American basket and spread options with a simple binomial tree

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Motivation

- Commodity, currency baskets consist of two or more assets, contain short positions (e.g., crack or crush spreads)
- Basket options are often American-style (or Asian-style)
- The valuation and hedging of basket options is challenging even in the Black-Scholes framework, because

  the sum of lognormal r.v.'s is not lognormal.
- Spreads can have negative values and negatively skewed distribution, so ***lognormal distribution cannot be used, even in approximation***.
- Existing approaches can only deal with baskets with positive weights or spreads between *two assets*.
- Numerical and Monte Carlo methods are slow and do not provide closed formulae for option price or greeks.

American basket options

Additional difficulties for American options:

- No closed form solution, truly path-dependent option

- Most common method: binomial tree (Cox Ross & Rubinstein'79)
- Monte-Carlo: not feasible, solution: Longstaff & Schwartz'01

Multi-asset situation:

- Binomial tree replaced by "binomial pyramid"
  → computationally not feasible for #assets > 2
- Implied binomial tree (Rubinstein): only positive weights
- Longstaff & Schwartz: involves complicated Hermite polynomials, can be slow

GLN approach (Borovkova et al.'07)

- Essentially a *moment-matching method,* in the spirit of the Wakeman method for Asian options (Turnbull and Wakeman (1991)).

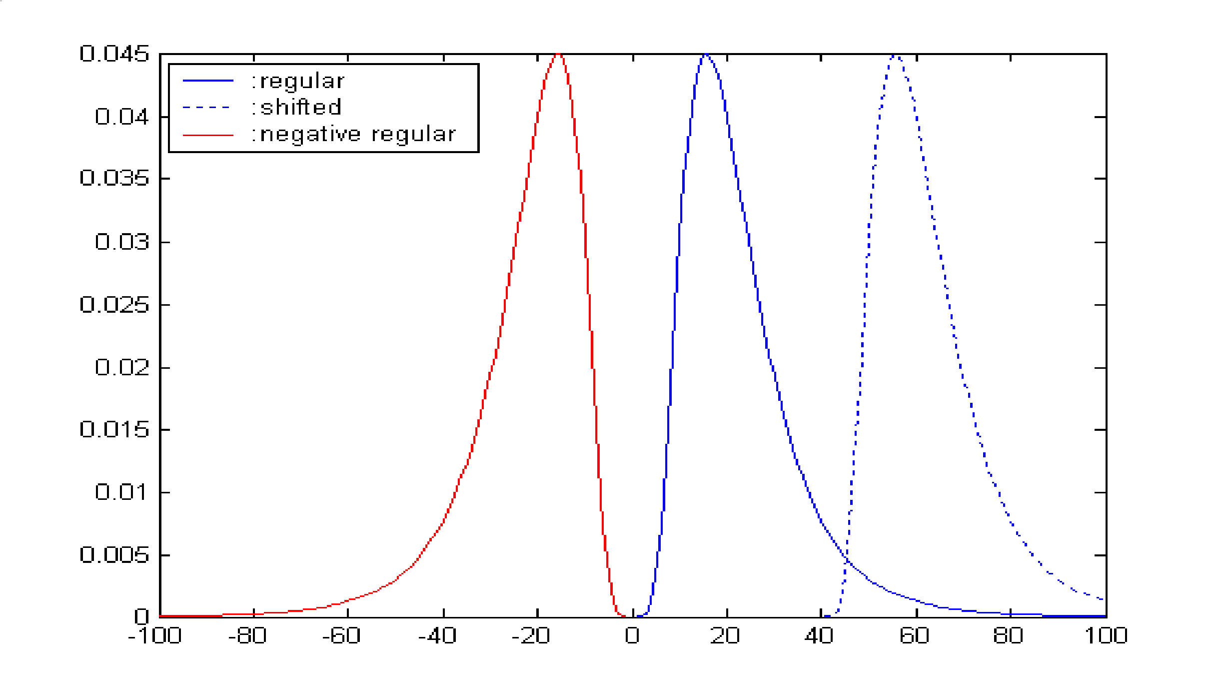
- Based on old and fundamental research on the good quality of approximation the sum of lognormal r.v.'s by a lognormal distribution (e.g., Mitchell (1968)).

