Credit Risk Premia and quadratic BSDEs with a single jump

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Introduction

Credit Risk Premium c defined via maximal expected utility

$$V^{\xi}(0) = \sup \left\{ EU(0 + G_T + \xi) \right\} = V^{\xi 1_{\{\tau > T\}}}(c).$$

ξ defaultable contingent claim, τ default time
$U(x) = -\exp(-\eta x)$ with risk aversion coef $\eta > 0$

Solve the BSDE

$$Y_t = \xi - \int_t^T Z_s dW_s - \int_t^T U_s dM_s + \int_t^T f(s, Z_s, U_s) ds$$

W Brownian Motion,
M martingale with **one single jump**
f has a **quadratic growth** in z.

Outline

Default Free market

Prices of k risky assets given by :

$$dS_t^i = S_t^i(\alpha_i(t)dt + \sigma_i(t)dW_t), \quad i = 1, \dots, k,$$

- W a d-dimensional Brownian motion, $\mathcal{F}_t$ its natural filtration,
- $\alpha_i(t)$ ith component of predictable map $\alpha : \Omega \times [0, T] \to \mathbb{R}^k$.
- $\sigma_i(t)$ ith row of predictable map $\sigma : \Omega \times [0, T] \to \mathbb{R}^{k \times d}$

No arbitrage : standard hypotheses, in particular : market price of risk $\vartheta = \sigma^*(\sigma\sigma^*)^{-1}\alpha$ bounded.

Defaultable Contingent Claim

- Default time τ : random time, no stopping time under $\mathcal{F}$
- Available information : $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(1\!\!1_{\tau \leq t})$, makes τ stopping time
- A defaultable contingent Claim F :

$$F = X_1 1_{\tau > T} + X_2(\tau) 1_{\tau \leq T}$$

X_1 is a bounded $\mathcal{F}_T$-measurable random variable,
X_2 is a $\mathcal{F}$-predictable process.

Hypothesis (H) :

Immersion property : Any square integrable $(\mathcal{F}, P)$-martingale is a square integrable $(\mathcal{G}, P)$-martingale

Notations :

- $D_t = 1_{\{\tau \leq t\}}$.
- K, the $(\mathcal{F}_t)$-*compensator* of D, such that $K_0 = 0$.
- $M_t = D_t - \int_0^t (1 - D_{s-}) dK_s$ $(\mathcal{G}_t)$-martingale.

k_s a $(\mathcal{F}_t)$-predictable non-negative bounded process and A a $(\mathcal{F}_t)$-predictable increasing process, $\{0, 1\}$-valued, such that

$$dK_s = k_s ds + dA_s.$$

Remark : A and M do not have any common jump.

Examples

1. $\tau_1 = \inf\{t, \int_0^t k_s ds > \Theta\}$, where k a bounded non-negative $(\mathcal{F}_t)$-predictable process and Θ an exponentially distributed random variable independent of the Brownian Motion W.

$$dK_t = k_t dt$$

2. $\tau = \tau_1 \wedge \tau_2$ with $\tau_2 = \inf\{t \geq 0 : S_t^i \leq a \text{ for one } 1 \leq i \leq k\}$ and τ_1 defined as previously.

$$dK_t = k_t dt + d\mathbb{1}_{\tau_2 \leq t}, \text{ in other words } A_t = \mathbb{1}_{\tau_2 \leq t}$$

Defaultable Zero-Coupon Bond

- Payoff of a defaultable zero-coupon bond is 1 if $\tau > T$, and 0 otherwise.
- Arbitrage free dynamics of a defaultable zero-coupon bond :

$$d\rho_t = \rho_{t-}(a_t dt + c_t dW_t - dM_t)$$

where (a_t, c_t) are $\mathbb{R} \times \mathbb{R}^d$-valued measurable processes.

