BiSPL Bio Imaging Signal Processing Lab

Compressive MUSIC for Diffuse Optical Tomography using Joint Sparsity

Jong Chul Ye, Ph.D
In collaboration with
Okyun Lee, Jong Min Kim, Yoram Bresler

KOREA ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY
KAIST
한국과학기술원
SINCE 1971

Bio-Imaging & Signal Processing Lab.,
Dept. Bio & Brain Engineering
Korea Advanced Institute of Science & Technology (KAIST)

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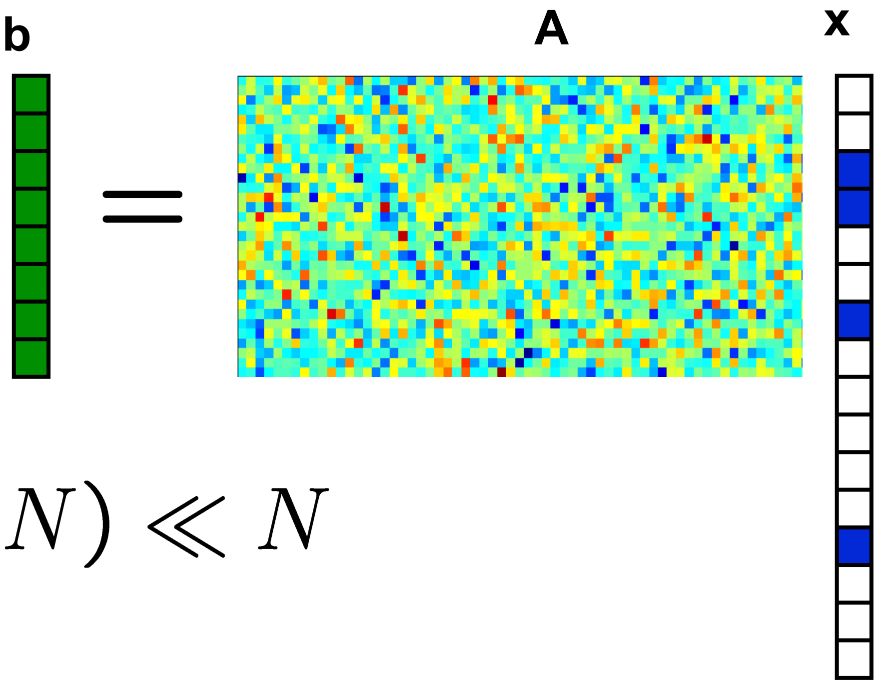
- *Compressive MUSIC (CS-MUSIC)*
- *Applications to DOT*
- *Summary*

