Simple Improvement Method for Upper Bound of American Option

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Introduction

Numerical Methods for Pricing American Option

1. Closed-Form Solution: It is difficult to find a closed-form solution.
2. Lattice Methods: When the condition is simple, the lattice methods give good approximated solutions.
3. Monte Carlo Simulation: When the condition is complicated, the Monte Carlo simulation is practical.

Monte Carlo simulation

Lower Bound: A stopping time gives a lower bound.
▷ The least-square method gives a good stopping time.
Longstaff and Schwartz (2001)
Upper Bound: A martingale gives an upper bound.
▷ Can we find a good martingale?

Setup

The saving account is the numeraire. All prices are discounted prices.

$T \in \mathbf{N}$	:	Fixed Maturity
$(\Omega, \mathcal{F}, P, \{\mathcal{F}_k; k = 0, 1, \ldots, T\})$	:	Filtered probability space
$S_k \ (k = 0, 1, \ldots, T)$	:	Price Process of Risky Asset
$H_k \ (k = 0, 1, \ldots, T)$	:	Payoff of American Option
$V_k \ (k = 0, 1, \ldots, T)$	:	Price of American Option

Assumption

- P is a unique equivalent martingale measure.
- $\mathcal{F}_k$ is a natural filtration generated by S. We write $E_k[\cdot] = E[\cdot|\mathcal{F}_k]$.
- H is an adapted process.

Definition 1 A supersolution is a supermartingale X satisfying

$$X_k \geq H_k, \quad k = 0, 1, \ldots, T-1$$

and the maturity condition, that is, $X_T = H_T$.

V is a minimum supersolution.

▷ Any supersolution is an upper bound process of the American option.

Main Problem

Suppose that a supersolution U is given. Note that U_0 is an upper bound.
Suppose that the lower bound process L of the continuation value is given.

$$L_k \leq \underbrace{E_k[V_{k+1}]}_{\text{continuation value}} \leq V_k \leq U_k, \quad k < T,$$

$$L_T = H_T \, (= U_T) \, .$$

We want to improve the upper bound U_0 in the Monte Carlo simulation.
Chen and Glasserman (2007) proposes an iterative method.

1. Using the supersolution U, a martingale is given by $M_k^U = \sum_{t=1}^{k} (U_t - E_{t-1}[U_t]) \, , \; k = 0, 1, \ldots, T$.
2. Using the martingale M, a new supersolution (= upper bound process) is given by $U_k^M = E_k[\max_{k \leq t \leq T}(H_t - M_t)] + M_k, \; k = 0, 1, \ldots, T$.

- The iterative improvement converges to the true price.
- The calculation of the conditional expectation is necessary at all times and all states for the Doob decomposition.
- The lower bound process is not used.

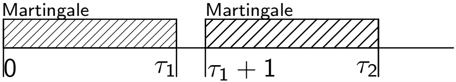
We want to find a computationally-efficient improvement method using L.

Basic Result

Let $\mathcal{T}^k$ be the set of the stopping times whose values are greater than or equal to k.

Theorem 1 Let $\tau_1, \tau_2 \in \mathcal{T}^0$ and $\tau_1 \leq \tau_2$. Suppose that V satisfies the martingale property in $[0, \tau_1] \cup [\tau_1 + 1, \tau_2]$, that is,

$$V_k = E_k[V_{k+1}], \quad k \in [0, \tau_1 - 1] \cup [\tau_1 + 1, \tau_2 - 1].$$



Let

$$w(\tau_1, \tau_2) = E[\max(H_{\tau_1}, E_{\tau_1}[U_{\tau_2}])].$$

Then

$$V_0 \leq \underbrace{w(\tau_1, \tau_2)}_{\text{New Upper Bound}} \leq U_0.$$

The problem is to find an appropriate pair of stopping times (τ_1, τ_2).

Methods 1, 2

We use the mathematical convention the minimum over the empty set is ∞, $\min(\emptyset) = +\infty$.

Lemma 1 Let $\tau_1^* = \min\{k \geq 0 | H_k > L_k\} \wedge T$. Then V satisfies the martingale property in $[0, \tau_1^*]$, that is, $V_k = E_k[V_{k+1}]$ for $k \in [0, \tau_1^* - 1]$.

Corollary 1 Let $w_L^1 = w(\tau_1^*, \tau_1^*)$. Then $V_0 \leq w_L^1 \leq U_0$.

Corollary 2 Let $w_L^2 = w(\tau_1^*, (\tau_1^* + 1) \wedge T)$. Then $V_0 \leq w_L^2 \leq w_L^1 \leq U_0$.

- $w_L^2 \leq w_L^1$. $\cdots$ w_L^2 is a better upper bound than w_L^1.
- When $U_k = E_k[\max_{k \leq t \leq T}(H_t - M_t)] + M_k$,
  $w_L^1 = E[\max_{\tau_1^* \leq t \leq T}(H_t - M_t)]$,
  $w_L^2 = E[\max\left(H_{\tau_1^*}, E_{\tau_1^*}[\max_{(\tau_1^*+1) \wedge T \leq t \leq T}(H_t - M_t)] + M_{\tau_1^*}\right)]$.
  - w_L^1 includes no conditional expectation per path.
  - w_L^2 requires only one conditional expectation per path.
  - The iterated method requires T conditional expectations per path.