- Basket distribution is approximated using a *generalized family of lognormal distributions:* shifted and negative shifted lognormals

The main attractions:

- applicable to baskets with several assets and negative weights
- Easily extended to Asian-style options (and now also to American options!)
- Allows to directly apply Black-Scholes formula (in European case)
- Provides closed form formulae for the option price and the greeks (approximate, European and Asian cases)
- Provides the way to build one binomial tree for the whole basket price process

Lognormal vs shifted and negative lognormal: two-parameter vs. three-parameter distribution



Assumptions

- Basket of *futures* on related commodities or currencies.
- The basket value at time of maturity T

$$B(T) = \sum_{i=1}^{N} a_i F_i(T)$$

where a_i : the weight of asset (futures contract) $i, a_i < or > 0$

N : the number of assets in the portfolio,

$F_i(T)$: the futures price *i* at the time of maturity .

- The futures in the basket and the basket option mature on the same date.

Individual assets' dynamics

Under the risk adjusted probability measure Q, the futures prices are martingales. The stochastic differential equations for $F_i(t)$ is

$$\frac{dF_i(t)}{F_i(t)} = \sigma_i dW_i(t), \; i = 1,2,3,\ldots,N$$

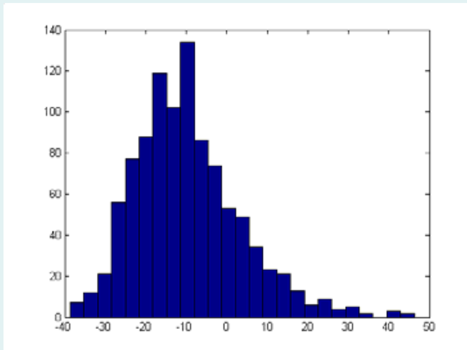
where

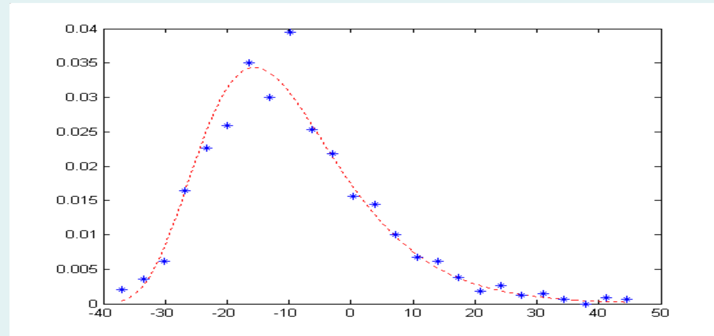
$F_i(t)$:the futures price *i* at time *t*

σ_i :the volatility of asset *i*

$W_i(t), W_j(t)$:the Brownian motions driving assets *i* and *j* with correlation $\rho_{i,j}$ $(dW_i dW_j = \rho_{i,j} dt)$

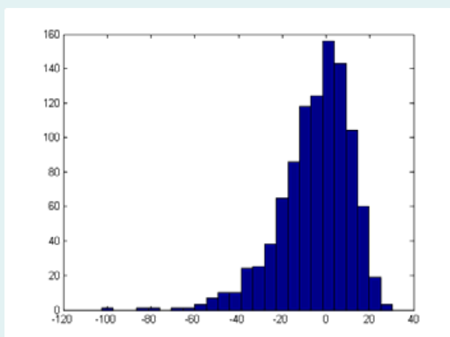
Examples of basket value distribution:

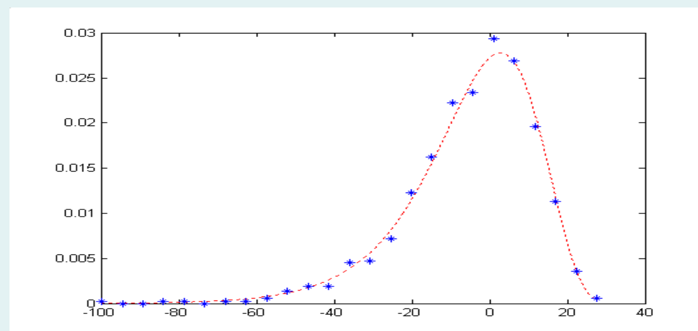




Shifted lognormal, $\tau < 0$

$$Fo = [100;90];\ \sigma = [0.2;0.3];\ a = [-1;1];\ X = -10;\ r = 3\%;\ T = 1\ year;\ \rho = 0.9$$