Compressed sensing

- *Incoherent projection*
- *Underdetermined system*
- *Sparse unknown vector*

$M \times 1$
measurements



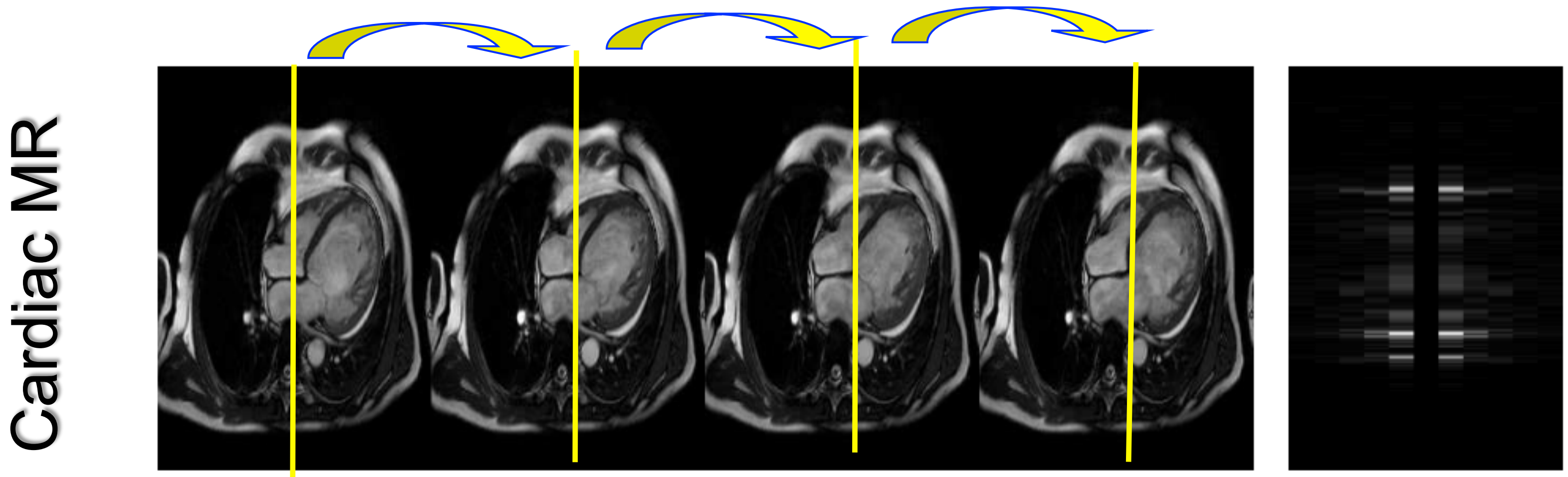
$N \times 1$
sparse
signal

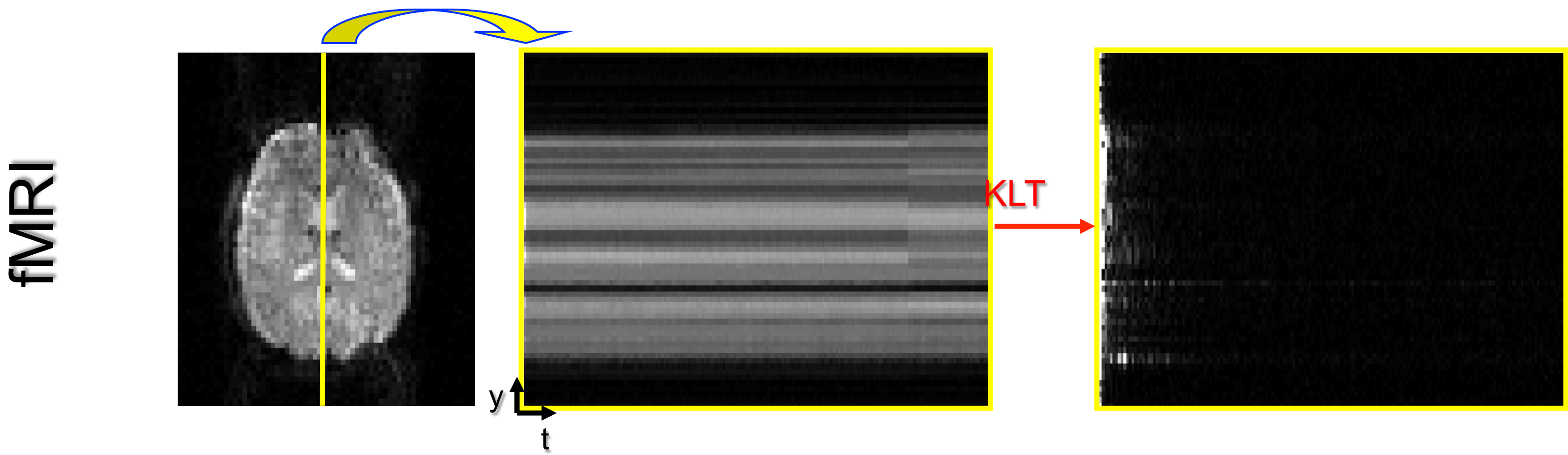
K
non-zeros

$$M \approx K \log(N) \ll N$$

Figure from Dr. Dror Baron

Compressed Sensing Dynamic MRI

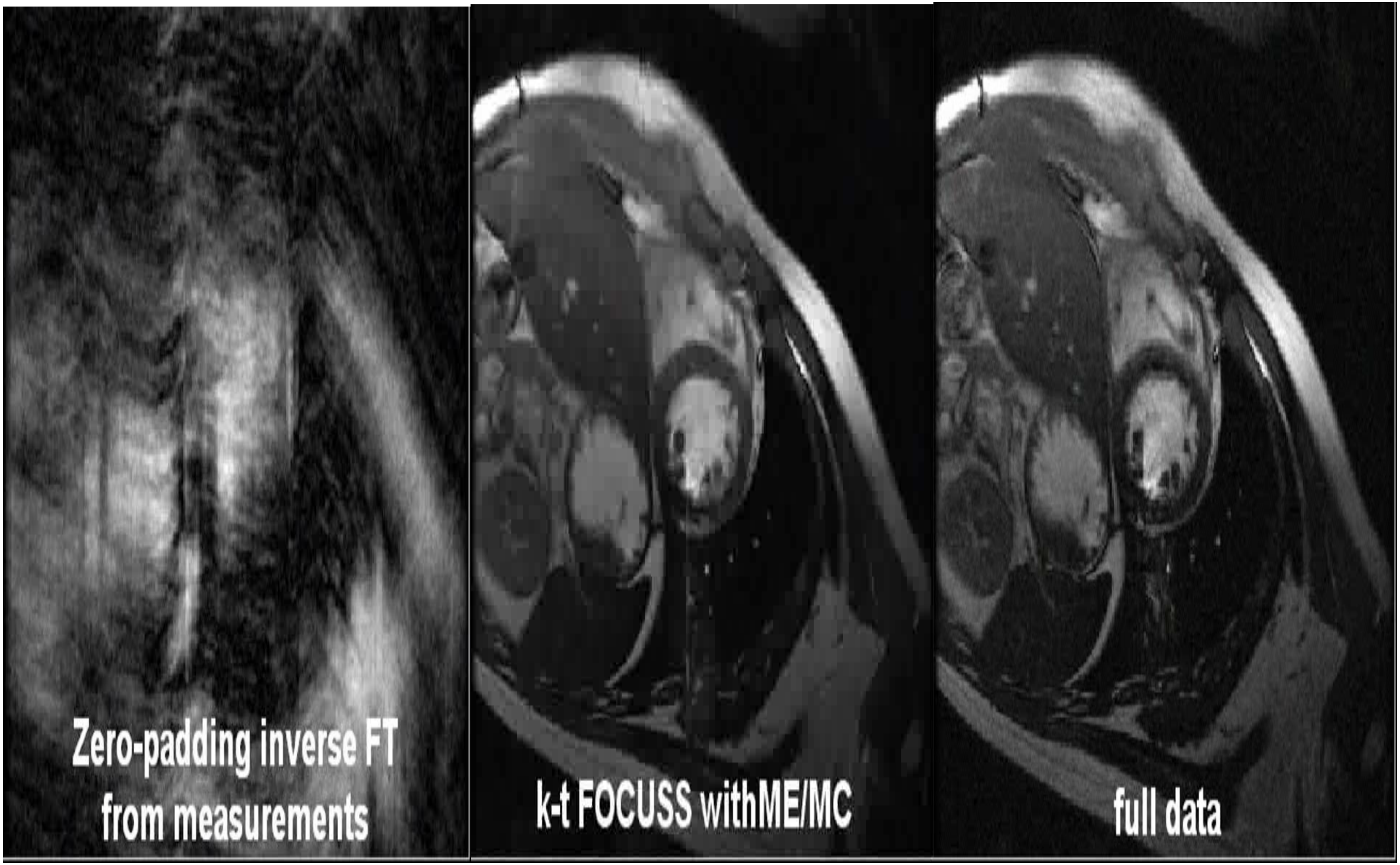




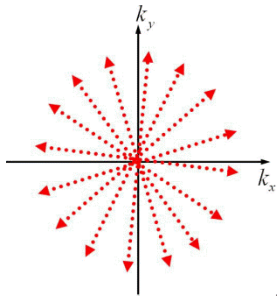
Jung et al, PMB 2007, MRM 2009, 2010, 2011, IJIT, 2011

