Convergence of price and sensitivities in Carr's randomization approximation globally and near barrier

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Motivation

Main objects: barrier options and first touch options with continuous monitoring under Lévy processes such as VG model, KoBoL (CGMY being a subclass), NIG, β-class

Most efficient tools:

I. approximations of pure jump Lévy processes of infinite intensity by diffusions with Poisson jumps

II. Carr's randomization approximation and operator form of the Wiener-Hopf method (BBL-methodology for short)

Qualitative differences between I and II

Near the boundary,

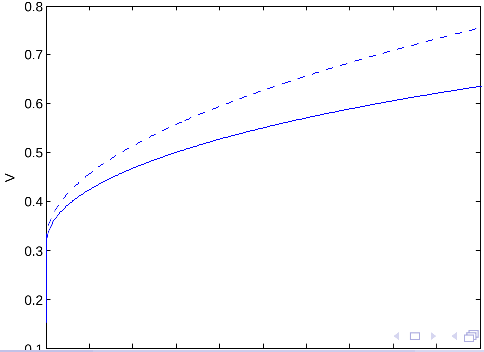
I. approximations of pure jump Lévy processes of infinite intensity by diffusions with Poisson jumps produces value functions of class C^∞ up to the barrier; in particular, sensitivities are bounded;
II. BBL-methodology produces rather irregular shapes near the boundary, and, typically, larger deltas and gammas

Price of the down-and-out barrier option, a process of finite variation, drift from the boundary

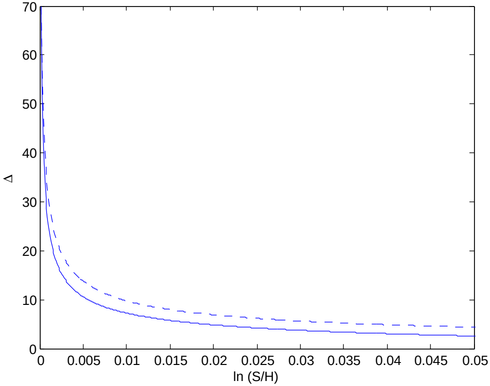
Process from S. Asmussen, D. Madan and M. Pistorius, *Pricing Equity Default Swaps under an approximation to the CGMY Lévy Model*, J. Comput. Finance **11** (2008), pp. 79–93.

KoBoL (a.k.a. CGMY) $\lambda_- = -10$, $\lambda_+ = 2$, $c = 0.5$, $\nu = 0.5$, $\mu \simeq 0.0295$; $r = 0.05$.

Solid line: Carr's randomization. Dashes: two-term asymptotic formula



Delta of the down-and-out barrier option, a process of finite variation, drift from the boundary



Tomorrow, there will be two talks:

Mitya Boyarchenko will explain the technical details and efficiency of the BBL-methodology, in application not only to single barrier options but double barrier options as well

Marco de Innocentis will demonstrate the theoretical results about the shapes of price and sensitivities, and the zoology of possible shapes

However, for practical purposes, it is useful to know if numerical methods based on the BBL-methodology really produce the correct shapes in the limit

The aims of the talk:

to prove that

1. Carr's randomization approximation converges to the true price in an appropriate Hölder norm (pointwise convergence was proved by Mitya Boyarchenko in 2008)
2. the leading term of asymptotics of the price in Carr's randomization approximation converges to the leading term of asymptotics of the price
3. the same holds for sensitivities
4. Richardson extrapolation of an arbitrary order is justified

Outline

Lévy processes: general definitions

L, infinitesimal generator, and ψ, characteristic exponent of $X = (X_t)$:

$$E\left[e^{i\xi X_t}\right] = e^{-t\psi(\xi)}, \quad Le^{i\xi x} = -\psi(\xi)e^{i\xi x}$$

Hence, $L = -\psi(D)$ is the PDO with the symbol $-\psi(\xi)$. In 1D,

$$Lu(x) = (2\pi)^{-1}\int_{\mathbb{R}} e^{ix\xi}(-\psi(\xi))\hat{u}(\xi)d\xi.$$

Explicit formulas (Lévy-Khintchine) are (also in 1D)