Investment strategies

An investment strategy (p, q) leads to the following gain at time t :

$$G_t^{p,q} = \int_0^t (p_s \vartheta_s + q_s a_s) ds + \int_0^t (p_s + q_s c_s) dW_s - \int_0^t q_s dM_s.$$

Let $\mathcal{A}$ denote the set of *admissible strategies* (p, q), satisfying

$$E \int_0^T |p_s|^2 ds + E \int_0^T |q_s|^2 ds < \infty.$$

Constraints are imposed to the investor.
A strategy $(p_t, q_t) \in C_t = C_t^1 \times C_t^2 \subset \mathbb{R}^k \sigma_t \times \mathbb{R}$ satisfying

$$(0,0) \in C_t^1 \times C_t^2 \text{ for all t,} \qquad C_t^1 \text{ is closed,} \qquad C_t^2 \text{ is bounded.}$$

If the investor has a defaultable position F in his portfolio, then his maximal expected utility is given by

$$V^F(v) = \sup_{(p,q)\in\mathcal{A}} \left\{ EU(v+G_T^{p,q}+F) : \ (p_s, q_s) \in C_s \text{ for all } s \in [0, T] \right\}.$$

Consider the following BSDE

$$Y_t = \xi - \int_t^T Z_s dW_s - \int_t^T U_s dM_s + \int_t^T f(s, Z_s, U_s) ds$$

where ξ is a bounded $\mathcal{G}_T$-measurable random variable, and generator $f : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ satisfies the following property :

(P1) There exist 3 predictable functions l, j and m such that

$$f(s, z, u) = [l(s, z) + j(s, u)](1 - D_{s^-}) + m(s, z) D_{s^-},$$

- There exists a constant $L \in \mathbb{R}_+$ such that $\forall\, z, z' \in \mathbb{R}^d$

$$|l(s, z) - l(s, z')| + |m(s, z) - m(s, z')| \leq L(1 + |z| + |z'|)|z - z'|,$$

- $j \geq 0$, and j is Lipschitz on $(-K, \infty)$ for every $K \in \mathbb{R}_+$,
- $l(., 0)$, $m(., 0)$ and $j(., 0)$ are bounded, say by $\Lambda \in \mathbb{R}_+$.

In addition, we will sometimes assume that the generator f satisfies also

(P2) There exists a continuous increasing function γ such that for all $s \in [0, T]$ and $u, u' \in [-n, n]$, $n \in \mathbb{N}$,

$$|j(s,u) - j(s,u')| \leq \gamma(n)\sqrt{k_s}|u - u'|,$$

Notations : $(\mathcal{J}_t)$ be an arbitrary filtration.

- $\mathcal{H}^2(\mathcal{J}_t)$ the set of all $(\mathcal{J}_t)$-predictable processes X_t satisfying $E\int_0^T |X_t|^2 ds < \infty$
- $\mathcal{H}^\infty(\mathcal{J}_t)$ the set of essentially bounded $(\mathcal{J}_t)$-predictable processes.
- $\mathcal{R}^\infty(\mathcal{J}_t)$ the set of all bounded $(\mathcal{J}_t)$-optional processes.

Existence

Theorem

Let $F = \xi 1_{\{\tau > T\}} + \zeta 1_{\{\tau \leq T\}}$ where ξ and ζ be two bounded $\mathcal{F}_T$-measurable random variables, and let f be a generator satisfying **(P1)**. *Then there exists a solution $(Y, Z, U) \in \mathcal{R}^\infty(\mathcal{G}_t) \times \mathcal{H}^2(\mathcal{G}_t) \times \mathcal{H}^\infty(\mathcal{G}_t)$ of*

$$Y_t = F - \int_t^T Z_s dW_s - \int_t^T U_s dM_s + \int_t^T f(s, Z_s, U_s) ds.$$

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Proof

Main Idea : Construct a solution of BSDE starting from two continuous quadratic BSDEs with terminal conditions ξ and ζ.

1. Take a solution $(\widehat{Y}, \widehat{Z}) \in \mathcal{H}^\infty(\mathcal{F}_t) \times \mathcal{H}^2(\mathcal{F}_t)$ of the BSDE

$$\widehat{Y}_t = \zeta - \int_t^T \widehat{Z}_s dW_s + \int_t^T m(s, \widehat{Z}_s) ds.$$

2. Define the $(\mathcal{F}_t)$-predictable stopping time

$$\tau_A = \inf\{t \geq 0 : A_t = 1\}, \ ((\mathcal{F}_t)\text{-predictable})$$

with convention $\inf \emptyset = \infty$.