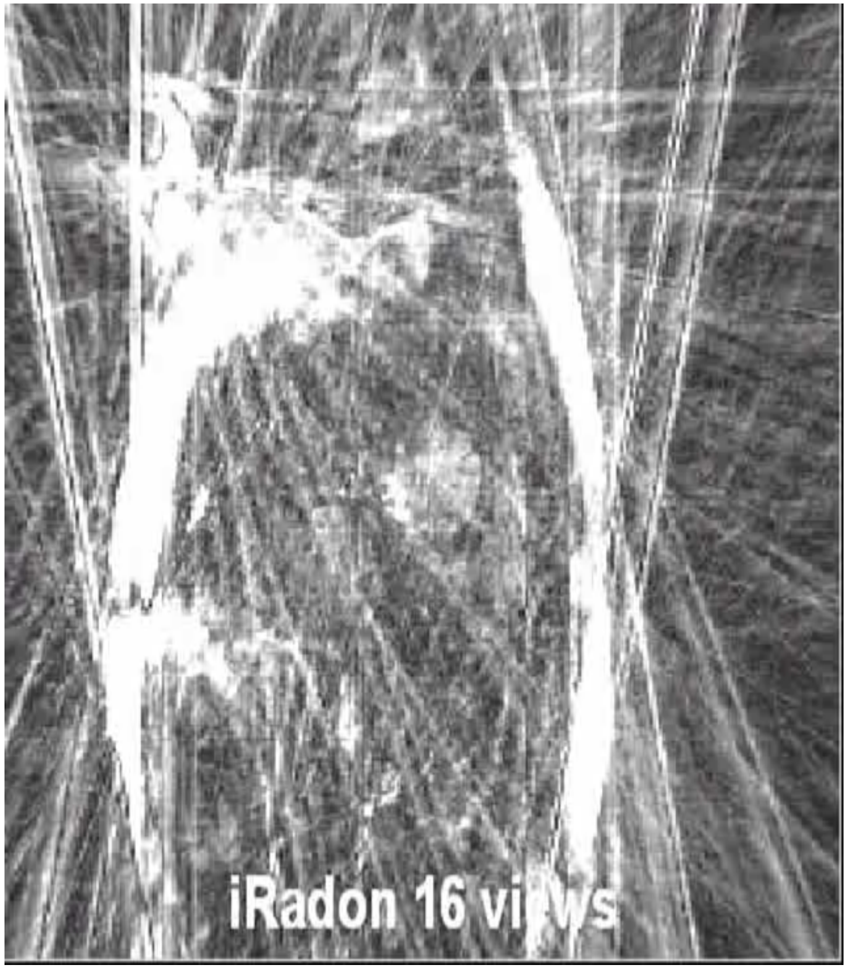
k-t FOCUSS for dynamic CS-MRI

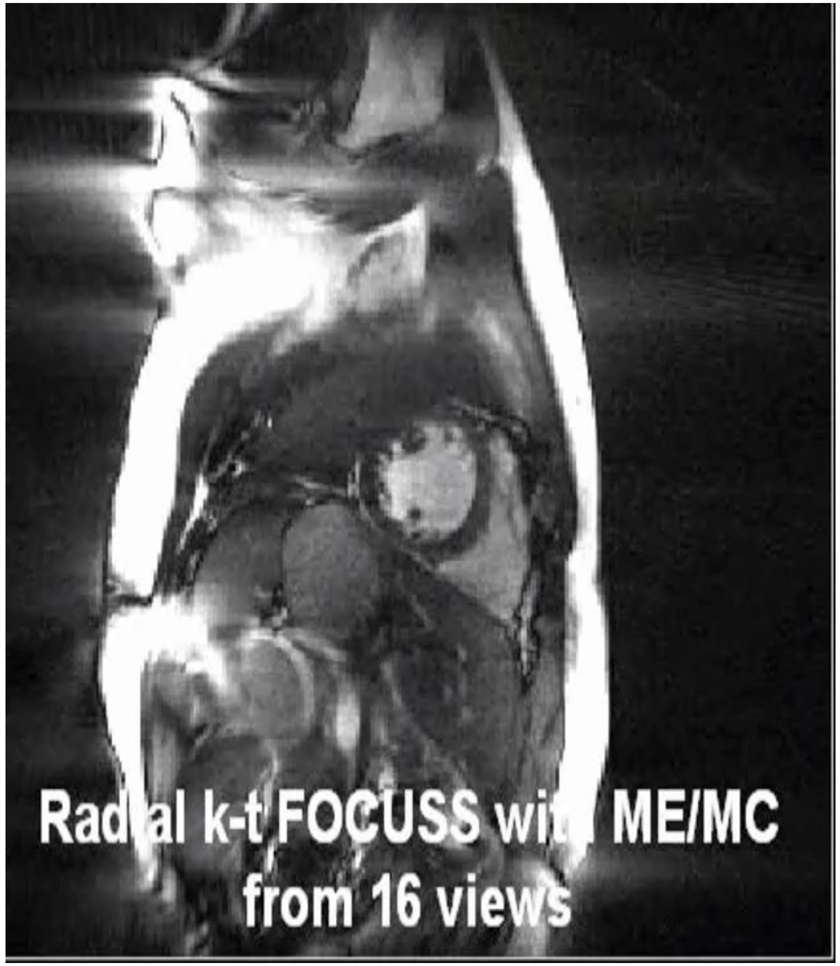
(Jung et al, PMB, 20007, Jung et MRM, 2009)



Radial k-t FOCUSS
(Jung et MRM, 2010)



