2)V \right)\right)dt$$

$$+\sigma_j\left(\sqrt{\theta(t)}\sum_{l=1}^{2d+1} a_{jl}^S dW_l^{rqS} + \omega_j\sqrt{\theta(t)}\sum_{l=1}^{d} a_{jl}\, dW_l^G + \sqrt{V}\rho_{j0}^V dW_0^H + \sqrt{(1-\omega_j^2)V}\sum_{l=1}^{d} a_{jl} dW_l^H \right)$$

Here $V(0)=V_0$, $dX_j(0)=\ln S_j(0), \quad j=1,...,d$, where a_l^r, a_{jl}^q and a_{jl}^S define historical correlations between r , q_j and X_j, a_{jl} define correlations between X_i and X_j, and ρ_{j0}^V are correlations between $V(t)$ and X_j.

To summarize, matrix A is from the decomposition of the constant correlation matrix $R=AA^T$ with the pair-wise historical correlations ρ_{lj}, $l, j=1,...,d$, for the basket constituents, interest rate and dividend yields as well as the calibrated risk neutral correlations ρ_{0j}, $j=1,...,d$, between the stochastic variance $V(t)$ and equity prices.

The stochastic variance $V(t)$ is normalized to 1 on average and represents a "common stochastic activity" of the market. Function $\theta(t)=\left(\theta_1(0,t_1), \theta_2(t_1,t_2),...,\theta_{m-1}(t_{m-2},t_{m-1}), \theta_m \equiv 1(t_{m-1},\infty)\right)$ is time-dependent (piece-wise constant) mean reversion level for the stochastic variance (also used in the Gaussian displacements for consistency with the limiting Black-Scholes case), $\sigma_j>0, \quad j=1,...,d$, are the stock average total volatilities, κ and η are constant mean reversion speed and volatility for the stochastic variance, $r(t)$ and $q_j(t)$ are stochastic risk free rate and equity dividend yields, $W_l^{rqS}, \quad l=1,...,2d+1$, $W_l^G, \quad l=1,...,d$, and $W_l^H, \quad l=0,...,d$, are independent standard Wiener processes for the Gaussian and Heston components, $0 \le \omega_j \le 1$ are weights for the displacements, and

$$\tilde{\omega}_j^2=\left[1-\rho_{j,0}^2-\sum_{l=1}^{2d+1}(a_{jl}^S)^2\right]w_j^2 .$$

Multi-Factor Affine Extended Heston Model with Displaced Stochastic Variance

The first special case of the general model is the case with deterministic interest rates and dividend yields. We will call this case the “Extended Heston Model with Displaced SV”. A globally affine system of SDEs for one common “normalized” Heston stochastic variance $V(t)$ and $d \geq 1$ stock log-prices $X_j(t) = \ln S_j(t), \quad j = 1,\ldots,d$, is as follows:

$$dI_v = V dt$$

$$dV = \kappa(\theta(t) - V)dt + \eta\sqrt{V}\, dW_0$$

$$dX_j = \left(r(t) - q_j(t) - \frac{\sigma_j^2}{2}\left(\tilde{\omega}_j^2\theta(t) + (1-\tilde{\omega}_j^2)V\right)\right) dt$$

$$+\sigma_j\left(\omega_j\sqrt{\theta(t)}\sum_{l=1}^{d} a_{jl}\, dW_l^G + \sqrt{V}\rho_{j0}^V dW_0^H + \sqrt{(1-\omega_j^2)V}\sum_{l=1}^{d} a_{jl} dW_l^H \right)$$

$$V(0) = V_0, \qquad dX_j(0) = \ln S_j(0), \quad j = 1,\ldots,d$$

Multi-Factor Affine Extended Quasi-Elliptical Heston Model

The second special case analyzed in this paper is a so-called Quasi-Elliptical Multi-Factor model with Displaced Stochastic Variance for zero Gaussian displacements. We will call this case the "Extended Quasi-Elliptical Heston Model". The name comes from the fact that a multivariate distribution of stock returns is "quasi-elliptical" for zero correlations ρ_{0j} between $V(t)$ and $X_j(t)$.