Negative shifted lognormal, $\tau > 0$

$$Fo = [105;100];\ \sigma = [0.3;0.2];\ a = [-1;1];\ X = -5;\ r = 3\%;\ T = 1\ year;\ \rho = 0.9$$

Moment matching

The first three moments and the skewness of basket distribution:

$$E\ B(t) = M_1(t) = \sum_{i=1}^{N} a_i F_i(0)$$

$$E\ (B(t))^2 = M_2(t) = \sum_{j=1}^{N}\sum_{i=1}^{N} a_i a_j F_i(0) F_j(0) \exp\left(\rho_{i,j}\sigma_i\sigma_j t\right)$$

$$E\ (B(t))^3 = M_3(t) =$$

$$= \sum_{k=1}^{N}\sum_{j=1}^{N}\sum_{i=1}^{N} a_i a_j a_k F_i(0) F_j(0) F_k(0) \exp\left(\rho_{i.j}\sigma_i\sigma_j t + \rho_{i.k}\sigma_i\sigma_k t + \rho_{j.k}\sigma_j\sigma_k t\right)$$

$$\eta_{B(t)} = \frac{E\left(B(t) - E(B(t))\right)^3}{\sigma^3_{B(t)}}$$

where $\sigma_{B(t)}$: standard deviation of basket at the time t

- If we assume the distribution of a basket $B(t)$ is shifted lognormal with parameters $m = m(t), s = s(t), \tau = \tau(t)$

  the parameters should satisfy non-linear equation system :

$$M_1 = \exp\left((m + \frac{1}{2} s^2 \right)$$

$$M_2 = \tau^2 + 2\tau \exp\left(m + \frac{1}{2} s^2 \right) + \exp\left(2m + 2s^2\right)$$

$$M_3 = \tau^3 + 3\tau^2 \exp\left(m + \frac{1}{2} s^2 \right) + 3\tau \exp\left(2m + 2s^2\right) + \exp\left(3m + \frac{9}{2} s^2 \right)$$

  (we omitted dependence of the moments and parameters on *t*)

- If we assume the distribution of a basket is negative shifted lognormal, the parameters should satisfy non-linear equation system above by changing M_1 to $-M_1$ and M_3 to $-M_3$

- NB: For basket *of futures*, we have an important simplification: zero drift implies *two unknown parameters* and not three

→ system of two equations can be solved explicitly → much better distribution matching !

Approximating distribution

Skewness	$\eta_t > 0$			$\eta_t < 0$
Approximating distribution	***Shifted lognormal***			***Negative shifted lognormal***

Note: we never use regular lognormal distribution as shifted lognormal always provides a better approximation
(improvement over Wakeman method for Asian and basket options)

Skewness sign remains the same throughout option's lifetime, increases in absolute value as $t \to T$

GLN approach for American options

- Need to approximate not just the *terminal basket distribution* by GLN, but the entire ***basket value process*** from time 0 to maturity by a GBM
- For this, we replace our original basket B by $B^*(t) = B(t) - \tau(t)$ or $-B(t) - \tau(t)$ depending on the sign of skewness.

