American Option Pricing Under Two Stochastic Volatility Processes

Jonathan Ziveyi
Joint work with Prof. Carl Chiarella

School of Finance and Economics
University of Technology, Sydney

6th World Congress of the
Bachelier Finance Society
23 June 2010

Outline

- Motivation
- Problem Statement
- General Solution
- Explicit Representation of the Density Function
- The American Call option Price
  - Numerical Results
  - Comparison with the GBM and MOL
- Possible Extensions

Motivation

- Asset returns are non normal and have fat tails, Blattberg & Gonedes (1974), Platen & Rendek (2008).
- Volatility changes with time and the changes are unpredictable, Scott (1987).
- Though persistent, volatility has a tendency of reverting to a long - run average.
- Range of research on European option pricing models under stochastic volatility; Hull & White (1987), Scott (1987), Wiggins (1987), Heston (1993), etc.
- Adolfsson at. el (2009) extend the Heston model to pricing American Options.
- Multifactor models suggest that asset returns are driven by many unpredictable processes.
- The need to consider two or more stochastic volatility processes.

The Problem Statement

- The risk - neutral dynamics of the driving processes, S, v_1 and v_2,

$$\boxed{\begin{aligned} dS &= (r-q)Sdt + \sqrt{v_1}Sd\tilde{W}_1 + \sqrt{v_2}Sd\tilde{W}_2, \\ dv_1 &= [\kappa_1\theta_1 - (\kappa_1+\lambda_1)v_1]dt + \rho_{13}\sigma_1\sqrt{v_1}d\tilde{W}_1 \\ &\quad + \sqrt{1-\rho_{13}^2}\sigma_1\sqrt{v_1}d\tilde{W}_3, \\ dv_2 &= [\kappa_2\theta_2 - (\kappa_2+\lambda_2)v_2]dt + \rho_{24}\sigma_2\sqrt{v_2}d\tilde{W}_2 \\ &\quad + \sqrt{1-\rho_{24}^2}\sigma_2\sqrt{v_2}d\tilde{W}_4. \qquad (1.1) \end{aligned}}$$

- $\tilde{W}_i$ for $i = 1, \cdots, 4$ are independent Wiener processes.
- ρ_{13} is correlation between $\tilde{W}_1$ and $\tilde{W}_3$ whilst ρ_{24} denotes the correlation between $\tilde{W}_2$ and $\tilde{W}_4$.
- No correlation between $\tilde{W}_1$ & $\tilde{W}_2$, and $\tilde{W}_3$ & $\tilde{W}_4$.

Pricing Partial Differential Equation

- We set $\tau = T - t$, $S = e^x$ and $C(\tau, x, v_1, \cdots, v_n)$ to be the pricing function. Exploiting the techniques of Jamshidian's (1992) yields the pricing PDE

$$\boxed{\frac{\partial C}{\partial \tau} = \mathcal{L}C - rC + \mathbb{1}_{x \geq \ln b(\tau, v_1, v_2)}(qe^x - rK), \qquad (1.2)}$$

where

$$\boxed{\begin{aligned}\mathcal{L} = {} & \left(r - q - \frac{1}{2}v_1 - \frac{1}{2}v_2\right)\frac{\partial}{\partial x} + \Phi_1 \frac{\partial}{\partial v_1} - \beta_1 v_1 \frac{\partial}{\partial v_1} + \Phi_2 \frac{\partial}{\partial v_2} \\ & - \beta_2 v_2 \frac{\partial}{\partial v_2} + \frac{1}{2} v_1 \frac{\partial^2}{\partial x^2} + \frac{1}{2} v_2 \frac{\partial^2}{\partial x^2} + \rho_{13}\sigma_1 v_1 \frac{\partial^2}{\partial x \partial v_1} \\ & + \rho_{14}\sigma_2 v_2 \frac{\partial^2}{\partial x \partial v_2} + \frac{1}{2}\sigma_1^2 v_1 \frac{\partial^2}{\partial v_1^2} + \frac{1}{2}\sigma_2^2 v_2 \frac{\partial^2}{\partial v_2^2}, \qquad (1.3)\end{aligned}}$$

and

$$\boxed{\Phi_1 = \kappa_1\theta_1, \quad \Phi_2 = \kappa_2\theta_2, \quad \beta_1 = \kappa_1 + \lambda_1 \quad \text{and} \quad \beta_2 = \kappa_2 + \lambda_2.}$$