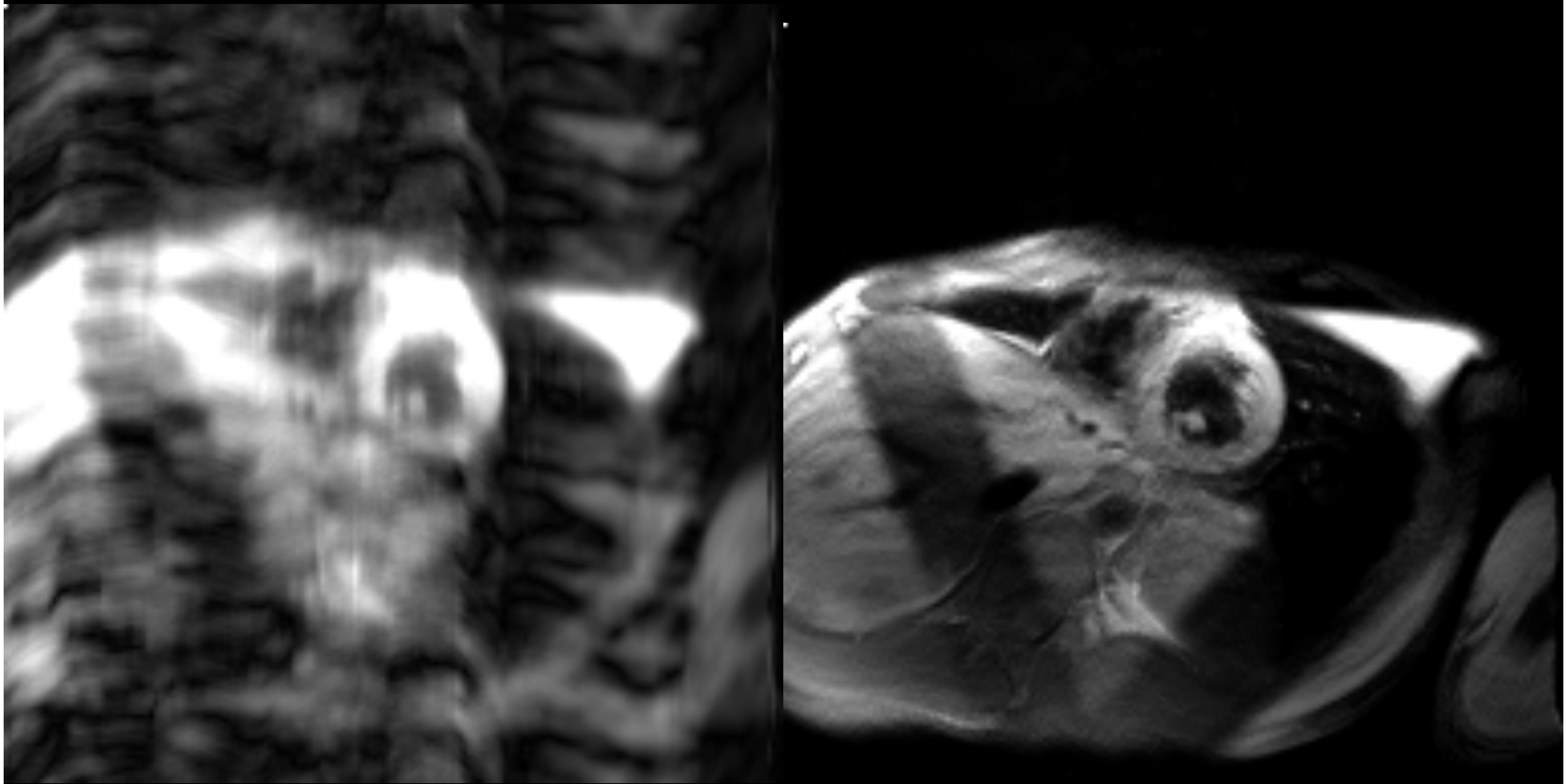




k-t FOCUSS for Cardiac T2 Mapping (Feng et al, 2011)