Note: $B^{(*)}(t)$ is lognormal with parameters $(m(t), s(t))$ if $B(t)$ is GLN $(m(t), s(t), \tau(t))$

(This is similar to displaced diffusions (M.Rubinstein, JF 1983))

So we can assume that $B^{(*)}(t)$ follows GBM with parameters (μ^*, σ^*) given by

$$\sigma^{*2} = \frac{s^2(t)}{t} \quad \text{and} \quad \mu^* = \frac{m(t) - \log B^*(0)}{t} + \frac{1}{2}\sigma^{*2} (= 0 \text{ for futures})$$

→ Parameters (σ^*, τ) are determined via matching the second and third central moments

→ NO numerical solution is required, just solve simple two equations, two unknowns system

→ Skewness near zero: Bachelier model (normal distribution rather than lognormal)

Parameters σ^*, τ : an example

$Basket:$

$Fo = [100;120]; \sigma = [0.2;0.3];$

$a = [-1;1]$

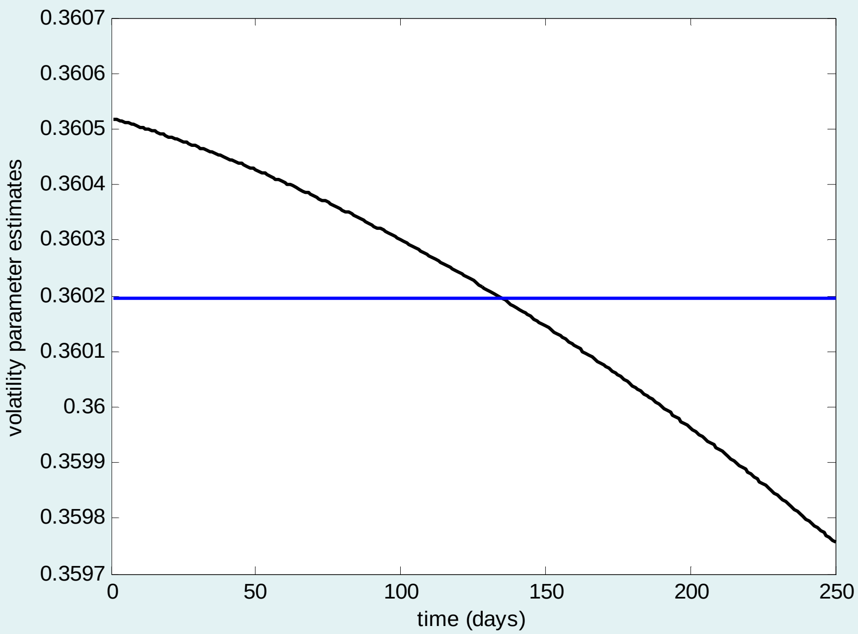
$\rho = 0.9;$

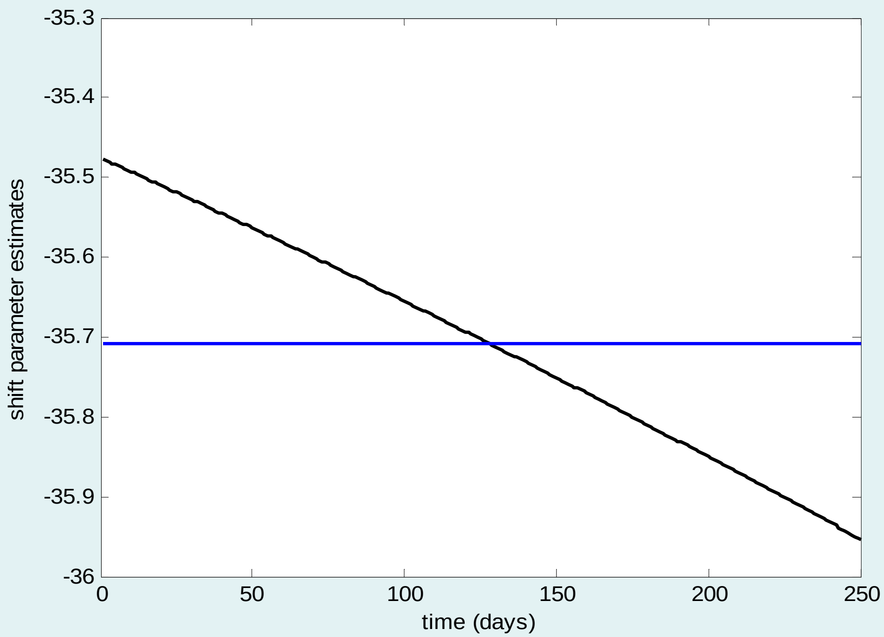
$T = 1\,year$

for all t: $\eta(t) > 0$, $\tau(t) < 0$

shifted lognormal approximation

Note: both parameters almost constant





Algorithm for building the binomial tree

- Build the binomial tree for the value $B^*(t)$ which follows GBM:
- At each time step on the tree, the value B^* moves either up to uB^*