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{\mathbb{R}\setminus 0}(1 - e^{iy\xi} + iy\xi\mathbb{1}_{|y|<1}(y))F(dy),$$

$$Lu(x) = \frac{\sigma^2}{2}u''(x) + \mu u'(x) + \int_{\mathbb{R}\setminus 0}(u(x+y) - \mathbb{1}_{|y|<1}(y)yu'(x) - u(x))F(dy),$$

where $F(dy)$, the Lévy density, satisfies

$$\int_{\mathbb{R}\setminus 0}\min\{|y|^2, 1\}F(dy) < \infty.$$

Examples of Lévy processes

a) KoBoL model (a.k.a. CGMY model and extended Koponen's family)

$$\psi(\xi) = -i\mu\xi + \Gamma(-\nu)\cdot\left[c_+((-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu) + c_-(\lambda_+^\nu - (\lambda_+ + i\xi)^\nu)\right], \quad (1)$$

where $\nu \in (0,2)$, $\nu \neq 1$, $c_\pm > 0$, $\lambda^- < 0 < \lambda^+$.

b) *Normal Tempered Stable Lévy processes* (NTS)

$$\psi(\xi) = -i\mu\xi + \delta\cdot\left[\left(\alpha^2 - (\beta + i\xi)^2\right)^{\nu/2} - (\alpha^2 - \beta^2)^{\nu/2}\right], \quad (2)$$

where $\nu \in (0,2)$, $\alpha > |\beta| > 0$, $\delta > 0$ and $\mu \in \mathbb{R}$. With $\nu = 1$, NTS is NIG.

c) VG model

$$\psi(\xi) = -i\mu\xi + c_+[\ln(-\lambda_- - i\xi) - \ln(-\lambda_-)] + c_-[\ln(\lambda_+ + i\xi) - \ln(\lambda_+)], \quad (3)$$

where $c_\pm > 0$, $\lambda^- < 0 < \lambda^+$, $\mu \in \mathbb{R}$.

Carr's randomization or analytical method of lines

Most efficient method for calculation of prices $V(T,x)$ of barrier options and American options with finite time horizon T: discretize time to expiry $(0=)t_0 < t_1 < \cdots < t_N (= T)$ but not the space variable.

The equivalent probabilistic version: **Carr's randomization**

Carr's randomization, cont-d

For American options

- put, BM - Carr and Faguet (1994), Carr (1998)
- Lévy processes and general payoff functions - Boyarchenko and Levendorskiǐ (2002)
- proof of convergence for wide classes of Markov processes - Bouchard, El Karoui, Touzi (2006)

For barrier options

- efficient procedure - Kudryavtsev and Levendorskiǐ (2007), M.Boyarchenko and Levendorskiǐ (2008);
- proof of convergence for Lévy processes of Types B and C - M.Boyarchenko (2008)
- in regime-switching Lévy models - M.Boyarchenko and S.Boyarchenko (2009)

Carr's randomization: down-and-out option

For the barrier option with payoff $G(X_T)_+$, barrier h and no rebate, the result is a sequence of stationary boundary problems, equivalently, a sequence of perpetual barrier options with the same barrier.

Algorithm of Carr's randomization approximation produces the sequence of approximations to the option value: $V^s(x) = V(t_{N-s}, x)$, $s = 0, 1, \dots, N$.

1. Set $V^0(x) = G(X_T)_+$
2. For $s = 0, 1, \dots, N-1$, set $\Delta_s = t_{s+1} - t_s$, $q^s = r + \Delta_s^{-1}$, and calculate

$$V^{s+1}(x) = \mathbb{E}^x \left[\int_0^{\tau_h^-} e^{-q^s t} \Delta_s^{-1} V^s(X_t) dt \right],$$

where τ_h^- is the hitting time of $(-\infty, h]$.

Need to calculate expressions of the form

$$V(x; g; h) := \mathbb{E}\left[\int_0^{\tau_h^-} e^{-qt} g(x + X_t) dt\right]$$

Standard approach: use the Wiener-Hopf factorization

EPV-operators and Wiener-Hopf factorization

Supremum and infimum processes

- $\overline{X}_t = \sup_{0\le s\le t} X_s$ - the supremum process
- $\underline{X}_t = \inf_{0\le s\le t} X_s$ - the infimum process

Normalized EPV operators under X, $\overline{X}$, and $\underline{X}$:

- $$(\mathcal{E}_q g)(x) := q\mathbb{E}^x \left[\int_0^{+\infty} e^{-qt} g(X_t)dt\right]$$
- $$(\mathcal{E}_q^+ g)(x) := q\mathbb{E}^x \left[\int_0^{+\infty} e^{-qt} g(\overline{X}_t)dt\right],$$
- $$(\mathcal{E}_q^- g)(x) := q\mathbb{E}^x \left[\int_0^{+\infty} e^{-qt} g(\underline{X}_t)dt\right].$$

Examples

Brownian Motion: EPV-operators are of the form

$$\mathcal{E}_q^+ u(x) = \beta^+ \int_0^{+\infty} e^{-\beta^+ y} u(x+y)dy,$$

$$\mathcal{E}_q^- u(x) = (-\beta^-) \int_{-\infty}^0 e^{-\beta^- y} u(x+y)dy,$$

where $\beta^- < 0 < \beta^+$ are the roots of $q + \psi(-i\beta) = 0$.

