ROM SIMULATION

Exact Moment Simulation using Random Orthogonal Matrices

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Outline

- $\boxed{\textit{ROM Simulation}}$
- ROM Simulation Properties
- Applications in Finance

Introduction: Motivation

Simulating a $N(\boldsymbol{\mu}_n, \boldsymbol{\Sigma}_n)$ multivariate sample $\mathbf{X}_{mn}$ (Monte Carlo):

- Simulate standard normal $\mathbf{Z}_{mn} = (z_{ij})$, where $z_{ij} = \Phi^{-1}(p_{ij})$, $p_{ij} \sim U[0,1]$ and Φ is the standard normal c.d.f
- Set $\mathbf{X}_{mn} = \mathbf{1}_m \boldsymbol{\mu}_n' + \mathbf{Z}_{mn}\mathbf{A}_n$ where $\mathbf{A}_n'\mathbf{A}_n = \boldsymbol{\Sigma}_n$ and $\mathbf{1}_m = (1, \ldots, 1)'$

Problem: Error in sample moments

$$\begin{aligned} M(\mathbf{X}_{mn}) &= m^{-1}\mathbf{1}_m'\mathbf{X}_{mn} \\ V(\mathbf{X}_{mn}) &= m^{-1}(\mathbf{X}_{mn} - \mathbf{1}_m M(\mathbf{X}_{mn}))'(\mathbf{X}_{mn} - \mathbf{1}_m M(\mathbf{X}_{mn})) \end{aligned}$$

- That is, particularly when m is small

$$\boxed{M(\mathbf{X}_{mn}) \neq \boldsymbol{\mu}_n' \quad \text{and} \quad V(\mathbf{X}_{mn}) \neq \boldsymbol{\Sigma}_n}$$

ROM Simulations

Solution: Replace $\mathbf{Z}_{mn}$ with an L-matrix satisfying

$$\mathbf{L}'_{nm}\mathbf{L}_{mn} = \mathbf{I}_n \quad \text{and} \quad \mathbf{1}'_m\mathbf{L}_{mn} = \mathbf{0}'_n$$

- e.g Apply Gram-Schmidt (GS) orthogonalisation

Exact Mean-Covariance Sample: ROM Simulations

$$\boxed{\mathbf{X}_{mn} = \mathbf{1}_m\boldsymbol{\mu}'_n + m^{\frac{1}{2}}\mathbf{Q}_m\mathbf{L}_{mn}\mathbf{R}_n\mathbf{A}_n}$$

where

- $\mathbf{A}'_n\mathbf{A}_n = \boldsymbol{\Sigma}_n$
- $\mathbf{R}_n$ is a random orthogonal matrix (ROM)
- $\mathbf{Q}_m$ is a permutation satisfying $\mathbf{1}'_m\mathbf{Q}_m = \mathbf{1}'_m$

ROM Simulated Paths

Different ROMs, applied to the same L-matrix, lead to different samples:

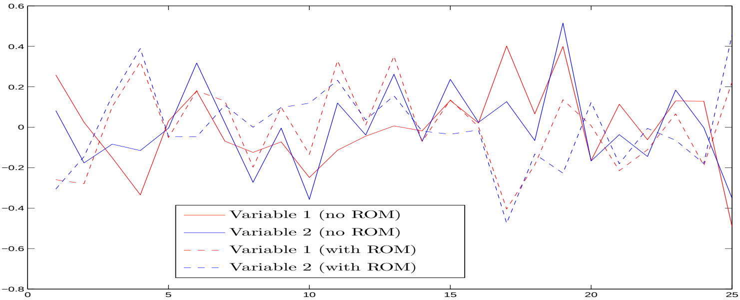

Figure: Both simulations based on the same multivariate normal sample. The solid lines show the paths from the first simulation (no ROM) and the dashed lines show the second simulation (with ROM). Path correlation is 0.75.