with probability q or down to dB^* with probability $1-q$, where

$$u = \exp((\mu^* - 1/2\sigma^{*2})\Delta t + \sigma^* \sqrt{\Delta t})$$
$$d = \exp((\mu^* - 1/2\sigma^{*2})\Delta t - \sigma^* \sqrt{\Delta t})$$
$$q = (\exp(\mu^* \Delta t - d))/(u - d)$$

- Translate the obtained binomial tree for B^* into the tree for B using

$B(t) = B^*(t) + \tau(t)$ for shifted lognormal case, or
$B(t) = -B^*(t) - \tau(t)$ for negative shifted lognormal case.

- This tree can be used now for valuing an American (or e.g. Bermudan) option on B, computing option's delta at each node and deciding on early exercise.

Simplification: constant shift parameter

- For constant shift parameter τ, can directly use the tree built for B^*:

- For shifted lognormal approximation, recall that $B^*(t) = B(t) - \tau$
- The payoff of an American call option on B with the strike price K is
  the payoff of a American call option on basket B^* with the strike $K - \tau$

- For negative shifted lognormal approximation, recall that $B^*(t) = -B(t) - \tau$
- The payoff of an American call option on B with strike K is
  the payoff of a American put option on basket B^* with the strike $-\tau - K$

- Both can be evaluated using the tree for B^*

	Basket 1	***Basket 2***	***Basket 3***		***Basket 4***	***Basket 5***
Futures price (Fo)	**[100;100]**	**[100;120]**	**[150;100]**		**[95;90;105]**	**[100;90;95]**
Volatility (σ)	**[0.3;0.2]**	**[0.2;0.3]**	**[0.3;0.2]**		**[0.2;0.3;0.25]**	**[0.25;0.3;0.2]**
Weights (a)	**[0.3;0.7]**	**[-1;1]**	**[-1;1]**		**[1; -0.8; -0.5]**	**[0.6;0.8; -1]**
Correlation (ρ)	**0.6**	**0.9**	**0.7**		$\rho_{1,2} = \rho_{2,3} =$ **0.9** $\rho_{1,3} =$ **0.8**	$\rho_{1,2} = \rho_{2,3} =$ **0.9** $\rho_{1,3} =$ **0.8**
Strike price (X)	**100**	**20**	**-50**		**-30**	**40**
skewness (η)	$\eta > 0$	$\eta > 0$	$\eta < 0$		$\eta < 0$	$\eta > 0$

T=1 year; r = 5 %

Simulation results

Basket	ATM put: Bin. Pyramid (N=150)	ATM put: Impl. BT (N=150)	ATM put: GLN (N=250)	OTM put (ATM-10): Bin. Pyramid (N=150)	OTM put (ATM-10): Impl.BT (N=150)	OTM put (ATM-10): GLN (N=150)
1	7.93	7.94	7.93	3.66	3.67	3.66
2	7.66	-	7.67	3.17	-	3.16
3	13.07	-	13.09	9.37	-	9.39
4	-	-	7.19	-	-	3.94
5	-	-	9.77	-	-	4.72

Other issues and further research

- **Delta-hedging performance**:
  - analyzed on the basis of Monte Carlo simulations
  - hedge error is in the order of 5-10% of the option price
  - almost the same as the hedge error for a single asset case with matching vol

- Other greeks (especially vegas)

- Quality of *generalized lognormal* approximation → need *theorems similar to those of Mitchell*

- Other applications:
  - *physics*: a wave propagating through a turbulent medium
  - *wireless communication*: attenuation due to shadowing in wireless channels
  - *health sciences*: incubation periods of diseases, e.g., anthrax inhalation
  - *analysis of handwriting*: direction control ~ weighted difference of lognormals
  - *ecology and chemistry*: particle size distribution, pollution