3. Let $(Y^g, Z^g) \in \mathcal{H}^\infty(\mathcal{F}_t) \times \mathcal{H}^2(\mathcal{F}_t)$ be a solution of the BSDE with generator l and terminal condition ξ. In particular, there exists a $K \in \mathbb{R}_+$ such that $\sup_{t\in[0,T]} Y_t^g \geq -K$, a.s.

3. Consider a BSDE with generator

$$\begin{aligned} h(s,y,z) &= f(s,z,(\widehat{Y}_s - y)1_{\{\tau_A \geq s\}}) + (\widehat{Y}_s - y)1_{\{\tau_A \geq s\}} k_s \\ &= l(s,z) + j(s,(\widehat{Y}_s - y)1_{\{\tau_A \geq s\}}) + (\widehat{Y}_s - y)1_{\{\tau_A \geq s\}} k_s. \end{aligned}$$

   and terminal condition $\psi = \xi 1_{\{\tau_A > T\}} + \zeta 1_{\{\tau_A \leq T\}}$. However h is not Lipschitz in y.

4. Introduce a modified generator h^a which is lipschitz.

$$h^a(s,y,z) = \begin{cases} h(s,y,z), & \text{if } y \geq -K \\ g(s,y,z) + j(s,\widehat{Y}_s + K)\left[1-(y+K)\right], & \text{otherwise} \end{cases}$$

   where $g(s,y,z) = l(s,z) + (\widehat{Y}_s - y)1_{\{\tau_A \geq s\}} k_s$.

5. Using a sandwich argument, show that there exist solutions of BSDEs with bounded terminal condition ξ and generator h denoted by $(Y^a, Z^a) \in \mathcal{H}^\infty(\mathcal{F}_t) \times \mathcal{H}^2(\mathcal{F}_t)$.

6. Finally, find a solution by setting

$$Y_t = \begin{cases} Y_t^a, & (\tau > t), \\ \widehat{Y}_t, & (\tau \leq t), \end{cases}$$

$$Z_t = \begin{cases} Z_t^a, & (\tau > t), \\ \widehat{Z}_t, & (\tau \leq t), \end{cases}$$

and

$$U_t = \begin{cases} \widehat{Y}_t - Y_t^a, & t \leq \tau, \\ 0, & t > \tau. \end{cases}$$

Existence II

Theorem

Let $(\bar{\zeta}(t))_{0\leq t\leq T}$ be a $(\mathcal{F}_t)$-predictable bounded process, such that $t \mapsto \bar{\zeta}(t)$ is almost surely right-continuous on $[0, T]$. Let f satisfy **(P1)** *and* **(P2)**. *Then there exists a solution $(Y, Z, U) \in \mathcal{R}^\infty(\mathcal{G}_t) \times \mathcal{H}^2(\mathcal{G}_t) \times \mathcal{H}^\infty(\mathcal{G}_t)$ of the BSDE*

$$\begin{aligned} Y_t &= \xi 1_{\{\tau>T\}} + \bar{\zeta}(\tau)1_{\{\tau\leq T\}} \\ &\quad - \int_t^T Z_s dW_s - \int_t^T U_s dM_s + \int_t^T f(s, Z_s, U_s)ds \end{aligned}$$

Proof

1. Assume that the process A is equal to zero, so $dK_s = k_s ds$.
2. Let τ_n, $n \in \mathbb{N}$, be the discrete approximation of the default time τ defined by

$$\tau_n(\omega) = \frac{k}{n} \quad \text{if} \quad \tau(\omega) \in \left] \frac{k-1}{n}, \frac{k}{n} \right], k \in \mathbb{Z}_+.$$

3. Observe that τ_n is a $(\mathcal{G}_t)$-stopping time.
4. Construct a Cauchy sequence (Y^n, Z^n, U^n) which converges to (Y, Z, U), solution of the BSDE.