$$dV = \kappa(\theta(t) - V)dt + \eta\sqrt{V}\,dW_0$$

$$dX_j = \left(r(t) - q_j(t) - \frac{\sigma_j^2}{2} V \right) dt + \sigma_j \sqrt{V} \left(\rho_{j0}^V dW_0^H + \sum_{l=1}^{d} a_{jl} dW_l^H \right)$$

$$V(0) = V_0, \qquad dX_j(0) = \ln S_j(0), \quad j = 1,\ldots,d$$

Price of a European option in the *Extended Heston Model with Displaced SV*

A closed-form pricing formula for European call option on stock $S_j(t)$ with strike K_j, maturity T and payoff $(S_j(T)-K_j)^+$ is as follows:

$$Call_j(t) = S_j(t)\, e^{-q_j(T)(T-t)} P_j^1 - K_j\, e^{-r(T)(T-t)} P_j^2$$

Here $P_j^n = P_j^n(S_j(t), K_j, T, t, V(t))$, $n=1,2$, are two Fourier transforms for $b_n = 2-n$, $\tau = T-t$:

$$P_j^n = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\frac{\exp\left\{i\,x\left[\ln(S_j(t)/K_j) + (r(T)-q_j(T))\,\tau\right] + A^{GH}(\tau, x-ib_n) + B_0^H(\tau, x-ib_n)\,\sigma_j^2\,V(t)\right\}}{i\,x}dx$$

$$A^{GH}(\tau,u) = A^G(\tau,u) + A^H(\tau,u), \qquad A^G(\tau,u) = -\frac{\sigma_j^2}{2}\tilde{\omega}_j^2\, u_j(i+u_j)\left(\sum_{l=1}^{M+1}\theta_{M-l+1}(\tau_l - \tau_{l-1})\right)$$

$$A^H(\tau,u) = \frac{\kappa}{\eta^2}\cdot\sum_{l=1}^{M}\theta_{M-l+1}\left[(\varsigma(u)-d(u))(\tau_l-\tau_{l-1}) - 2\ln\frac{G(u)\,e^{-d(u)\tau_l}-1}{G(u)\,e^{-d(u)\tau_{l-1}}-1}\right]$$

$$B_0^H(\tau,u) = \frac{\varsigma(u)-d(u)}{2\gamma}\cdot\frac{e^{-d(u)\tau}-1}{G(u)\,e^{-d(u)\tau}-1}, \qquad G(u) = \frac{\varsigma(u)-d(u)}{\varsigma(u)+d(u)}$$

$$d(u) = \sqrt{\varsigma(u)^2 - 4\gamma\,\xi(u)}, \quad \varsigma(u) = \kappa - i\eta\sigma_j\,\rho_j\,u, \quad \xi(u) = \frac{1}{2}i\,u(1-\tilde{\omega}_j^2)(i\,u-1),$$

$$\gamma = \frac{1}{2}\eta^2\sigma_j^2, \quad \tau_l = T - t_{M-l}, \quad l = 1,\ldots,M-1, \quad \tau_0 = 0, \quad \tau_M = \tau = T-t \ \left(T\in(t_{M-1,}t_M)\right)$$

Price of a variance swap in the *Extended Heston Model with Displaced SV*

The variance swap is a forward contract on the realized annualized variance:

$$Variance\ Swap\ at\ T = N \times A \times \left\{ \frac{1}{N} \sum_{i=1}^{N} \left\{ \log\left(\frac{S_j(t_i)}{S_j(t_{i-1})} \right) \right\}^2 - \left\{ \frac{1}{N} \log\left(\frac{S_j(t_N)}{S_j(t_0)} \right) \right\}^2 \right\} - N \times K_j^{\mathrm{var}}$$

Here N is the notional amount of the swap, A is the annualization factor and K_j^{var} is the strike price. The drift term in the above payoff may or may not appear. The price of the variance swap in continuous time is defined as:

$$K^{\mathrm{var}} = \frac{1}{T-t} E\left\{ \int_t^T V(s)ds \right\}$$

The corresponding variance swap price formula K_j^{var} for the individual stock $S_j(t)$ extends the standard Heston model price formula (see, for example, Gatheral (2006)):

$$K_j^{\mathrm{var}} = \sigma_j^2 \left\{ \tilde{\omega}_j^2 \left[\sum_{l=1}^{M} \theta_{M-l+1} \left(\frac{\tau_l - \tau_{l-1}}{\tau} \right) \right] + (1-\tilde{\omega}_j^2) \left[\frac{1-e^{-\kappa\tau}}{\tau\kappa} V(t) + \sum_{l=1}^{M} \theta_{M-l+1} \left(\frac{\tau_l - \tau_{l-1}}{\tau} + \frac{e^{-\kappa\tau_l} - e^{-\kappa\tau_{l-1}}}{\tau\kappa} \right) \right] \right\}$$

Geometric Average Basket Option Price in the *Extended Quasi-Elliptical Heston Model*