L-Matrix Types

Parametric: Orthogonalisation of a zero mean parametric sample

- ROM Simulation $\longrightarrow$ (small) adjustment to Monte Carlo

Data Specific: Orthogonalise a collection of data (mean deviations)

- ROM Simulation $\longrightarrow$ infinitely many "historical samples"

Deterministic: Orthogonalise linearly independent vectors

$$\mathbf{v}_j = (\underbrace{0, \ldots, 0}_{j-1}, \underbrace{1, -1, \ldots, 1, -1}_{2k}, 0, \ldots, 0)' \xrightarrow{GS} \mathbf{L}_{mn}^{k}$$

- $\mathbf{L}_{mn}^{1}$ relates to Helmertian (1876) matrices; $k > 1$ are new
- ROM Simulation $\longrightarrow$ target higher multivariate moments

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Multivariate Skewness and Kurtosis

We employ the multivariate measures introduced by Mardia (1970)

- Skewness:

$$\tau_M(\mathbf{X}_{mn}) = m^{-2}\sum_{i=1}^{m}\sum_{j=1}^{m}\left\{(\mathbf{x}_i - \boldsymbol{\mu}_n')V(\mathbf{X}_{mn})^{-1}(\mathbf{x}_j - \boldsymbol{\mu}_n')'\right\}^3$$

- Kurtosis:

$$\kappa_M(\mathbf{X}_{mn}) = m^{-1}\sum_{i=1}^{m}\left\{(\mathbf{x}_i - \boldsymbol{\mu}_n')V(\mathbf{X}_{mn})^{-1}(\mathbf{x}_i - \boldsymbol{\mu}_n')'\right\}^2$$

- Key Property is invariance under non-singular affine transformations:

$$\begin{aligned}\tau_M(\mathbf{X}_{mn}) &= \tau_M(\mathbf{1}_m\mathbf{b}_n' + \mathbf{X}_{mn}\mathbf{B}_n)\\ \kappa_M(\mathbf{X}_{mn}) &= \kappa_M(\mathbf{1}_m\mathbf{b}_n' + \mathbf{X}_{mn}\mathbf{B}_n)\end{aligned}$$

Skewness and Kurtosis of ROM Simulations

ROM simulations are (random) affine transformations of L-matrices

Parametric: Multivariate normal case

$$\begin{aligned} \mathbb{E}[\tau_M(\mathbf{L}_{mn})] &= n(n+1)(n+2)m^{-1} \\ \mathbb{E}[\kappa_M(\mathbf{L}_{mn})] &= n(n+2)(m-1)(m+1)^{-1} \end{aligned}$$

Data Specific: ROM simulation moments identical to historical data

Deterministic: When $k = 1$

$$\begin{aligned} \tau_M(\mathbf{L}^1_{mn}) &= n\left[(m-3) + (m-n)^{-1}\right] \\ \kappa_M(\mathbf{L}^1_{mn}) &= n\left[(m-2) + (m-n)^{-1}\right] \end{aligned}$$

- When $k > 1 \longrightarrow$ Moments available numerically
- Calibrate m (and k) for "moment targeting"

Orthogonal Matrices

Recall: ROM simulation equation $\boxed{\mathbf{X}_{mn} = \mathbf{1}_m\boldsymbol{\mu}' + m^{\frac{1}{2}}\mathbf{Q}_m\mathbf{L}_{mn}\mathbf{R}_n\mathbf{A}_n}$

- $\mathbf{Q}_m$ are (cyclic) permutation matrices
- $\mathbf{R}_n$ are combinations of the following random orthogonal matrix types

(1) Sign Matrices: $\mathbf{R}_n = \mathrm{diag}\left\{(-1)^{d_1}, \ldots, (-1)^{d_n}\right\}$

- where $d_k \sim Bin(1, p_k)$

(2) Upper Hessenberg Rotations: $\mathbf{R}_n = \mathbf{G}_n(\theta_1)\mathbf{G}_n(\theta_2)\ldots\mathbf{G}_n(\theta_{n-1})$

- where $\mathbf{G}_n(\theta_j)$ are Givens (1958) rotations

From a random skew-symmetric matrix, satisfying $\mathbf{S}_n' = -\mathbf{S}_n$, we form

(3) Cayley (1846) Rotations: $\mathbf{R}_n = (\mathbf{I}_n - \mathbf{S}_n)^{-1}(\mathbf{I}_n + \mathbf{S}_n)$