Transition Probability Density PDE

- Eqn (1.3) is solved subject to the initial and boundary conditions,

$$C(0, x, v_1, v_2) = (e^x - K)^+, \quad -\infty < x < \infty, \tag{1.4}$$

$$b(\tau, v_1, v_2) - K = C(\tau, b(\tau, v_1, v_2), v_1, v_2). \tag{1.5}$$

- Smooth pasting condition can also be imposed depending on the particular problem considered.
- After effecting the transformation $S_j = e^{x_j}$ the transition density function $U(\tau, x, v_1, v_2)$ for the SDE system (1.1) is a solution of the backward Kolmogorov PDE,

$$\frac{\partial U}{\partial \tau} = \mathcal{L}U \tag{1.6}$$

- Equation (1.6) is to be solved subject to the initial condition,

$$U(0, x, v_1, v_2; x_0, v_{1,0}, v_{2,0}) = \delta(x - x_0)\delta(v_1 - v_{1,0})\delta(v_2 - v_{2,0}). \tag{1.7}$$

Useful Result from PDE Theory

- Duhamel's principle states that the solution to the one dimensional inhomogeneous parabolic PDE of the form,

$$\boxed{\frac{\partial U}{\partial \tau} = \mathcal{L}U + f(\tau, x),}$$

subject to the initial condition,

$$\boxed{U(0, x) = \phi(x),}$$

can be represented as,

$$\begin{aligned} U(\tau, x) = & \int_{-\infty}^{\infty} \phi(y) U(\tau, x - y) dy \\ & + \int_0^{\tau} \int_{-\infty}^{\infty} f(\xi, y) U(\tau - \xi, x - y) dy d\xi. \end{aligned} \tag{1.8}$$

- Here, $\mathcal{L}$ is a parabolic partial differential operator.

General Solution of Inhomogeneous PDE

- By use of Duhamel's principle, the solution of the American call option pricing PDE (1.3) can be represented as,

$$C(\tau, x, v_1, v_2) = C_E(\tau, x, v_1, v_2) + C_P(\tau, x, v_1, v_2), \quad (1.9)$$

where,

$$C_E(\tau, x, v_1, v_2) = e^{-r\tau} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty (e^u - K)^+ \times U(\tau; x, v_1, v_2; u, w_1, w_2) du dw_1 dw_2,$$

$$C_P(\tau, x, v_1, v_2) = \int_0^\tau e^{-r(\tau-\xi)} \int_0^\infty \int_0^\infty \int_{\ln b(\xi, w_1, w_2)}^\infty (qe^u - rK) \times U(\tau - \xi; x, v_1, v_2; u, w_1, w_2) du dw_1 dw_2 d\xi.$$

- The 1^{st} part of (1.9) is the European Option component and the 2^{nd} is the Early Exercise premium.

Explicit Form of the Transition Density Function

- A Fourier transform is applied to the log S variable followed by Laplace transforms to the v variables of the PDE (1.6) and solving the resulting system of PDE by the method of characteristics.
- Once the system of PDEs is solved, we use the tabulated results in Abramowitz and Stegun (1964) to find the inverse Laplace transform of the resulting solution.
- Application of the inverse Fourier transform to the resulting solution yields the transition density function as,

$$\begin{aligned} U(\tau, \log S, v_1, v_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\eta \ln S} \Bigg\{ e^{i\eta S_0 - i\eta(r-q)} \prod_{j=1}^{2} \Bigg[\exp\left\{ \left(\frac{\Theta_j - \Omega_j}{\sigma_j^2} \right) \left(v_j - v_{j,0} + \Phi_j \tau \right) \right\} \\ \times \exp\left\{ - \left(\frac{2\Omega_j}{\sigma_j^2 (e^{\Omega_j \tau} - 1)} \right) \left(v_{j,0} e^{\Omega_j \tau} + v_j \right) \right\} \frac{2\Omega_j e^{\Omega_j \tau}}{\sigma_j^2 (e^{\Omega_j \tau} - 1)} \left(\frac{v_{j,0} e^{\Omega_j \tau}}{v_j} \right)^{\frac{\Phi_j}{\sigma_j^2} - \frac{1}{2}} \\ \times I_{\frac{2\Phi_j}{\sigma_j^2} - 1} \left(\frac{4\Omega_j}{\sigma_j^2 (e^{\Omega_j \tau} - 1)} (v_j v_{j,0} e^{\Omega_j \tau})^{\frac{1}{2}} \right) \Bigg] \Bigg\} d\eta. \qquad (1.10) \end{aligned}$$