The affine model allows for a closed-form pricing formula for a Geometric Average Basket (GAB) European option with the payoff of the form:

$$(\prod_j S_j^{\beta_j} - K)^+ = (\exp(\sum_j \beta_j X_j) - K)^+$$

The solution can be presented in the following form:

$$Call_{GAB}(t) = \prod_j S_j^{\beta_j} \cdot \exp(-\sum_{j=1}^N \beta_j Q_j \tau) \cdot P^1 - K e^{-r(T)(T-t)} P^2$$

Here $P^n = P^n(S_j(t), K, T, t, V(t))$, $n = 1,2$, are two Fourier transforms for $b_n = 2-n$, $\tau = T-t$:

$$P^n = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\frac{\exp\left\{ i\,x\left[\ln\,(\prod_j S_j^{\beta_j}/K) + (r(\tau) - \sum_{j=1}^N \beta_j Q_j(\tau))\,\tau\right] + A^H(\tau, \beta(x - ib_n)) + B_0^H(\tau, \beta(x - ib_n))\,V(t)\right\}}{i\,x} dx$$

$$A^H(\tau,u) = \frac{\kappa}{\eta^2} \cdot \sum_{l=1}^M \theta_{M-l+1}\left[(\varsigma(u) - d(u))(\tau_l - \tau_{l-1}) - 2\ln\frac{G(u)e^{-d(u)\tau_l} - 1}{G(u)e^{-d(u)\tau_{l-1}} - 1}\right],$$

$$B_0^H(\tau,u) = \frac{\varsigma(u) - d(u)}{2\gamma} \cdot \frac{e^{-d(u)\tau} - 1}{G(u)e^{-d(u)\tau} - 1}, \qquad G(u) = \frac{\varsigma(u) - d(u)}{\varsigma(u) + d(u)},$$

$$d(u) = \sqrt{\varsigma(u)^2 - 4\gamma\xi(u)}, \quad \varsigma(u) = \kappa - i\eta\sum_{j=1}^N \sigma_j \rho_j u_j, \quad \gamma = \frac{1}{2}\eta^2\sigma_j^2, \quad \xi(u) = -\frac{1}{2}\sum_{j=1}^N \sigma_j^2 u_j - \frac{1}{2}\sum_{l=1}^N \sum_{k=1}^N \sigma_l \sigma_k \rho_{lk} u_l u_k,$$

$$\tau_l = T - t_{M-l}, \; l = 1,\dots,M-1, \; \tau_0 = 0, \; \tau_M = \tau = T - t, \; T \in (t_{M-1}, t_M)$$

Parameter Calibration

For the model calibration, we consider one set of parameters, $V_0, \kappa, \eta, \theta(t)$ for the normalized common stochastic variance and different parameters σ_j, ρ_{0j} , $\tilde{\omega}_j$ for each basket component $S_j(t)$.
The calibration is achieved by solving an optimization problem of the weighted least squares fit into the market implied volatilities and, where available, the variance swap prices (e.g., VIX Term Structure):

$$\min_x \|F(x)\|_2^2 = \min_x \sum_j \sum_{k,z} \left(wIV_{jk} \left[\frac{ivol_{jk}^H - ivol_{jk}^M}{ivol_{jk}^M} \right]^2 + wVS_{jz} \left[\frac{VarSwap_{jz}^H - VarSwap_{jz}^M}{VarSwap_{jz}^M} \right]^2 \right)$$

Tikhonov regularization was implemented to improve stability of calibration.

Test 1 (Fig. 1). Joint fit into the S&P 500 implied volatilities and VIX Term Structure significantly improves the variance swap term structure approximation without affecting the quality of the implied volatility approximation. The S&P 500 &VIX joint calibration with time dependent $\theta(t)$ decreased RMSE for VIX Term Structure by 50% over constant θ. The calibrated Heston parameters and RMSE are presented below ($\theta(t)$ is on **Fig. 1)**:

$V_0 = 0.044, \kappa = 3.786, \eta = 1.448, \rho = -0.721, \sigma = 0.340$, RMSE=0.015, Relative RMSE=0.049.

Test 2 (Fig. 2a-2c,Table 1) compares the calibration of Modified Quasi-Elliptical Heston model and Affine Extended Heston Model with Displaced Stochastic Variance for a basket of stocks (HD, MON,& MSFT). The use of displacements $\tilde{\omega}_j$ decreased the objective functional by 17% and improved the stability of $\theta(t)$.

