Sample Path Large Deviations and Optimal Importance Sampling for Stochastic Volatility Models

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Motivation

Research goal

- Find a robust method to improve Monte Carlo simulation performance when valuating path dependent options.
- Valid for stochastic volatility models.

To achieve this goal

- Use sample path Large Deviations Principles (LDP) to identify an *asymptotically optimal* importance sampling change of drift.
- Problem : standard LDP results do not apply to stochastic volatility models.
  - Volatility can degenerate, local Lipschitz condition violated.

Secondary research goal

- Prove LDP for common stochastic volatility models (e.g. Heston, Hull & White).

Talk Outline

Focus on path dependent option pricing.

- Problem setup.
- Review of Importance Sampling.
- Overview on constructing Asymptotically Optimal changes of drift.
    - Valid for general diffusions.
- Specification to Heston stochastic volatility model.
- Numerical example
    - Asian put option in Heston model.

Path Dependent Option Pricing

Setup:

- $S = \{S_t; 0 \leq t \leq T\}$: Price process
- $G = G(S)$: Path dependent option payoff

Closed-form solution for $E_P[G]$ not easily calculated.

Primary example

- Heston model:

$$\begin{aligned} \frac{dS_t}{S_t} &= rdt + \sqrt{v_t}dW_t \\ dv_t &= \kappa(\theta - v_t)\,dt + \xi\sqrt{v_t}dB_t \\ d\langle W, B\rangle_t &= \rho dt \end{aligned}$$

- Asian put option:

$$G(S) = \left(K - \frac{1}{T}\int_0^T S_t dt\right)^+$$

Monte Carlo Simulation

To calculate $E_P[G]$, run a Monte Carlo simulation.

- Robust : only have to replicate price/volatility dynamics.

Problem : simulation inefficient if G only pays off in rare events.

- $G \neq 0$ a "Large Deviation" from the norm.
- Asian put : $K >> S_0$.

Estimating confidence intervals is difficult.

- Simulation variance artificially low.

Improving the Monte Carlo Simulation

Goal : Run an effective Monte Carlo simulation by using *Importance Sampling*.

- Change simulation measure from P to Q and change option payoff from G to $G\frac{dP}{dQ}$ so that

$$E_Q\left[G\frac{dP}{dQ}\right] = E_P[G]$$

Variance under Q:

$$\mathrm{Var}_Q\left[G\frac{dP}{dQ}\right] = E_P\left[G^2\frac{dP}{dQ}\right] - E_P[G]^2$$

Optimization problem : $\min_{Q\in\mathcal{A}} E_P\left[G^2\frac{dP}{dQ}\right]$

- $\mathcal{A}$ an appropriate family of equivalent measures.

Example

Arithmetic average Asian put option

$$G(S) = \left(K - \frac{1}{T}\int_0^T S_t dt\right)^+$$

in the Heston model when $K >> S_0$.

Change of measure corresponds to *two* changes in drift:

- One for the volatility v.
- One for the asset price S.

Change the drift so option is more in the money.

- Compensate for change in drift by including the "scaling factor" in the option payoff.

Optimization Considerations

General optimization problem ill-posed : zero variance achieved for

$$\frac{dQ}{dP} = \frac{G}{E_P\left[G\right]}$$

- Not allowable because $E_P\left[G\right]$ unknown in the first place.

Questions:

- How to adjust notion of optimality?
- How to choose an appropriate family of measures $\mathcal{A}$?
- How to provide an optimal answer for a large class of functionals G?

Previous Work

Glasserman, Heidelberger, Shahabuddin (1999): use LDP to find an efficient change of measure.

- Work in Black-Scholes model. Partition $[0, T]$ to reduce to a finite dimensional problem.
- Approximate $E_P\left[G^2 \frac{dP}{dQ}\right]$ by taking an asymptotic expansion as noise parameter goes away.
- Solve an associated minimization problem.

Guasoni, R. (2008) : extend methodology to continuous time in Black-Scholes model.

- Find an optimal continuous change of drift.
- Characterize optimal change of drift via an Euler-Lagrange equation, possibly with an explicit solution.