(4) Exponential Rotations: $\mathbf{R}_n = \exp(\mathbf{S}_n)$

ROM Simulation Densities: Rotation Effects

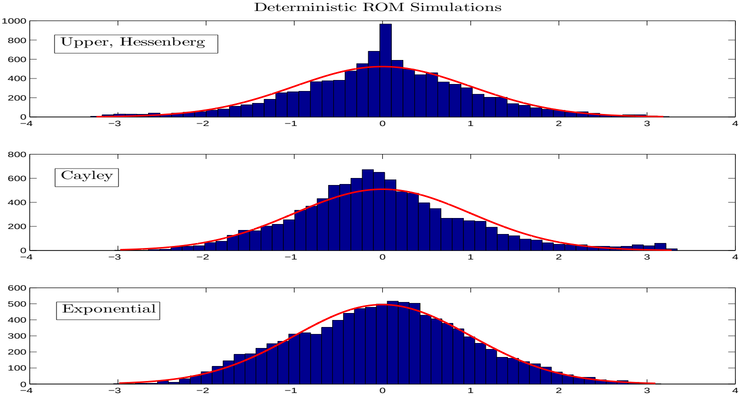

Figure: Histograms for the 5th marginal density of a ROM simulation involving deterministic L-matrices ($m = 15$, $n = 10$, $k = 2$). Over 10,000 observations are used for each simulation. Marginals are compared to scaled normal distributions.

ROM Simulation Densities: Sign Matrix Effects

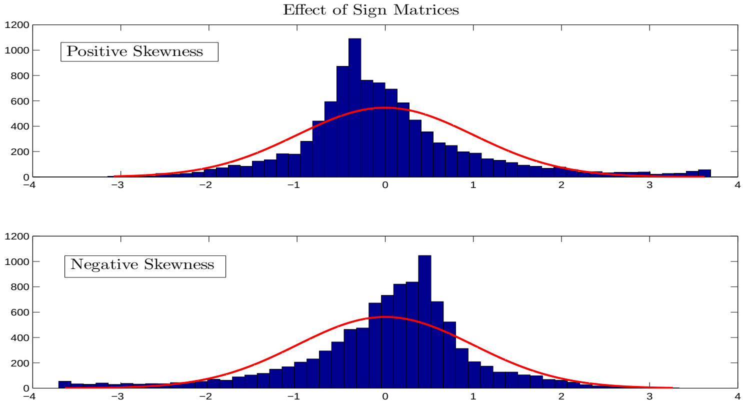

Figure: Histograms for the 5th marginal distribution of two ROM simulations involving deterministic L-matrices ($m = 15$, $n = 10$, $k = 0$) and Cayley matrices. In the lower figure sign matrices are used to induce negative skew.

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Portfolio Value-at-Risk (VaR)

The level $\mathrm{VaR}(\alpha, h)$ represents the h-day portfolio loss that we assume is exceeded with probability α

- If portfolio h-day returns are normally distributed then

$$\mathrm{VaR}(\alpha, h) = -\Phi^{-1}(\alpha)\sigma_h - \mu_h$$

- Portfolio losses are typically non-normal (leptokurtic)
- Common to simulate losses (returns) and calculate empirical quantiles

$$\mathrm{VaR}(\alpha, h) = -q_\alpha(\mathbf{r}_m^h)$$

where $\mathbf{r}_m^h$ is a vector containing m scenarios for portfolio h-day returns

- Historical "simulation" is particularly popular

An MSCI Country Index Portfolio

We consider a portfolio whose assets individually track the $n = 45$ country indices in the MSCI All Country World Index

- Portfolio return r_π is weighted average of asset returns $\mathbf{x} = (x_1, \ldots, x_n)$

$$r_\pi = \pi(\mathbf{x}) = \sum_{i=1}^{n} w_i x_i \quad \text{where} \quad \sum_{i=1}^{n} w_i = 1$$

