Errors from discrete hedging in exponential Lévy models: the L^2 approach

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Introduction

- We consider the problem of approximating the stochastic integral

$$\int_0^T F_{t-} dS_t \qquad \text{with} \qquad \int_0^T F_{h\lfloor t/h \rfloor} dS_t$$

  where h denotes the distance between discretization points,
  - S is an exponential Lévy process that represents the stock and
  - F is a Lévy-Itô process that represents the hedging strategy.
- Denote by ϵ_T^h the difference between the integral and its approximation

$$\epsilon_T^h = \int_0^T (F_{t-} - F_{h\lfloor t/h \rfloor}) dS_t \, .$$

- We investigate the L^2-error $E[(\epsilon_T^h)^2]$ as h approaches zero.

Hedging

- Let C denote the price of an option with pay-off function G and assume that $r = 0$, i.e.

$$C(t, S_t) = E^Q[G(S_T)|\mathcal{F}_t] \, .$$

- Find a process F such that the difference (“the hedging error of the first type”)

$$G(S_T) - C(0, S_0) - \int_0^T F_{t-} dS_t$$

  becomes small.
- Hedging errors of the first type have been analyzed in several papers in the context of exponential Lévy processes (see e.g. Cont et al. (2007) and Hubalek et al. (2006)).

Discrete time hedging

- In practical situations it is impossible to follow the process F since it requires that the hedge portfolio is rebalanced continuously.
- Assume that the hedge portfolio is rebalanced at equidistant points in time. Then the true hedging error is given by

$$G(S_T) - C(0, S_0) - \int_0^T F_{h\lfloor t/h \rfloor} dS_t,$$

which may be decomposed as

$$\begin{aligned} & G(S_T) - C(0, S_0) - \int_0^T F_{h\lfloor t/h \rfloor} dS_t \\ &= \underbrace{G(S_T) - C(0, S_0) - \int_0^T F_t dS_t}_{\text{Hedging error of the first type}} + \underbrace{\int_0^T F_t dS_t - \int_0^T F_{h\lfloor t/h \rfloor} dS_t}_{\text{Hedging error of the second type}} . \end{aligned}$$

Hedging errors of the second type

Hedging errors of the second type have been analyzed

- in the context of complete markets, i.e.

  $$\text{"Hedging error of the first kind"}=0,$$

  in a couple of papers, e.g.
  - Zhang (1999), European options: $\lim_{h\downarrow 0} h^{-1} E[(\epsilon_T^h)^2]$ converges to a non zero finite limit.
  - Gobet and Temam (2001), digital options: $\lim_{h\downarrow 0} h^{-\frac{1}{2}} E[(\epsilon_T^h)^2]$ converges to a non zero finite limit.
  - Geiss (2002), digital options: the order of convergence may be increased using a nonequidistant time net.
- in the context of incomplete markets
  - Tankov and Voltchkova (2009) studied the hedging error in a market with jumps from the point of view of weak convergence.
    In particular they showed that if the underlying process contains no diffusion part then $h^{-\frac{1}{2}} \epsilon_T^h \to 0$ in probability as $h \downarrow 0$.

Market model

The stock is modeled by $S_t = e^{X_t}$ where X is a Lévy model

- The process X has characteristic triplet (a^2, ν, γ). Furthermore denote

$$\phi_t(u) = E[e^{iuX_t}], \qquad A = a^2 + \int_{\mathbb{R}} (e^z - 1)^2 \nu(dz)\,.$$

- There exists an equivalent measure Q such that X is a Lévy process also under Q but with characteristic triplet $(a^2, \bar{\nu}, \bar{\gamma})$. Furthermore denote

$$\bar{\phi}_t(u) = E^Q[e^{iuX_t}], \qquad \bar{A} = a^2 + \int_{\mathbb{R}} (e^z - 1)^2 \bar{\nu}(dz)\,.$$

- The process S is of the from

$$S_t = 1 + \int_0^t bS_u du + \int_0^t aS_u dW_u + \int_0^t S_{u-} \int_{\mathbb{R}} (e^z - 1)\tilde{J}(du \times dz)\,.$$

L^2 convergence of the hedging error

- It is assumed that the hedging strategy may be expressed using the following integral representation

$$F_t = F_0 + \int_0^t \mu_u du + \int_0^t \sigma_u dW_u + \int_0^t \int_{\mathbb{R}} \gamma_{u-}(z)\tilde{J}(du \times dz). \quad (1)$$

- Denote $\underline{\eta}(t) = \sup\{T_i, T_i < t\}$ and $\overline{\eta}(t) = \inf\{T_i, T_i \geq t\}$, then

$$\epsilon_T^h = \int_0^T (F_{t-} - F_{\underline{\eta}(t)})dS_t.$$

- Choose a function $r(h) : (0, \infty) \to (0, \infty)$ with $\lim_{h\downarrow 0} r(h) = 0$ (the rate of convergence to zero of the hedging error).
- We shall see that under suitable assumptions $E[(\epsilon_T^h)^2/r(h)]$ converges to a finite nonzero limit when $h \downarrow 0$.