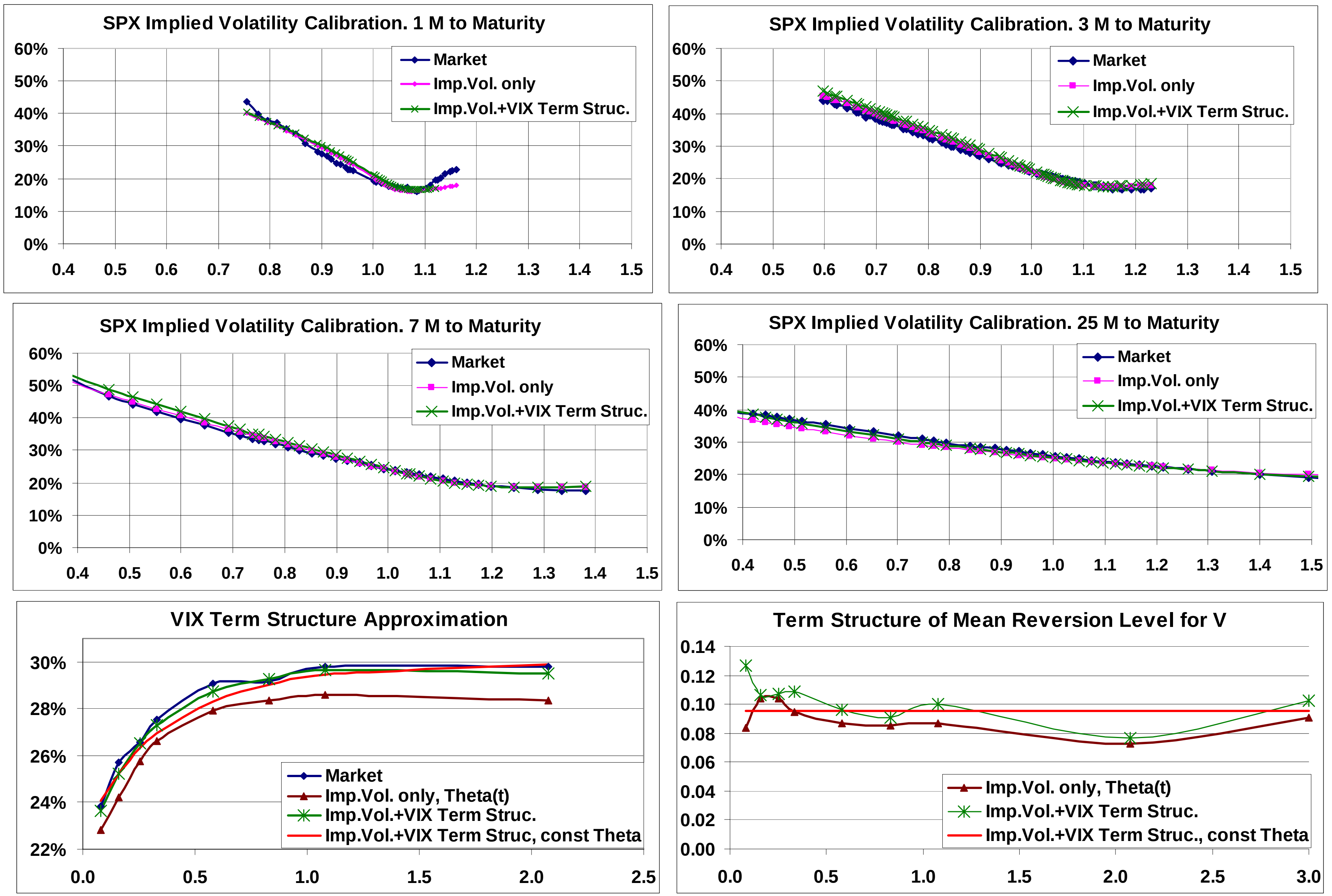

Fig. 1. Calibration to S&P 500 Implied Volatilities with and without Fitting into VIX Term Structure