- Correlations between the assets returns are key
- Multivariate kurtosis is also important
- Let $\mathbf{X}_{mn}$ denote m (simulated) scenarios on the n assets, then

$$\mathbf{X}_{mn} \xrightarrow{\pi} \mathbf{r}_m$$

where $\mathbf{r}_m$ is a vector of portfolio scenarios

Scenario Generation Techniques for VaR

Two year historical window $\longrightarrow$ target moments $\boldsymbol{\mu}_n$, $\mathbf{S}_n$, κ_M

We generate scenarios $\mathbf{X}_{mn}$ using six different techniques:

(1) - (3) ROM Simulations

- Type I L-matrices used to target κ_M
- Three types of random orthogonal matrices

(4) Multivariate Normal (Monte Carlo and analytic)

(5) Multivariate Student-t (Monte Carlo, $\nu = 6$ degrees of freedom)

(6) Historical simulation (two years of observations)

Note: Limited data are available for historical quantile estimation

Estimate portfolio VaR $\longrightarrow$ roll window forward and repeat

Daily VaR for Equally Weighted Portfolio

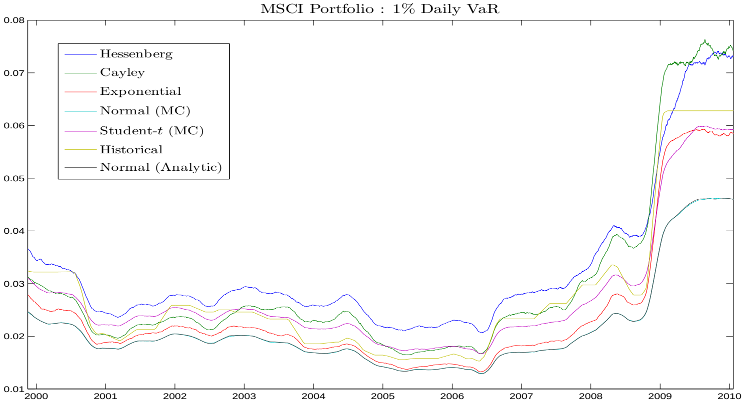

Figure: Evolution of daily VaR, given as a percentage of the portfolio value

Daily VaR Exceedances

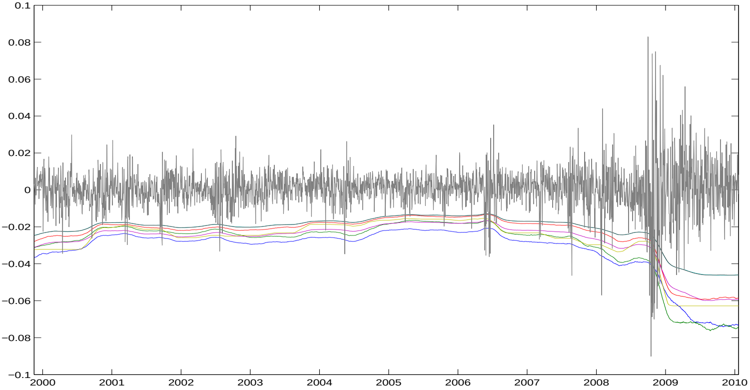

Figure: Equally weighted daily portfolio returns, plotted with negative VaR

Proportion of Exceedances for Equally Weighted Portfolio

	Exceedances		Coverage Tests	
	m_1	m_1/m	Uncond.	Cond.
Hessenberg	24	0.89%	0.36	6.03%
Cayley	32	1.18%	0.86	4.45%
Exponential	56	2.07%	23.89	47.49%
Normal (MC)	73	2.70%	53.80	84.48%
Student-t (MC)	41	1.52%	6.27	24.23%
Historical	37	1.37%	3.31	18.39%
Normal (Analytic)	74	2.73%	55.84	85.82%

Table: m is the total number of out-of-sample returns (2647 daily). The 1% critical values are 6.63 for the Unconditional test and 9.21 for the Conditional test

Summary

- Exact mean-covariance samples are generated from L-matrices
- Orthogonal matrices can be used to randomise these samples
- Different simulation properties associated with different types of orthogonal matrix
- Target higher order moments with deterministic L-matrices
- ROM simulation is a useful scenario generation technique
  - Portfolio Value-at-Risk
  - Portfolio allocation optimisations