Kou's model: EPV-operators are of the form

$$\mathcal{E}_q^+ u(x) = \sum_{j=1,2} a_j^+ \beta_j^+ \int_0^{+\infty} e^{-\beta_j^+ y} u(x+y)dy,$$

$$\mathcal{E}_q^- u(x) = \sum_{j=1,2} a_j^- (-\beta_j^-) \int_{-\infty}^0 e^{-\beta_j^- y} u(x+y)dy,$$

where $\beta_2^- < \lambda^- < \beta_1^- < 0 < \beta_1^+ < \lambda^+ < \beta_2^+$ are the roots of the "characteristic equation" $q + \psi(-i\beta) = 0$, and $a_j^\pm > 0$ are constants.

Wiener-Hopf factorization formula

Three versions:

1. Let $T_q \sim \mathrm{Exp}(q)$ be the exponential random variable of mean q^{-1}, independent of process X. For $\xi \in \mathbb{R}$,

$$\mathbb{E}[e^{i\xi X_{T_q}}] = \mathbb{E}[e^{i\xi \overline{X}_{T_q}}]\mathbb{E}[e^{i\xi \underline{X}_{T_q}}];$$

2. For $\xi \in \mathbb{R}$,

$$\frac{q}{q+\psi(\xi)} = \phi_q^+(\xi)\phi_q^-(\xi),$$

   where $\phi_q^\pm(\xi)$ admits the analytic continuation into the corresponding half-plane and does not vanish there

3. $\mathcal{E}_q = \mathcal{E}_q^- \mathcal{E}_q^+ = \mathcal{E}_q^+ \mathcal{E}_q^-$.

3 is valid in appropriate function spaces, and can be either proved as 1 or deduced from 2 because $\mathcal{E}_q = q(q+\psi(D))^{-1}$, $\mathcal{E}_q^\pm = \phi_q^\pm(D)$.

Perpetual barrier down-and-out option

Theorem 1

Let g be a measurable locally bounded function satisfying certain conditions on growth at ∞. Then for any h,

$$V(x; g; h) = q^{-1}\mathcal{E}_q^{-}\mathbb{1}_{(h,+\infty)}\mathcal{E}_q^{+}g(x). \tag{4}$$

The proof is based on the following result

Lemma 2

Let X and T_q be as above. Then

(a) *the random variables $\overline{X}_{T_q}$ and $X_{T_q} - \overline{X}_{T_q}$ are independent (deep!); and*

(b) *the random variables $\underline{X}_{T_q}$ and $X_{T_q} - \overline{X}_{T_q}$ are identical in law.*

Price of the down-and-out option with payoff $G(X_T)$

Laplace transform w.r.t. T

$$\begin{aligned}\hat{V}(q,x) &= \int_0^{+\infty} e^{-qt}\mathbb{E}^x\left[e^{-rt}G(X_t)\mathbb{1}_{\{\tau_h^- > t\}}\right]dt \\ &= \mathbb{E}^x\left[\int_0^{\tau_h^-} e^{-(q+r)t}G(X_t)dt\right].\end{aligned}$$

Inverse Laplace transform and WHF: for $\sigma > 0$,

$$\begin{aligned}V(T,x) &= \frac{e^{-rT}}{2\pi i}\int_{\mathrm{Re}\, q=\sigma} e^{qT}q^{-1}\left(\mathcal{E}_q^-\mathbb{1}_{(h,+\infty)}\mathcal{E}_q^+G\right)(x)dq \\ &= \frac{e^{-rT}}{2\pi i(-T)^k}\int_{\mathrm{Re}\, q=\sigma} e^{qT}\partial_q^k\left(q^{-1}\mathcal{E}_q^-\mathbb{1}_{(h,+\infty)}\mathcal{E}_q^+\right)G(x)dq.(5)\end{aligned}$$