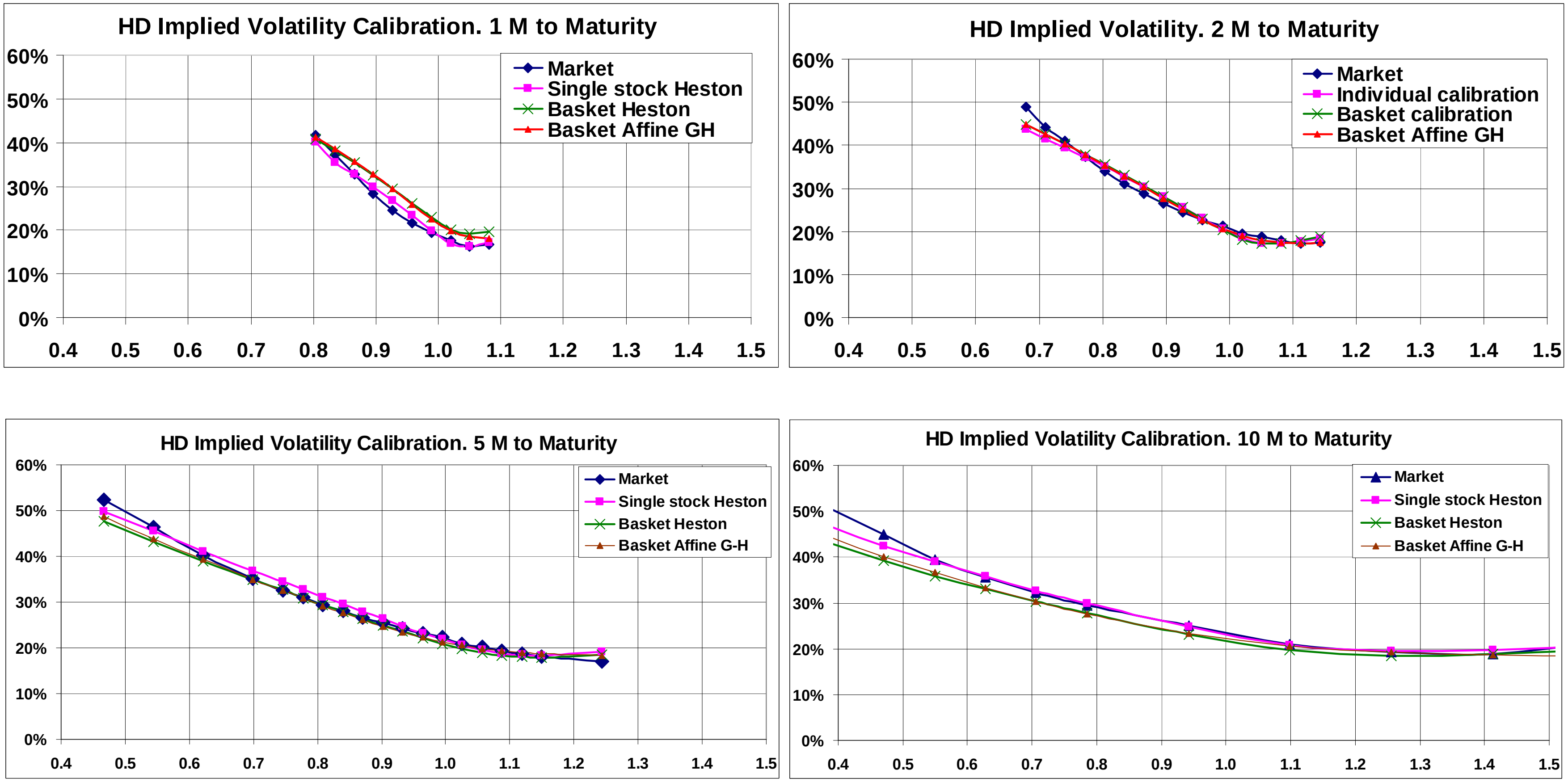

Figure 2a. Basket Calibration Results for Affine Extended Heston Model for Home Depot