The American Call Option Price

- By letting $V(\tau, S, v_1, v_2) \equiv C(\tau, \log S, v_1, v_2)$ and approximating the early exercise boundary with the expression,

$$\boxed{\ln b(\tau, v_1, v_2) \approx b_0(\tau) + b_1(\tau) v_1 + b_2(\tau) v_2, \qquad (1.11)}$$

the value of the American call option can be expressed as,

$$\boxed{V(\tau, S, v_1, v_2) \approx V_E(\tau, S, v_1, v_2) + V_P^A(\tau, S, v_1, v_2), \qquad (1.12)}$$

- The component on the RHS can be represented as,

$$\boxed{V_E(\tau, S, v_1, v_2) = e^{-q\tau} S P_1(\tau, S, v_1, v_2; K) - e^{-r\tau} K P_2(\tau, S, v_1, v_2; K), \qquad (1.13)}$$

where,

$$\boxed{P_j(\tau, S, v_1, v_2; K) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\left(\frac{g_j(\tau, S, v_1, v_2; \eta) e^{-i\eta \ln K}}{i\eta} \right) d\eta, \qquad (1.14)}$$

for $j = 1, 2$.

The Early Exercise Premium Component

- The early exercise premium is given as

$$\boxed{\begin{aligned} V_P^A(\tau, S, v_1, v_2) = \int_0^\tau [&q e^{-q(\tau-\xi)} S \bar{P}_1^A(\tau-\xi, S, v_1, v_2; b_0(\xi), b_1(\xi), b_2(\xi)) \\ &- r e^{-r(\tau-\xi)} K \bar{P}_2^A(\tau-\xi, S, v_1, v_2; b_0(\xi), b_1(\xi), b_2(\xi))] d\xi, \end{aligned} \quad (1.15)}$$

with,

$$\boxed{\begin{aligned} \bar{P}_j^A(\tau-\xi, S, v_1, v_2; b_0(\xi), b_1(\xi), b_2(\xi)) &= \frac{1}{2} \quad (1.16) \\ &+ \frac{1}{\pi}\int_0^\infty Re\left(\frac{\bar{g}_j(\tau-\xi, S, v_1, v_2; \eta, b_1(\xi), b_2(\xi)) e^{-i\eta b_0(\xi)}}{i\eta}\right) d\eta, \end{aligned}}$$

for $j = 1, 2$.

Iterative Equations

- Given equation (1.11), the value – matching condition can be expressed as,

$$e^{b_0(\tau)+b_1(\tau)v_1+b_2(\tau)v_2} - K = V(\tau, e^{b_0(\tau)+b_1(\tau)v_1+b_2(\tau)v_2}, v_1, v_2). \tag{1.17}$$

- The implicit time functions are found by solving the system,

$$\begin{aligned}
b_0(\tau) &= \ln[V(\tau, e^{b_0(\tau)+b_1(\tau)v_1+b_2(\tau)v_2}, v_1, v_2) + K] - b_1(\tau)v_1 - b_2(\tau)v_2, \\
b_1(\tau) &= \frac{1}{v_1}\left(\ln[V(\tau, e^{b_0(\tau)+b_1(\tau)v_1+b_2(\tau)v_2}, v_1, v_2) + K] - b_0(\tau) - b_2(\tau)v_2\right) \\
b_2(\tau) &= \frac{1}{v_2}\left(\ln[V(\tau, e^{b_0(\tau)+b_1(\tau)v_1+b_2(\tau)v_2}, v_1, v_2) + K] - b_0(\tau) - b_1(\tau)v_1\right)
\end{aligned} \tag{1.18}$$

Numerical Implementation

- In implementing equations (1.12) and the system (1.18), we treat the American option as a Bermudan option.
- The time interval is partitioned into M – equally spaced subintervals of length $h = T/M$.
- It has been shown in Kim (1990) that the early exercise boundary at maturity is,

$$\boxed{b(0, v_1, v_2) = \max\left(\frac{r}{q}K, K\right). \qquad (1.19)}$$

By comparing coefficients, we can readily deduce that,

$$\boxed{b_0(0) = \max\left(K, \frac{r}{q}K\right), \quad b_1(0) = 0, \quad \text{and}, \quad b_2(0) = 0. \qquad (1.20)}$$

Numerical Implementation cont...