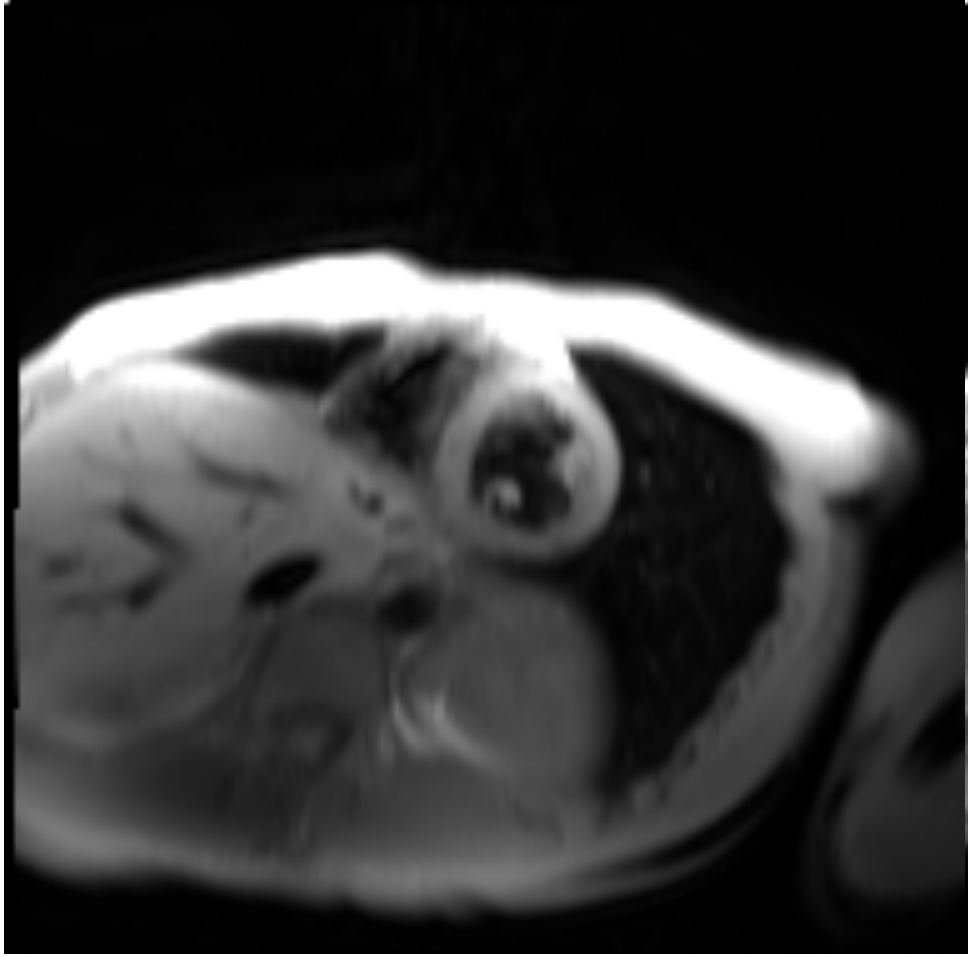
6 x accel. Conventional method

6 x accel. k-t FOCUSS

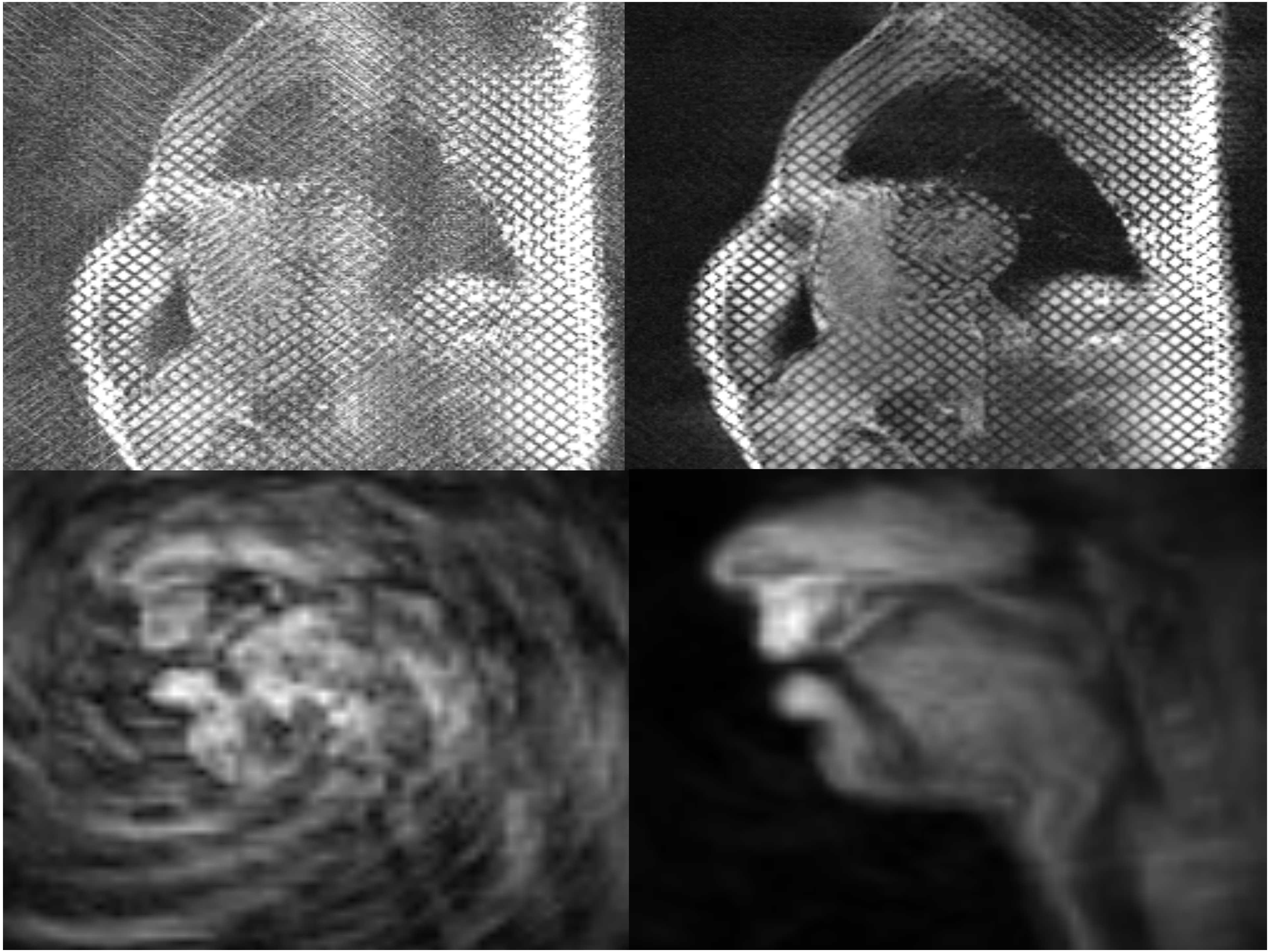


k-t FOCUSS for Cardiac T2 Mapping (Feng et al, 2011)