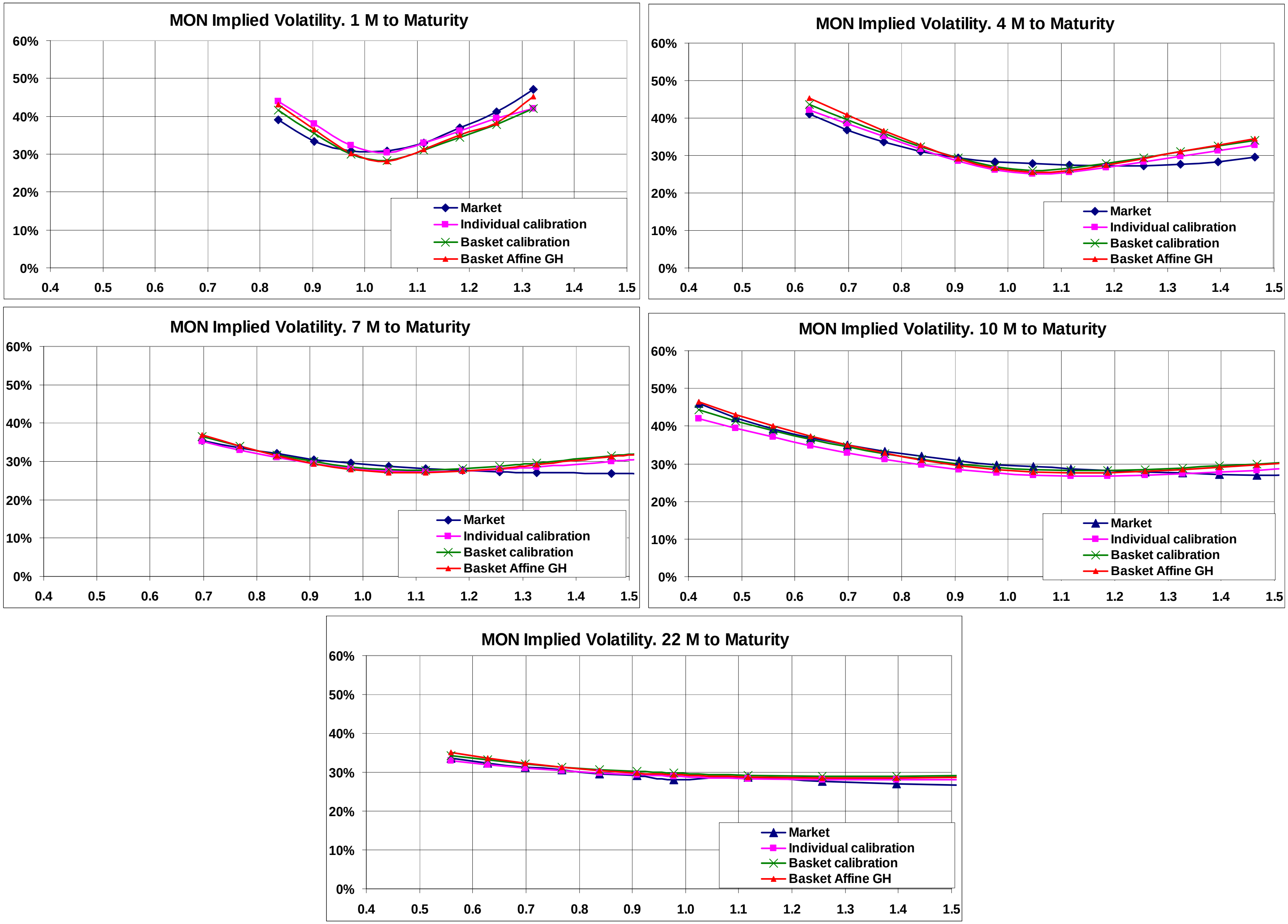

Figure 2b. Basket Calibration Results for Affine Extended Heston Model for Monsanto