- The discretized version of the pricing function is,

$$\boxed{V(hm, S, v_1, v_2) = V_E(hm, S, v_1, v_2) + V_P^A(hm, S, v_1, v_2). \quad (1.21)}$$

- At each time we determine the three unknown boundary terms, $b_0^m = b_0(hm)$, $b_1^m = b_1(hm)$ and $b_2^m = b_2(hm)$.
- We solve iteratively,

$$\begin{aligned}
b_{0,k}^m &= \ln[V(hm, e^{b_{0,k}^m + b_{1,k-1}^m v_1 + b_{2,k-1}^m v_2}, v_1, v_2) + K] - b_{1,k-1}^m v_1 - b_{2,k-1}^m v_2, \\
b_{1,k}^m &= \frac{1}{v_1}\Big(\ln[V(hm, e^{b_{0,k}^m + b_{1,k}^m v_1 + b_{2,k-1}^m v_2}, v_1, v_2) + K] - b_{0,k}^m - b_{2,k-1}^m v_2 \Big), \\
b_{2,k}^m &= \frac{1}{v_2}\Big(\ln[V(hm, e^{b_{0,k}^m + b_{1,k}^m v_1 + b_{2,k}^m v_2}, v_1, v_2) + K] - b_{0,k}^m - b_{1,k}^m v_1 \Big).
\end{aligned} \quad (1.22)$$

- We continuously repeat the iterative process until a tolerance level is reached.

Numerical Results

Parameter	Value	v_1 – Parameter	Value	v_2 – Parameter	Value
K	100	θ_1	6%	θ_2	8%
r	3%	κ_1	3	κ_2	4
q	5%	σ_1	10%	σ_2	11%
T	0.5	ρ_{12}	± 0.5	ρ_{13}	± 0.5
M	200	λ_1	0	λ_2	0
		$v_1^{\max}$	20%	$v_2^{\max}$	20%

Table: Parameters used for the American call option. The v_1 column contains are parameters for the first variance process whilst the v_2 column contains parameters for the second variance process.

Early Exercise Boundary Surface

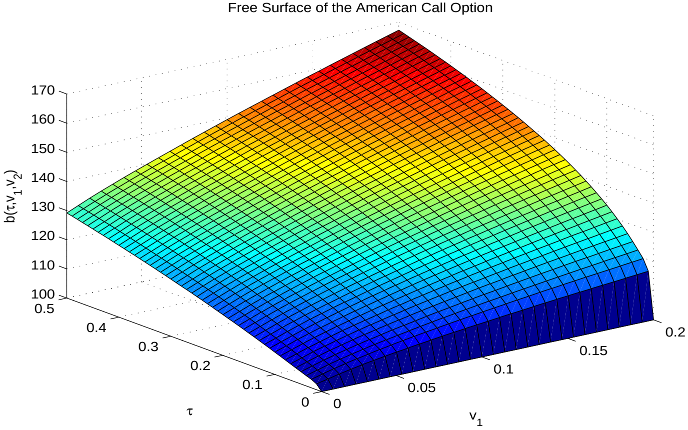

Figure: Early Exercise Surface of the American Call option when $v_2 = 0.67\%$, $\rho_{13} = 0.5$ and $\rho_{24} = 0.5$. All other parameters are as presented in Table 1.1.

Effects of Stochastic Volatility on the Early Exercise Boundary

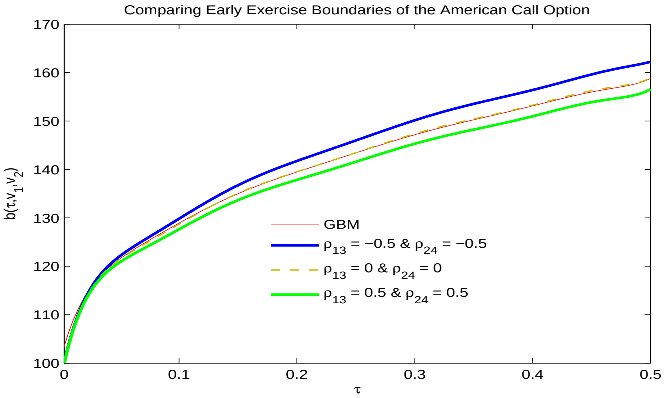

Figure: Exploring the effects of stochastic volatility on the early exercise boundary of the American call option for varying correlation coefficients when $\sigma_{GBM} = 0.3742$, $v_1 = 6\%$ and $v_2 = 8\%$. All other parameters are provided in Table 1.1.