General limit theorem

Theorem 1

Assume that the hedging strategy F is of the form (1) *and*

$$\lim_{h\downarrow 0}\frac{h}{r(h)}E\left[\int_0^T S_t^2(\overline{\eta}(t)-t)\left(\mu_t^2+\int_{\mathbb{R}}\gamma_t^2(z)\nu(dz)\right)dt\right]=0\,. \qquad (2)$$

Then

$$\lim_{h\downarrow 0}\frac{1}{r(h)}E\left[\left(\epsilon_T^h\right)^2\right]$$
$$=\lim_{h\downarrow 0}\frac{A}{r(h)}E\left[\int_0^T S_t^2(\overline{\eta}(t)-t)\left(\sigma_t^2+\int_{\mathbb{R}}\gamma_t^2(z)e^{2z}\nu(dz)\right)dt\right],$$

whenever the limit on the right-hand side exists.

The regular regime

Corollary 2

Assume that (2) *is satisfied and*

$$E\left[\int_0^T S_t^2\left(\sigma_t^2+\int_{\mathbb{R}}\gamma_t^2(z)e^{2z}\nu(dz)\right)dt\right]<\infty.$$

Then

$$\lim_{h\downarrow 0}\frac{1}{h}E\left[\left(\int_0^T \epsilon_T^h\right)^2\right]=\frac{A}{2}E\left[\int_0^T S_t^2\left(\sigma_t^2+\int_{\mathbb{R}}\gamma_t^2(z)e^{2z}\nu(dz)\right)dt\right].$$

The highest convergence rate that can be obtained in this setting is $r(h)=h$. When $r(h)=h$ we say that the convergence takes place in the regular regime.

Option pricing

- Option prices may be calculated using Fourier inversion (see e.g. Hubalek et al. (2006) or Tankov (2009)).
- For some $R \in \mathbf{R}$

$$C(t, S_t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(u + iR)\bar{\phi}_{T-t}(-u - iR) S_t^{R-iu} du,$$

where

$$\hat{g}(u) := \int_{\mathbb{R}} e^{iux} g(x) dx.$$

- Examples
  - Digital option:

$$\hat{g}(u + iR) = \frac{K^{iu-R}}{R - iu}.$$

  - European option:

$$\hat{g}(u + iR) = \frac{K^{iu+1-R}}{(R - iu)(R - 1 - iu)}.$$

Hedging strategies

We consider two hedging strategies

- Delta hedging:

$$F_t = \frac{\partial C(t, S_t)}{\partial S}.$$

- Quadratic hedging: the solution to

$$\arg\min_F E^Q\left[\left(G(S_T) - C(0, S_0) - \int_0^T F_t dS_t\right)^2\right]$$

  is given by the Kunita-Watanabe decomposition and can be explicitly written as

$$F_t = \frac{d\langle C, S\rangle_t^Q}{d\langle S, S\rangle_t^Q}.$$

Hedging strategies

The strategies may be calculated using Fourier inversion.

- Delta hedging

$$F_t = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(u + iR)\bar{\phi}_{T-t}(-u - iR)(R - iu)S_t^{R-iu-1} du.$$

- Quadratic hedging

$$F_t = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(u + iR)\bar{\phi}_{T-t}(-u - iR)S_t^{R-iu-1}\Upsilon(u) du$$

where $\Upsilon(u) = \frac{\bar{\psi}(-u-i(R+1))-\bar{\psi}(-u-iR)-\bar{\psi}(-i)}{\bar{\psi}(-2i)-2\bar{\psi}(-i)}$.