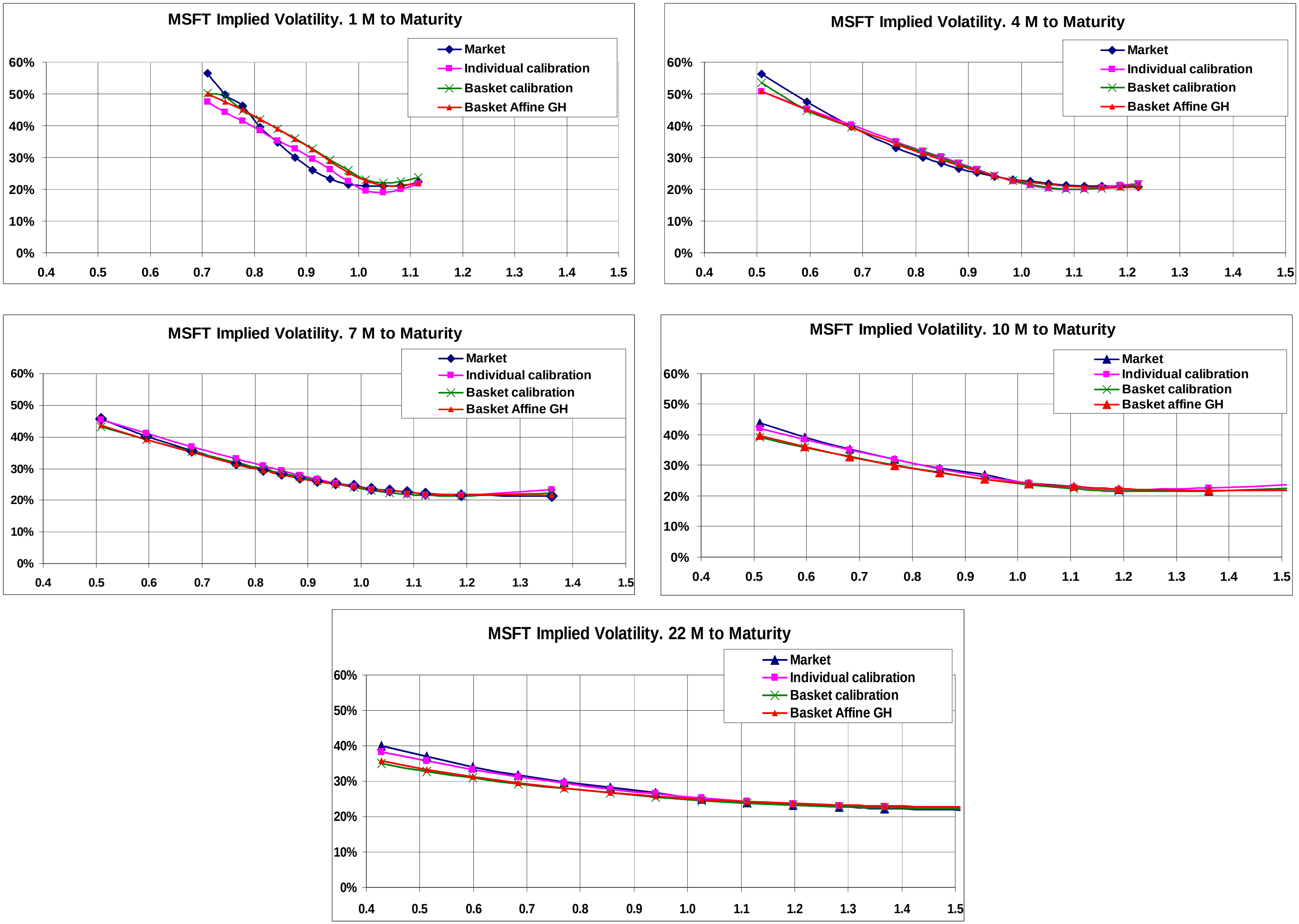

Figure 2c. Basket Calibration Results for Affine Extended Heston Model for Microsoft

Calibration Results for the USD Basket of HD, MON and MSFT		
Calibrated parameters	Modified Quasi-Elliptical Heston model with time dependent Theta(t)	**Affine Extended Heston model** with time dependent Theta(t) and displacements
V0	0.966	1.335
Kappa	10.163	7.639
Eta	9.989	9.975
Rho HD	-0.573	-0.621
Rho MON	-0.223	-0.251
Rho MSFT	-0.494	-0.534
Sigma HD	0.245	0.253
Sigma MON	0.315	0.320
Sigma MSFT	0.270	0.276
Displacement for HD	0.000	0.400
Displacement for MON	0.000	0.107
Displacement for MSFT	0.000	0.401
Theta 1 M	1.324	0.533
Theta 2 M	0.405	0.740
Theta 4 M	1.178	1.010
Theta 5 M	1.245	1.272
Theta 3 Y	1.000	1.000
Final min functional	0.818	0.678
RMSE	0.025	0.024
Relative RMSE	0.081	0.076

Table 1. Comparison of calibration results for USD basket with and without Gaussian displacements

Basket Pricing and Calibration Results

Test 3 demonstrates the multi-factor model calibration and pricing results for an arithmetic basket option $(\sum_j \beta_j X_j - K)^+$ of ETF's (XFN, XEG, XMA, and XIT) representing four major sub-indices (98.5%) of the Toronto Stock Exchange Index. The index itself is represented by the ETF with the ticker XIU. The quality of calibration was tested by comparison of the market prices for the XIU European call options for various maturities (considered as options on the basket) with the simulated basket option prices (using fixed historical equity correlations and model calibrated parameters). The Monte Carlo simulation was based on the methods from Andersen (2008). On average, the absolute difference in the theoretical and market basket option prices was 5.2%. Then, the historical equity correlations were adjusted to better fit into the index option prices. The obtained "implied" equity correlations were higher than the historical.

Basket Component Name	Market price	Basket weight
XIU	15.42	0.00%
XFN	18.07	33.72%
XEG	16.40	31.17%
XMA	15.97	22.32%
XIT	5.77	11.30%

Table 2. XIU basket composition and weights

Historical correlations	XFN	XEG	XMA	XIT
XFN	1.000	0.642	0.262	0.592
XEG	0.642	1.000	0.693	0.406
XMA	0.262	0.693	1.000	0.224
XIT	0.592	0.406	0.224	1.000
Implied correlations	**XFN**	**XEG**	**XMA**	**XIT**
XFN	1.000	0.756	0.3971	0.6747
XEG	0.756	1.000	0.7943	0.5656
XMA	0.397	0.794	1.0000	0.3679
XIT	0.675	0.566	0.368	1.000

Table 3. Implied versus historical equity correlations for the basket of ETF's

Historical correlations, constant theta	Historical correlations with term structure of theta	Implied correlations with constant theta	Implied correlations with term structure of theta
5.22%	2.10%	1.65%	1.30%

Table 4. Abs. average error for the XIU Index European option price vs. Basket option price

Test 4 compares closed form price with Monte Carlo simulated price for Geometric Average Basket option.