The calculations of w_L^1 and w_L^2 spend much less time than that of the iterative method. The proposed methods are more efficient.

Method 3

Lemma 2 Let $\tau_2^* = \min\{k > \tau_1^* | H_k > L_k\} \wedge T$. Then V satisfies the martingale property in $[\tau_1^* + 1, \tau_2^*]$, that is, $V_k = E_k[V_{k+1}]$ for $k \in [\tau_1^* + 1, \tau_2^* - 1]$.

Corollary 3 Let

$$w_L^3 = w(\tau_1^*, \tau_2^*).$$

Then

$$V_0 \leq w_L^3 \leq w_L^2 \leq U_0.$$

- w_L^3 is the best upper bound of the three proposed methods.
- When $U_k = E_k[\max_{k \leq t \leq T}(H_t - M_t)] + M_k$,

$$w_L^3 = E[\max\left(H_{\tau_1^*}, E_{\tau_1^*}[\max_{\tau_2^* \leq t \leq T}(H_t - M_t)] + M_{\tau_1^*}\right)].$$

We have to calculate τ_2^*. When the lower bound process can be calculated by an analytic formula, the calculation of τ_2^* is not time-consuming and then the amount of calculation of w_L^3 is as much as that of w_L^2.

Lower Bound Effect

Lemma 3 Let $\tau_a, \tau_b \in \mathcal{T}^0$. If $\tau_a \leq \tau_b$, then

$$w(\tau_a, \tau_a) \geq w(\tau_b, \tau_b),$$
$$w(\tau_a, (\tau_a + 1) \wedge T) \geq w(\tau_b, (\tau_b + 1) \wedge T).$$

Proposition 1 Let L^a and L^b be lower bound processes. Suppose that

$L_k^a \leq L_k^b, \quad k = 0, 1, \ldots, T.$ L^b is a better lower bound process than L^a.

Then

$$w_{L^a}^1 \geq w_{L^b}^1,$$
$$w_{L^a}^2 \geq w_{L^b}^2,$$
$$w_{L^a}^3 \geq w_{L^b}^3.$$

The better a lower bound process is, the greater improvement of upper bound can be expected.

European Option Based Model

Let V^E be the price process of the European option satisfying $V_k^E = E_k[H_T]$.

$$M_k = V_k^E - V_0^E,$$
$$U_k = E_k[\max_{k\le t\le T}(H_t - M_t)] + M_k.$$

We call this model the European option based model.

Proposition 2 Consider the European option based model with $L = V^E$. If $\tau \in \mathcal{T}^0$ satisfies $\tau < \tau_1^*$, then

$$U_0 = w(\tau,\tau) = w(\tau,(\tau+1)\wedge T).$$

If L is smaller than V^E, it fails to improve the upper bound.

Proposition 3 In the European option based model, if $L = V^E$, then we have

$$U_0 = w_L^1 \ge w_L^2 = w_L^3.$$

- V^E is the worst lower bound which may improve the upper bound.
- We check whether $w_L^2 = w_L^3$ generated by V^E can improve the upper bound by the numerical analysis.

Simulation Condition

- The price process is given by the Black Scholes Model, that is,

$$S_k = S_{k-1} \exp\left(-\frac{\sigma^2}{2}\triangle t + \sigma\sqrt{\triangle t}\xi_k\right), \ k = 1, \ldots, T,$$

$$H_k = \max\left(Ke^{-rk\triangle t} - S_k, 0\right), \ k = 0, 1, \ldots, T,$$

where $\xi_1, \ldots, \xi_T$ are independent and standard normally distributed.

- Let $L = V^E$, that is,

$$L_k = K\Phi(d(k, T, K, 0)) - S_k\Phi(d(k, T, K, \sigma^2)), \quad k = 0, 1, \ldots, T - 1$$

where $\Phi(\cdot)$ is the standard normal distribution function and

$$d(k, T, K, r) = \frac{1}{\sigma\sqrt{(T-k)\triangle t}}\left(\log\frac{K}{S_k} - \left(r - \frac{1}{2}\sigma^2\right)(T-k)\triangle t\right).$$

- $S_0 = 100, \ r = 0.04, \ \sigma = 0.3, \ \triangle t = 0.01, \ T = 50, 100, 150.$
- The number of paths for calculating the expectation is 2,500.
- The number of paths for calculating the conditional expectation is 500.
- The antithetic sampling is used.

Better Lower Bound

- Let $L_T^a = L_T^b = H_T$ and for $k = 0, 1, \ldots, T-1$,

$$L_k^a = \max_{t_0 > k} \left(\sup_{\tau \in \mathcal{T}_{t_0, T}} E_k[H_\tau] \right), \quad L_k^b = \sup_{\tau \in \mathcal{T}^{k+1}} E_k[H_\tau]$$

  where $\mathcal{T}_{t_0, T}$ is the set of the stopping times whose values are t_0 or T.