Both strategies may (under some additional assumptions) be expressed using the integral representation

$$F_t = F_0 + \int_0^t \mu_u du + \int_0^t \sigma_u dW_u + \int_0^t \int_{\mathbb{R}} \gamma_{u-}(z)\tilde{J}(du \times dz).$$

European options

Theorem 3 (European-like options)

Assume that

- *F follows the delta or the quadratic hedging strategy,*
- *the pay-off function is of European type.*

Then, for most parametric models found in the literature

$$\lim_{h \downarrow 0} \frac{1}{h} E\left[\left(\epsilon_T^h\right)^2\right] = \frac{A}{2} E\left[\int_0^T S_t^2 \left(\sigma_t^2 + \int_{\mathbb{R}} \gamma_t(z) e^{2z} \nu(dz)\right) dt\right].$$

Digital options

Assumption on the process X

(H-α) The Lévy measure ν has a density satisfying

$$\nu(x) = \frac{f(x)}{|x|^{1+\alpha}}, \quad \lim_{x\to 0+} f(x) = f_+, \quad \lim_{x \to 0-} f(x) = f_-,$$

for some constants $f_- > 0$ and $f_+ > 0$.

Theorem 4 (Delta hedging, digital options)

Assume that

- *F follows the delta hedging strategy,*
- $G(S_T) = 1_{S_T \geq K}$,
- *H-α is satisfied with $\alpha \in (1, 2)$.*

Then

$$\lim_{h \downarrow 0} \frac{1}{h^{1-\frac{1}{\alpha}}} E\left[\left(\epsilon_T^h\right)^2\right] = ADp_T(\log K),$$

where D only depends on α, f_+ and f_-.

Theorem 5 (Martingale quadratic hedging, digital options)

Assume that

- F *follows the quadratic hedging strategy,*
- $G(S_T) = 1_{S_T \geq K}$.

Then

(i) *if H-α is satisfied with* $\alpha \in (0, 3/2)$

$$\lim_{h \downarrow 0} \frac{1}{h} E\left[\left(\epsilon_T^h\right)^2\right] = \frac{A}{2} E\left[\int_0^T S_t^2 \left(\sigma_t^2 + \int_{\mathbb{R}} \gamma_t(z) e^{2z} \nu(dz)\right) dt\right].$$

(ii) *if H-α is satisfied with* $\alpha \in (3/2, 2)$

$$\lim_{h \downarrow 0} \frac{1}{h^{\frac{3}{\alpha}-1}} E\left[\left(\epsilon_T^h\right)^2\right] = \frac{AQ}{\bar{A}^2} p_T(\log K)$$

where Q only depends on α, f_+ and f_-.

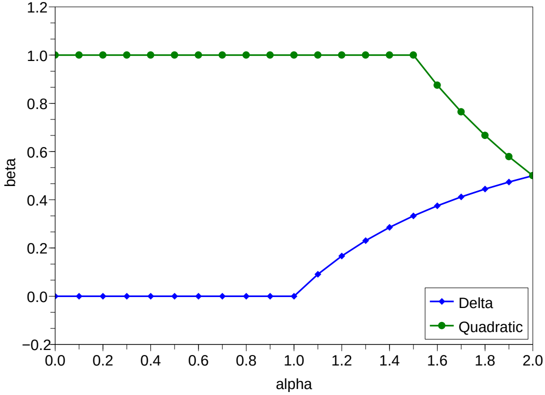

Figure: Convergence rate of the expected squared discretization error to zero as function of the stability index α for a digital option. The rate is given by $r(h) = h^{\beta}$, where β is plotted in the graph.

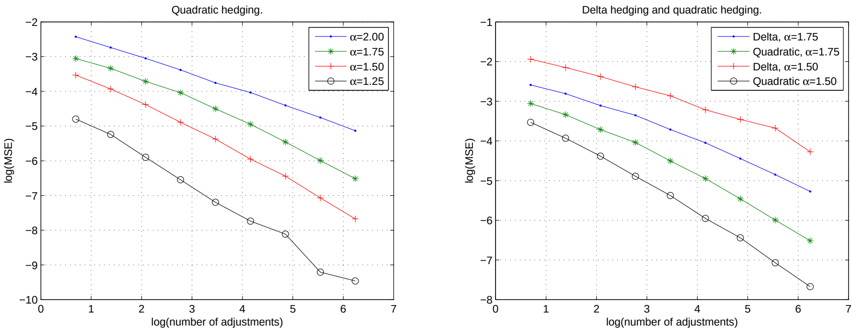

Figure: Convergence of the discretization error to zero for a digital option in the CGMY model. Left: quadratic hedging. Right: quadratic hedging vs. delta hedging.

Conclusions

- The limit $\lim_{h\downarrow 0} h^{-1} E[(\epsilon_T^h)^2]$ is positive in all cases and may be infinite. Note that for pure jump processes the rate of L^2 convergence is different from the rate of convergence in probability.
- The rate of convergence for digital options depends on the hedging strategy. The discretization error for the delta hedging strategy converges slower to zero than for the quadratic hedging strategy.
- For digital options the rate of convergence of the discretization error depends on the fine properties of the Lévy measure near zero.