The test focus is to verify the analytical expression for the Geometric Average Basket option in the *Extended Quasi-Elliptical Heston* and use the obtained analytical solution as a control variate in pricing of the Arithmetic Average Basket option. The approach is to test homogeneous basket first with the same weights and initial stock prices, but different correlations. After that, the obtained analytical price for the Geometric Average Basket option is used as a control variate for the homogeneous Arithmetic Average Basket option.

The optimal coefficient b^* that minimizes the variance of the $Y_1,\ldots,Y_n$ outputs from n replications of a simulation given another output $X_1,\ldots,X_n$ with the known expectation $E[X]$ is as follows (Glasserman, 2004)

$$b^* = \frac{\sigma_Y}{\sigma_X}\rho_{XY} = \frac{Cov[X,Y]}{Var[X]}$$

The ratio of the variance of the optimally controlled estimator to that of the uncontrolled estimator is

$$\frac{Var[\bar{Y} - b^*(\bar{X} - E[X])]}{Var[\bar{Y}]} = 1 - \rho_{XY}^2$$

Where $\bar{X}$ and $\bar{Y}$ are sample means.

The test results demonstrate significant improvement in the accuracy, achieving the average variance ratio of 0.075 (for $b^* = 1.34$). For the non-homogeneous basket of ETF's (XFN, XEG, XMA, and XIT), the results are not as good as for the case of homogeneous basket, but still satisfactory (resulting in the average variance ratio of 0.289 for $b^* = 1.5$).

Geometric Average Basket Option Analytical price for homogeneous basket			
Option Maturity, y.	**ITM**	**ATM**	**OTM**
0.1452	1.2222	0.5683	0.1614
0.3178	1.4207	0.8018	0.3555
0.5671	1.6200	1.0380	0.5859

Option Maturity, y.	**100K simulations**			**1M simulations**		
	ITM	**ATM**	**OTM**	**ITM**	**ATM**	**OTM**
Geomteric Basket Average Option, MC QE Price						
0.1452	1.2242	0.5686	0.1609	1.2230	0.5687	0.1619
0.3178	1.4246	0.8041	0.3578	1.4222	0.8031	0.3566
0.5671	1.6224	1.0398	0.5870	1.6209	1.0392	0.5872
Arithmetic Basket Average Option, MC QE Price						
0.1452	1.2487	0.5850	0.1686	1.2473	0.5850	0.1696
0.3178	1.4735	0.8404	0.3806	1.4707	0.8392	0.3795
0.5671	1.7064	1.1060	0.6352	1.7044	1.1053	0.6354
Arithmetic Basket Average Option with Geometric Basket Average as control variate						
0.1452	1.2463	0.5845	0.1690	1.2460	0.5844	0.1690
0.3178	1.4681	0.8373	0.3780	1.4678	0.8371	0.3780
0.5671	1.6999	1.1016	0.6326	1.6995	1.1016	0.6327

Geometric Average Basket Option Analytical price for non-homogeneous basket			
Option Maturity, y.	**ITM**	**ATM**	**OTM**
0.1452	0.7619	0.1415	0.0473
0.3178	1.0408	0.3585	0.1950
0.5671	1.2828	0.6015	0.4059

Option Maturity, y.	**100K simulations**			**1M simulations**		
	ITM	**ATM**	**OTM**	**ITM**	**ATM**	**OTM**
Geomteric Basket Average Option, MC QE Price						
0.1452	0.7634	0.1415	0.0473	0.7625	0.1421	0.0478
0.3178	1.0431	0.3606	0.1969	1.0420	0.3597	0.1960
0.5671	1.2847	0.6024	0.4069	1.2837	0.6027	0.4072
Arithmetic Basket Average Option, MC QE Price						
0.1452	1.6037	0.6365	0.3620	1.6010	0.6357	0.3619
0.3178	1.8995	0.9714	0.6758	1.8965	0.9704	0.6751
0.5671	2.1521	1.2774	0.9837	2.1498	1.2766	0.9838
Arithmetic Basket Average Option with Geometric Basket Average as control variate						
0.1452	1.6020	0.6364	0.3620	1.6001	0.6350	0.3614
0.3178	1.8961	0.9690	0.6737	1.8942	0.9690	0.6739
0.5671	2.1470	1.2751	0.9816	2.1457	1.2738	0.9815