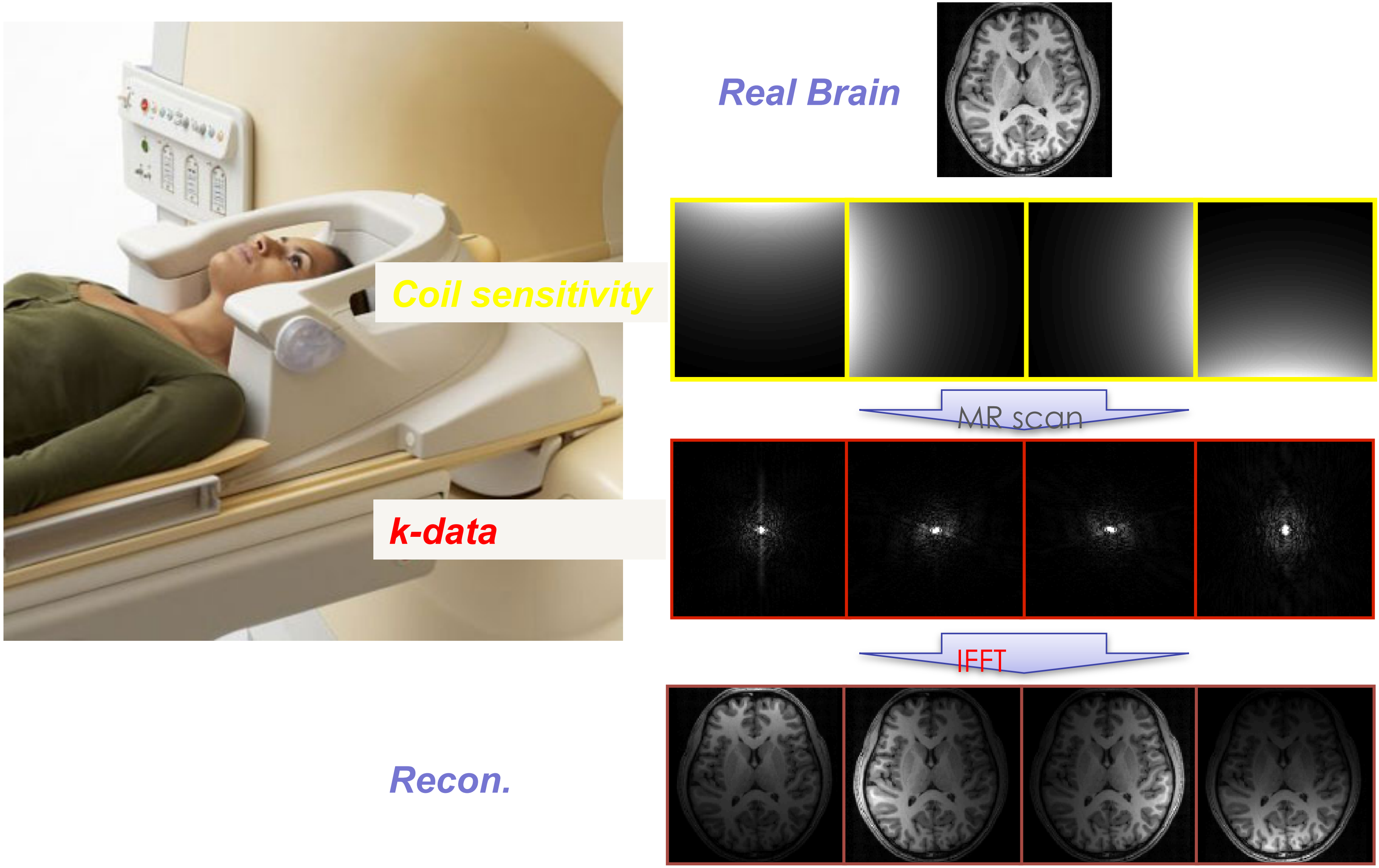
1.8 x accel. GRAPPA



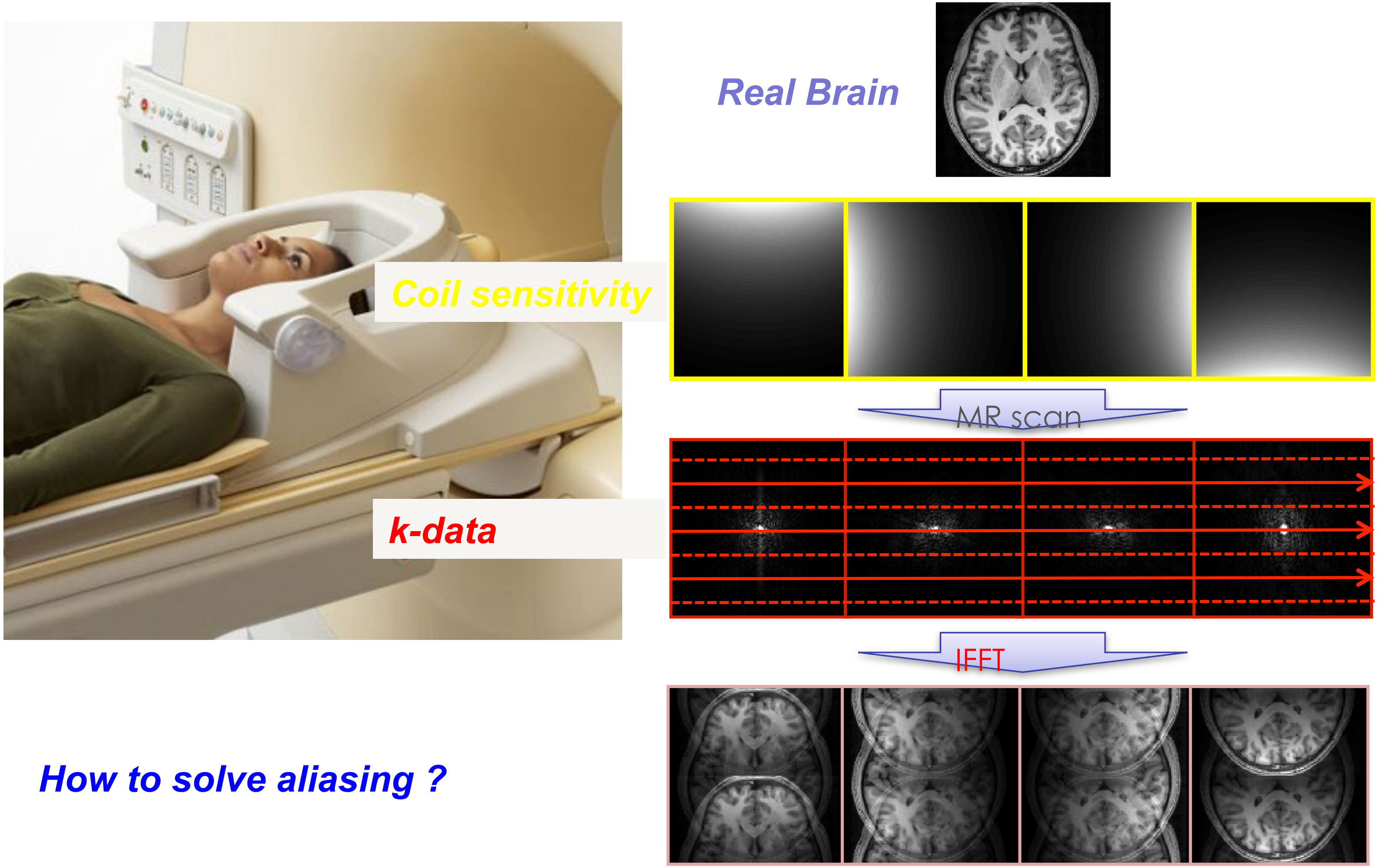
6 x accel. k-t FOCUSS



CS + Parallel Imaging



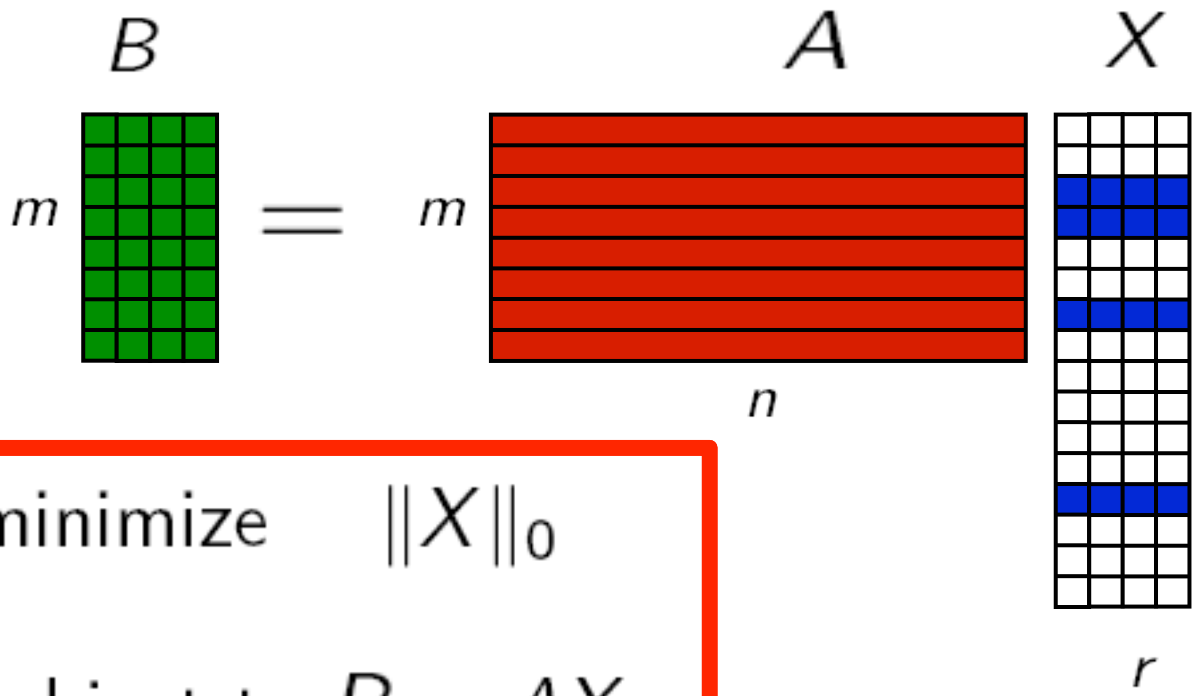
CS + Parallel Imaging



How to solve aliasing ?