- L^a can be calculated by the analytic formula since

$$\begin{aligned} \sup_{\tau \in \mathcal{T}_{t_1, T}} E_{t_0}[H_\tau] &= K\Phi(d(t_0, t_1, S_{t_1}^*, 0)) - S_{t_0}\Phi(d(t_0, t_1, S_{t_1}^*, \sigma^2)) \\ &+ K\Phi_2(-d(t_0, t_1, S_{t_1}^*, 0), d(t_0, T, K, 0); \frac{t_1 - t_0}{T - t_0}) \\ &- S_{t_0}\Phi_2(-d(t_0, t_1, S_{t_1}^*, \sigma^2), d(t_0, T, K, \sigma^2); \frac{t_1 - t_0}{T - t_0}) \end{aligned}$$

  where $\Phi_2(\cdot, \cdot; \rho)$ is the standard bivariate normal distribution function.
  $S_{t_1}^*$ is a solution of

$$K\Phi(d(t_1, T, K, 0)) - S_{t_1}^*\Phi(d(t_1, T, K, \sigma^2)) = Ke^{-rt_1 \triangle t} - S_{t_1}^*.$$

- L^b is used in order to estimate the maximum improvement.
  Note that L^b can be calculated by the lattice tree.

Numerical Result (Lower Bound Effect)

$K = 90$ (OTM)

T	U_0	w_L^3	$w_{L^a}^3$	$w_{L^b}^3$	V_0
50	3.471(0.002)	3.469(0.002)	3.465(0.002)	3.463(0.002)	3.460
100	5.861(0.006)	5.856(0.006)	5.845(0.006)	5.821(0.006)	5.806
150	7.618(0.010)	7.612(0.009)	7.584(0.010)	7.542(0.010)	7.509

$K = 100$ (ATM)

T	U_0	w_L^3	$w_{L^a}^3$	$w_{L^b}^3$	V_0
50	7.612(0.004)	7.608(0.004)	7.596(0.004)	7.581(0.004)	7.579
100	10.334(0.009)	10.327(0.008)	10.299(0.009)	10.254(0.009)	10.223
150	12.274(0.015)	12.268(0.013)	12.225(0.014)	12.123(0.014)	12.064

$K = 110$ (ITM)

T	U_0	w_L^3	$w_{L^a}^3$	$w_{L^b}^3$	V_0
50	13.704(0.006)	13.696(0.006)	13.671(0.006)	13.629(0.006)	13.616
100	16.253(0.013)	16.241(0.011)	16.195(0.012)	16.089(0.012)	16.037
150	18.151(0.019)	18.145(0.016)	18.066(0.018)	17.888(0.019)	17.782

1. $U_0 > w_L^3 > w_{L^a}^3 > w_{L^b}^3 > V_0$. $\cdots$ $L \leq L^a \leq L^b$, Lower Bound Effect
2. $w_{L^b}^3 > V_0$. The proposed methods can improve the upper bound efficiently but cannot attain the true price.

Bermudan Max Call Option on five Assets

- Suppose that the price processes S^i for $i = 1, \ldots, 5$ are given by $S_0^i = S_0$,

$$S_k^i = S_{k-1}^i \exp\left(\left(-q - \frac{\sigma^2}{2}\right)\triangle t + \sigma\sqrt{\triangle t}\xi_k^i\right), \quad k = 1, \ldots, T.$$

- $H_k = \max\left(\max_{1 \le i \le 5} S_k^i - Ke^{-rk\triangle t}, 0\right), \; k = 0, 1, \ldots, T.$
- $K = 100, \; q = 0.1, \; \sigma = 0.2, \; r = 0.05, \; T = \frac{3}{\triangle t}.$
- The number of paths for calculating the expectation and the conditional expectation are 250, 000 and 500 respectively.
- An upper bound process is generated by the single European options.
- A lower bound process is based on the least square method.
- The true price V_0 is the point estimate in Broadie and Glasserman (2004).

$\triangle t$	S_0	U_0	w_L^1	w_L^2	V_0
1/2	90	17.572 (0.015)	16.866 (0.015)	16.496 (0.014)	16.474
1/2	100	28.038 (0.019)	26.645 (0.020)	25.997 (0.019)	25.920
1/2	110	39.721 (0.023)	37.545 (0.024)	36.615 (0.023)	36.497
1/3	90	17.804 (0.014)	17.033 (0.014)	16.677 (0.013)	16.659
1/3	100	28.296 (0.018)	26.855 (0.018)	26.264 (0.017)	26.158
1/3	110	39.956 (0.021)	37.816 (0.022)	36.994 (0.021)	36.782

Concluding Remarks

We have proposed a simple and computationally tractable improvement method for the upper bound of American options.

- The method is based on two stopping times. The stopping times are generated from a lower bound process of the continuation value.
- A better, namely higher lower bound process gives a greater improvement of the upper bound.
- Our method can be used together with the approximation of lower bound process by the least square method.

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