Early Exercise Boundary Comparisons

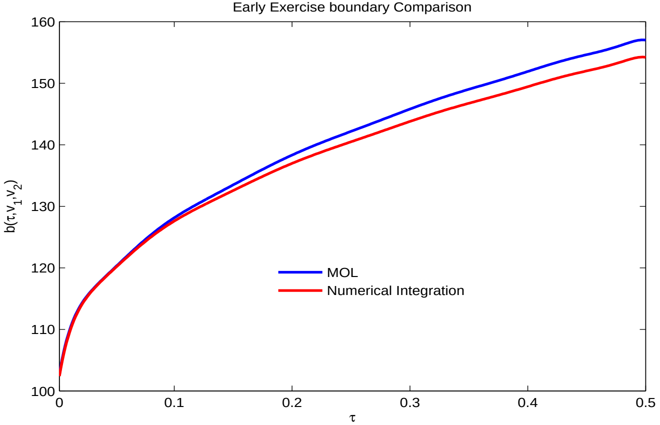

Figure: Comparing early exercise boundaries from the MOL and Numerical integration approach when the two instantaneous variances are fixed. Here, $v_1 = 0.67\%$, $v_2 = 13.33\%$, $\rho_{13} = 0.5$ and $\rho_{24} = 0.5$ with all other parameters as given in Table 1.1.

Price Comparisons

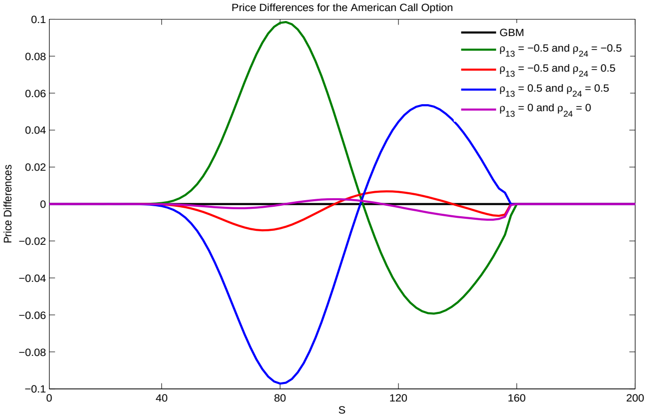

Figure: Option prices from the Geometric Brownian motion minus option prices from the Stochastic volatility model for varying correlation coefficients. Here, $\sigma_{GBM} = 0.3742$, $v_1 = 6\%$ and $v_2 = 8\%$ with all other parameters provided in Table 1.1.

Price Comparisons

S	Numerical Integration	MOL	GBM
60	0.2036	0.2029	0.1850
80	2.4088	2.400	2.4154
100	9.8082	9.7918	9.9452
120	23.1069	23.0920	23.3006
140	40.4756	40.4686	40.5922
160	60	60	60
180	80	80	80
200	100	100	100

Table: American call option price comparisons when $v_1 = 0.67\%$, $v_2 = 13.33\%$, $\rho_{13} = 0.5$, $\rho_{24} = 0.5$. We have taken GBM volatility to be $\sigma_{GBM} = 0.3741657$ and this is found by using the formula $\sigma_{GBM} = \sqrt{\theta_1 + \theta_2}$.

Summary

- We have derived the integral representation of an American call option when the underlying asset is driven by two stochastic variance processes.
- An explicit form of the transition density function has been provided.
- We approximated the three dimensional early – exercise boundary by a multivariate log – linear function.
- Numerical results and comparisons with alternative methods have been presented.

Possible Extensions

- Incorporating more than two stochastic volatility processes.
- Generalizing to Multiple assets under Multiple stochastic volatility.
- Performing empirical studies on multifactor models.