Compressed Sensing for Joint Sparse Signals

<u>Multiple measurement vector (MMV) problem</u>



$$\text{minimize} \quad \|X\|_0$$

$$\text{subject to } B = AX.$$

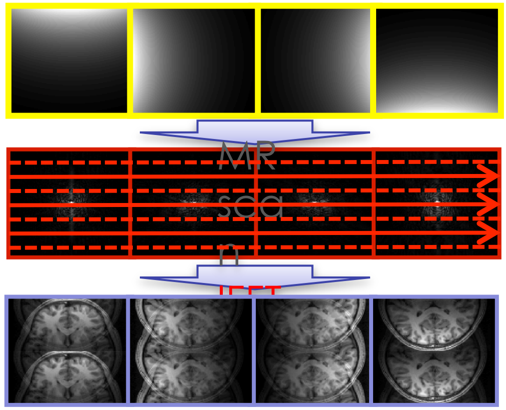
$\|X\|_0$ denote the number of nonzero rows

r : number of snapshots

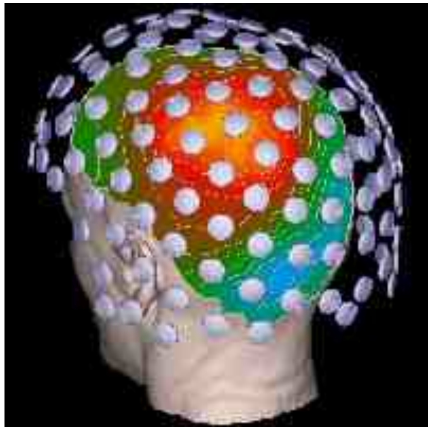
m: number of sensor elements

MMV for Medical Imaging

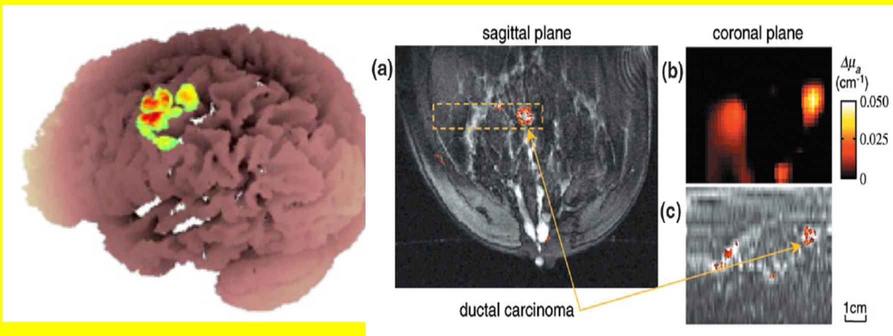
- Parallel MRI + CS



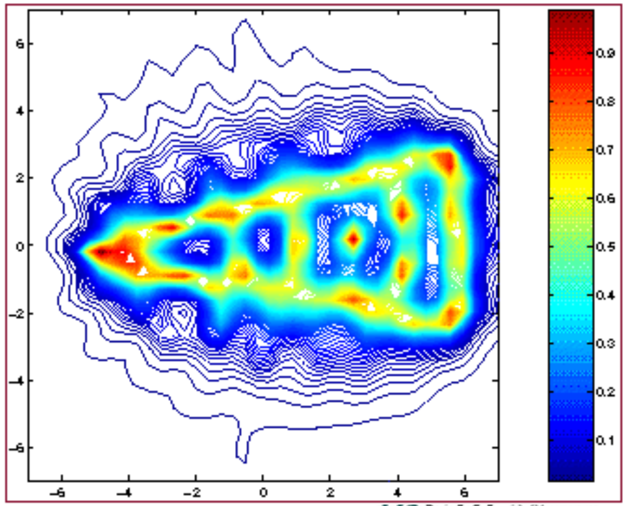
- EEG/MEG



- Diffuse optical tomography

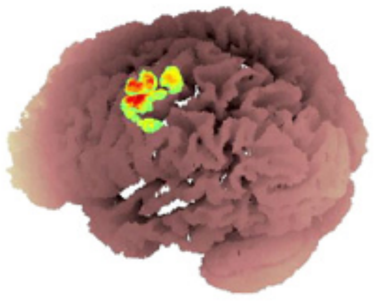


- Wave inverse scattering

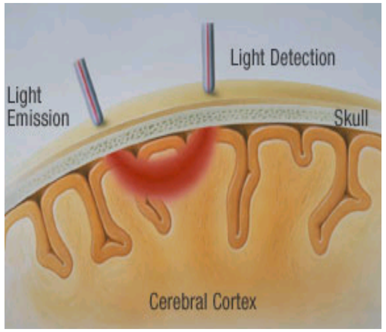


Applications of DOT

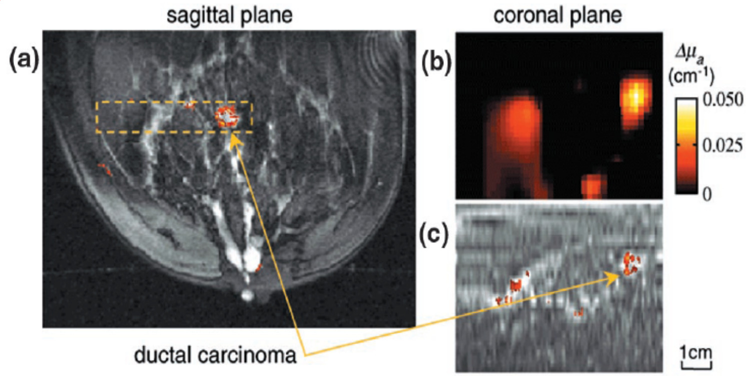
Main applications: Molecular imaging, Neuroimaging