Price in Carr's randomization approximation

Boundary problem for V^{s+1}, $s \geq 0$

$$\frac{V^s(x) - V^{s+1}(x)}{T/N} + (L - r)V^{s+1}(x) = 0, \quad x > h, \tag{6}$$

s.t. $V^{s+1}(x) = 0, x \leq h$

Boundary problem for $\hat{V}(z, \cdot) := \sum_{s=0}^{\infty} z^s V^{s+1}$

$$(r + (N/T)(1 - z) - L)\hat{V}(z, x) = (N/T)G(x), \quad x > h, \tag{7}$$

s.t. $\hat{V}(z, x) = 0, x \leq h$

Price in Carr's randomization approximation, cont-d

Set $q = r + (N/T)(1 - z)$. If $q > 0$ is large,

$$\hat{V}(z, x) = (N/T)q^{-1}\mathcal{E}_q^- \mathbb{1}_{(h,+\infty)}\mathcal{E}_q^+ G(x). \tag{8}$$

Both sides being analytical functions of z, (8) holds for z in the disc $|z| < e^{rT/N}$. Hence, we can recover

$$V^N(x) = \frac{1}{2\pi i}\int_{|z|=1} z^{-N}\Delta_t^{-1}q^{-1}\mathcal{E}_q^- \mathbb{1}_{(h,+\infty)}\mathcal{E}_q^+ G(x)dz. \tag{9}$$

Change the variable $z = 1 + T(r - q)/N$, $dz = -\Delta_t dq$, and denote by $\mathcal{C}_N$ the contour $\{q \mid |1 + (T/N)(r - q)| = 1\}$. Then

$$V^N(x) = -\frac{1}{2\pi i}\int_{\mathcal{C}_N}(1 + T(r - q)/N)^{-N}q^{-1}\mathcal{E}_q^- \mathbb{1}_{(h,+\infty)}\mathcal{E}_q^+ G(x)dq. \tag{10}$$

Convergence of Carr's randomization

We integrate by parts in (10) to get

$$V^N(x) = -\frac{N^k(N-k-1)!}{(N-1)!}\cdot\frac{1}{2\pi i(-T)^k} \times \int_{\mathcal{C}_N}(1+T(r-q)/N)^{k-N}\partial_q^k\left(q^{-1}\mathcal{E}_q^-\mathbb{1}_{(h,+\infty)}\mathcal{E}_q^+\right)G(x)dq, \qquad (11)$$

and compare with (5):

$$V(T,x) = \frac{1}{2\pi i(-T)^k}\int_{\mathrm{Re}\,q=\sigma}e^{(q-r)T}\partial_q^k\left(q^{-1}\mathcal{E}_q^-\mathbb{1}_{(h,+\infty)}\mathcal{E}_q^+\right)G(x)dq.$$

Idea of the proof of convergence

1. Fix $\epsilon \in (0, 1/2)$ and k. For $|q| \le N^{\epsilon}$,

$$(1 + T(r - q)/N)^{k-N} = e^{(q-r)T} + o(1)$$

2. If k is sufficiently large, then the Fourier transform of

$$f_k(q, x) = \partial_q^k \left(q^{-1} \mathcal{E}_q^{-} \mathbb{1}_{(h,+\infty)} \mathcal{E}_q^{+} \right) G(x)$$

   admits a bound

$$|\hat{f}_k(q, \xi)| \le C|q|^{-1-s}|\xi|^{-1-\rho}$$

   where $s, \rho > 0$ (with some modification for processes of finite variation with the drift pointing from the barrier)

3. In the region $|q| \le N^{\epsilon}$, the contour $\mathcal{C}_N$ can be deformed into $\operatorname{Re} q = \sigma$.

Convergence of Carr's randomization approximations to the leading term and sensitivities

First, derive explicit formulas using WHF - for the leading term and sensitivities, exact formulas and in Carr's randomization approximation

After that, use an argument similar to the argument above.

Richardson extrapolation

Assume that, for large N, and several multiples of N, $N_j = n_j N$, $j = 1, 2, \ldots, m$, Carr's randomization approximations $V^{N_j}(x)$ are calculated.

Then we can use the extrapolation formula of order m in $1/N$-line, to calculate $W(0) := V(T, x)$ given $W(1/N_j) := V^{N_j}(x)$, $j = 1, 2, \ldots, m$.

In particular, the linear extrapolation gives $V(T, x) = 2V^{2N}(x) - V^N(x)$

The quadratic extrapolation used by Carr:
$V(T, x) = 0.5V^N(x) - 4V^{2N}(x) + 9.5V^{3N}(x)$.

The reader can easily derive extrapolations of higher order; however, we have found that quite often even linear extrapolation with moderate $N = 40$ for $T = 1$ gives the result with the relative error less than 1 percent.