Asymptotic Optimality - General Idea

For now, consider the optimization problem:

$$\inf_{Q \in \mathcal{A}} E_P \left[G(X)^2 \frac{dP}{dQ} \right]$$

X is a d-dimensional diffusion satisfying

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t; \qquad X_0 = x$$

where $b : \mathbb{R}^d \mapsto \mathbb{R}^d$, $\sigma : \mathbb{R}^d \mapsto \mathbb{R}^{d \times d}$

Construct $\mathcal{A}$ by taking Cameron-Martin-Girsanov changes of measure:

$$\mathcal{A} = \left\{ P^h \;\middle|\; \frac{dP^h}{dP} = \exp\left(\int_0^T u(h)'_t dW_t - \frac{1}{2} \int_0^T \|u(h)_t\|^2 dt \right), h \in \mathbb{H}_T^x \right\}$$

where

$$u(h)_t = \sigma^{-1}(h_t) \left(\dot{h}_t - b(h_t) \right)$$

$$\mathbb{H}_T^x = \left\{ h \;\middle|\; h(0) = x, \int_0^T \|u(h)_t\|^2 dt < \infty \right\}$$

Asymptotic Optimality (2)

Imbed X into the family of diffusions (for $0 < \varepsilon \le 1$):

$$dX_t^\varepsilon = b(X_t^\varepsilon)dt + \sqrt{\varepsilon}\sigma(X_t^\varepsilon)dW_t; \qquad X_0^\varepsilon = x$$

For $h \in \mathbb{H}_T^X$ set

$$H^h(X, W) = 2\log G(X) - \int_0^T u(h)_t' dW_t + \frac{1}{2}\int_0^T \|u(h)_t\|^2 dt$$

With $W^\varepsilon = \sqrt{\varepsilon}W$

$$E_P\left[G^2(X)\frac{dP}{dP^h}\right] = E_P\left[\exp\left(\frac{1}{\varepsilon}H^h(X^\varepsilon, W^\varepsilon)\right)\right]$$

at $\varepsilon = 1$. The small noise approximation is

$$L(h) = \limsup_{\varepsilon \downarrow 0} \varepsilon \log E_P\left[\exp\left(\frac{1}{\varepsilon}H^h(X^\varepsilon, W^\varepsilon)\right)\right]$$

$\hat{h}$ is *asymptotically optimal* if

$$\hat{h} = \underset{\{h \in \mathbb{H}_T^X\}}{\operatorname{argmin}} \; L(h)$$

Asymptotic Optimality and LDP

As $\varepsilon \downarrow 0$, $(X^\varepsilon, W^\varepsilon)$ "converges" to $(\phi_t, 0)$ where ϕ solves

$$\dot{\phi}_t = b(\phi_t), \qquad \phi_0 = x$$

Sample path LDP identify precise rate of convergence for the law of $(X^\varepsilon, W^\varepsilon)$ to $\delta_{(\phi,0)}$.

Classical result (Freidlin-Wentzell): for $H : C[0,T]^2 \mapsto \mathbb{R}$ bounded, continuous (supremum norm topology)

$$\lim_{\varepsilon \downarrow 0} \varepsilon \log E\left[e^{-\frac{1}{\varepsilon} H(X^\varepsilon, W^\varepsilon)}\right] = -\inf_{\{(\phi,\psi)\in C[0,T]^2\}} \left(H(\phi,\psi) + I(\phi,\psi)\right)$$

- Valid for b, σ bounded, Lipschitz (some relaxation OK)
- *Rate function*:

$$I(\phi,\psi) = \begin{cases} \frac{1}{2}\int_0^T \|u(\phi)_t\|^2 dt & \phi \in \mathbb{H}_T^X, \psi = u(\phi) \\ \infty & \text{else} \end{cases}$$

Variational Considerations

Freidlin-Wentzell asymptotics imply

$$L(h) = \sup_{\{\phi \in \mathbb{H}_T^{\mathbf{x}}\}} \left(2 \log G(\phi) + \frac{1}{2} \int_0^T \|u(h)_t - u(\phi)_t\|^2 dt - \int_0^T \|u(\phi)_t\|^2 dt \right)$$

Asymptotically optimal change of measure found by solving

$$\inf_{\{h \in \mathbb{H}_T^{\mathbf{x}}\}} \sup_{\{\phi \in \mathbb{H}_T^{\mathbf{x}}\}} \left(2 \log G(\phi) + \frac{1}{2} \int_0^T \|u(h)_t - u(\phi)_t\|^2 dt - \int_0^T \|u(\phi)_t\|^2 dt \right) \tag{1}$$

A lower bound:

$$\sup_{\{\phi \in \mathbb{H}_T^{\mathbf{x}}\}} \left(2 \log G(\phi) - \int_0^T \|u(\phi)_t\|^2 dt \right) \tag{2}$$

Practical plan:

- Solve (2) and find maximizer $\hat{\phi}$.
- With $\hat{h} = \hat{\phi}$, see if $L(\hat{h})$ equals value in (2).

Interpretation

For any family Q^ε of equivalent measures

$$\liminf_{\varepsilon\downarrow 0} \varepsilon \log E_P\left[G(X^\varepsilon)^{2/\varepsilon}\frac{dP}{dQ^\varepsilon}\right] \geq 2\liminf_{\varepsilon\downarrow 0} \varepsilon \log E_P\left[G(X^\varepsilon)^{1/\varepsilon}\right]$$

$$= \sup_{\{\phi\in\mathbb{H}_T^{\mathbf{x}}\}}\left(2\log G(\phi) - \int_0^T \|u(\phi)_t\|^2 dt\right)$$

- If practical plan works, $\hat{h}$ is robust.