Uniqueness

Theorem

Let ξ be a bounded $\mathcal{G}_T$-measurable random variable an f a generator satisfying **(P1)** *and* **(P2)**, *then the BSDE has a unique solution in $\mathcal{R}^\infty(\mathcal{G}_t) \times \mathcal{H}^2(\mathcal{G}_t) \times \mathcal{H}^\infty(\mathcal{G}_t)$.*

Sketch of the proof : Use a priori estimates.

Credit Risk Premia

- No tradable defaultable asset and so $C_t = C_t^1 \times \{0\}$.
- ξ a *bounded* $\mathcal{F}_T$-measurable random variable (value of a position if no default occurs).
- $V^{\xi}(v)$ and $V^{\xi 1_{\{\tau > T\}}}(v)$ maximal expected utilities of an investor with initial wealth v, and endowment ξ and $\xi 1_{\{\tau > T\}}$.

Definition

The indifference credit risk premium is the amount c defined as the unique real number satisfying

$$V^{\xi}(0) = V^{\xi 1_{\{\tau > T\}}}(c).$$

Remark : As U is exponential function, c doesn't depend on the initial wealth of the investor.

Proposition

The indifference credit risk premium satisfies

$$c = \bar{Y}_0,$$

where $(\bar{Y}, \bar{Z}, \bar{U})$ is the solution of the BSDE

$$\bar{Y}_t = \xi 1_{\{\tau \leq T\}} - \int_t^T \bar{Z}_s d\widehat{W}_s - \int_t^T \bar{U}_s dM_s + \int_t^T h(s, \bar{Z}_s, \bar{U}) ds, \quad (1)$$

with generator $h(t, z, u) = -\vartheta z + \frac{1}{\eta}(1 - D_s) k_s [e^{\eta u} - 1 - \eta u]$, $\frac{d\widehat{P}}{dP} = \mathcal{E}(\frac{1}{2}\eta \int_0^{\cdot} \gamma_s dW_s)_T$ and $\widehat{W}_t = W_t + \int_0^t \gamma_s ds$ is a Brownian motion with respect to $\widehat{P}$.

Analytic representation for a defaultable Put Option

We suppose that our financial market consists in one tradable asset with dynamics

$$dS_t = S_t \alpha dt + S_t \sigma dW_t.$$

Assume $k = d = 1$ and $C_t = \mathbb{R} \times \{0\}$ (no contraints and no defaultable asset).
Credit risk is the only source for market incompleteness.
Suppose the compensator K satisfies $dK_t = k(S_t)dt + dA_t$ (for ex. k is a positive continuous function).

Let u be the solution of

$$u_t + \frac{1}{2}\sigma^2 x^2 u_{xx} = 0, \quad u(T, x) = (C - x)^+.$$

Proposition

Conditionally to $S_t = x$ and $\tau > t$, the credit risk premium at time t of a defaultable put option with strike C and maturity $T > t$ is given by $v(t, x)$, where v is the solution of the following PDE

$$v_t + \frac{1}{2}\sigma^2 x^2 v_{xx} + \frac{k(x)}{\eta}(e^{\eta(u-v)} - 1) = 0, \quad v(T, x) = 0.$$

Remark : Notice that the PDE does not depend on the drift parameter μ, which is almost impossible to estimate in practice.

- Uses the fact that solutions of BSDEs can be represented in terms of solutions of PDEs
- Up to default time, the solution of the problem coincides with the solution of a well-chosen BSDE.

Conclusion

- Studied a new class of BSDE with 1 possible jump.
- Obtained Existence and Uniqueness for a BSDE with a jump at a random time and satisfying a quadratic growth condition.
- Derived BSDE representations of the maximal expected exponential utility of investors endowed with defaultable contingent claims.
- Same arguments can be used to determine indifference value of the defaultable contingent claim, allow to write hedging formulas in terms of derivatives of the indifference value w.r.t. the market price process.

The paper is available on Arxiv http ://fr.arxiv.org/abs/0907.1221

Thank you for your attention.