Consider when $X = W$. Euler-Lagrange equation for (2):

$$D_\eta\left(2\log G(\phi) - \int_0^T \|\dot{\phi}_t\|^2 dt\right) = 0 \qquad D_\eta : \text{Gâteaux derivative towards } \eta$$

If G is Fréchet differentiable, using a Taylor expansion

$$E_{P^{\hat{h}}}\left[G(W)\frac{dP}{dP^{\hat{h}}}\right] = G(\phi)\exp\left(-\frac{1}{2}\int_0^T \|\dot{\phi}_t\|^2 dt\right) E_{P^{\hat{h}}}\left[\exp\left(R(W)\right)\right]$$

where $R(W)$ contains no linear terms.

- Variance due to linear part of $\log(G)$ eliminated.

Application to Heston Model

In the Heston model, $X = (S, v)$, $W = (B, Z)$ and

$$b(s,v) = \begin{pmatrix} rs \\ \kappa(\theta - v) \end{pmatrix} \qquad \sigma(s,v) = \begin{pmatrix} \rho s\sqrt{v} & \bar{\rho} s\sqrt{v} \\ \xi\sqrt{v} & 0 \end{pmatrix}$$

where $\bar{\rho} = \sqrt{1-\rho^2}$.

BIG PROBLEM : σ is neither elliptic nor locally Lipschitz.

- Freidlin Wentzell LDP must be extended.

Fortunately:

- If v satisfies a LDP by itself, then so does (S, v). (R. (2010))
- v satisfies LDP (Donati-Martin, Rouault, Yor, Zani (2004))

Application (2) - Questions

Does the Freidlin-Wentzell result apply to the unbounded and discontinuous function

$$H^h(X, W) = 2 \log G(X) - \int_0^T u(h)'_t dW_t + \frac{1}{2} \int_0^T \|u(h)_t\|^2 dt \ ?$$

- Yes, if G bounded from above and h smooth enough.

$$\int_0^T u(h)'_t dW_t = u(h)_T W_T - \int_0^T \dot{u}(h)'_t W_t dt$$

Do the variational problems in (1) and (2) admit maximizers?

- Yes, if G is continuous and bounded from above. (R. (2010))
  - Transfer problem to $L^2[0, T]$ via $u : \mathbb{H}_T^{(S,v)} \mapsto L^2[0, T]$.
  - u^{-1}, G weakly continuous, functionals in (1), (2) coercive.

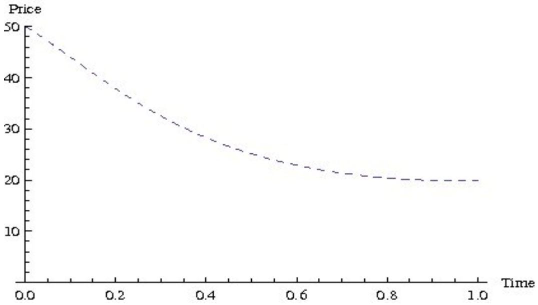
Numerical Example

For the Asian put option, the following parameter values are considered (Heston (1993))

$$\kappa = 2, \theta = 0.09, \xi = 0.2, v_0 = 0.04,$$
$$r = 0.05, T = 1, S_0 = 50, K = 30, \rho = -0.5.$$

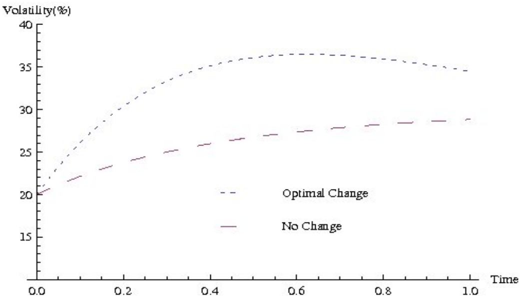
Asymptotic Optimality holds for $\hat{h}$ solving (2) with these values.

Optimal price drift



Numerical Example (2)

Optimal volatility drift



Interpretation:

- Under P the option is out of the money.
- To bring the option into the money either
  - The "average" price path must come down.
  - The "average" volatility must go up.

Future Work

Run numerical simulations to see actual variance reduction.

- Black-Scholes model : $5X - 10X$ variance reduction typical. Does this carry over?

Apply methodology to options which depend more directly on volatility.

- Out of the money call or put : variance reduction obtained primarily by changing price drift.
- What about for a straddle option? No obvious direction to move the price.

Derive LDP for other stochastic volatility models.

- SABR, CEV

THANK YOU!

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