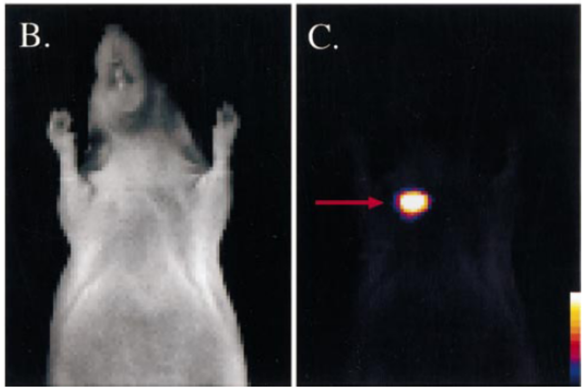
A. Custo et al. NeuroImage. 2010



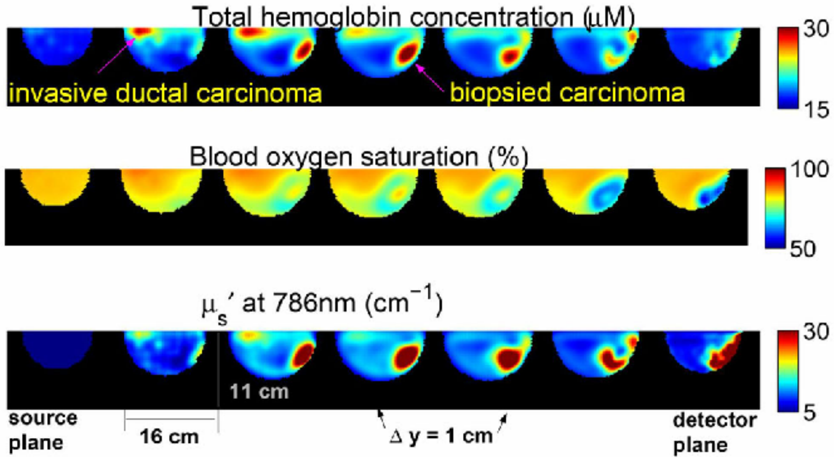
Hitachi NIRS System



V. Ntziachristos et al. Breast Cancer Res. 2001

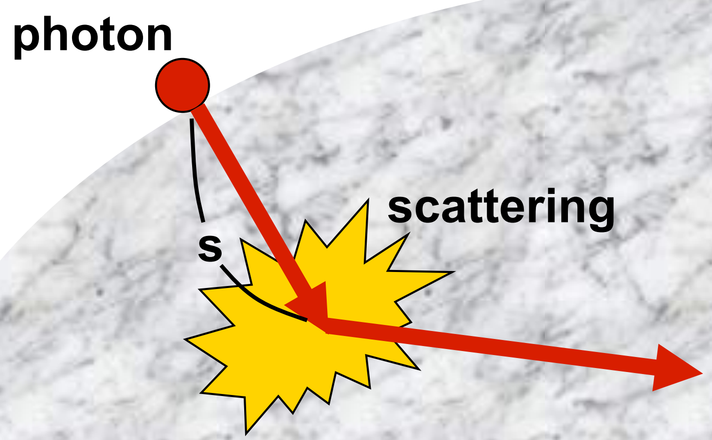


R. Weissleder et al. Radiology. 2001



A. Yodh Group

Diffusion Equation



mean path length of 's' : $\dfrac{1}{\mu_s}$

a photon is absorbed after moving : $\dfrac{1}{\mu_a}$

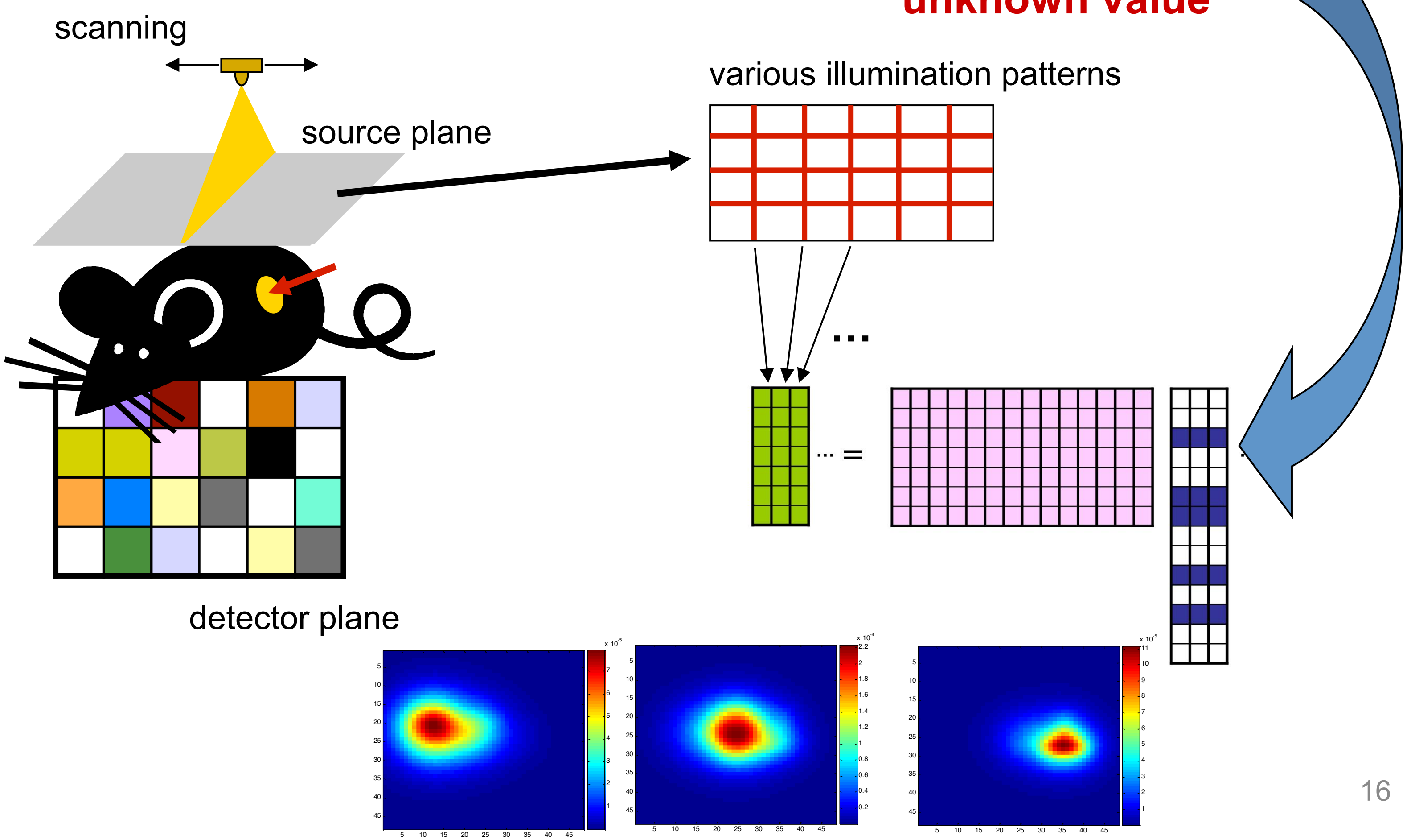
The movement of photon can be described by

$$\frac{1}{c}\frac{\partial u(\mathbf{x},t)}{\partial t} = \nabla \cdot D(\mathbf{x}) \nabla u(\mathbf{x},t) - \mu_a(\mathbf{x}) u(\mathbf{x},t) + f(\mathbf{x},t)$$

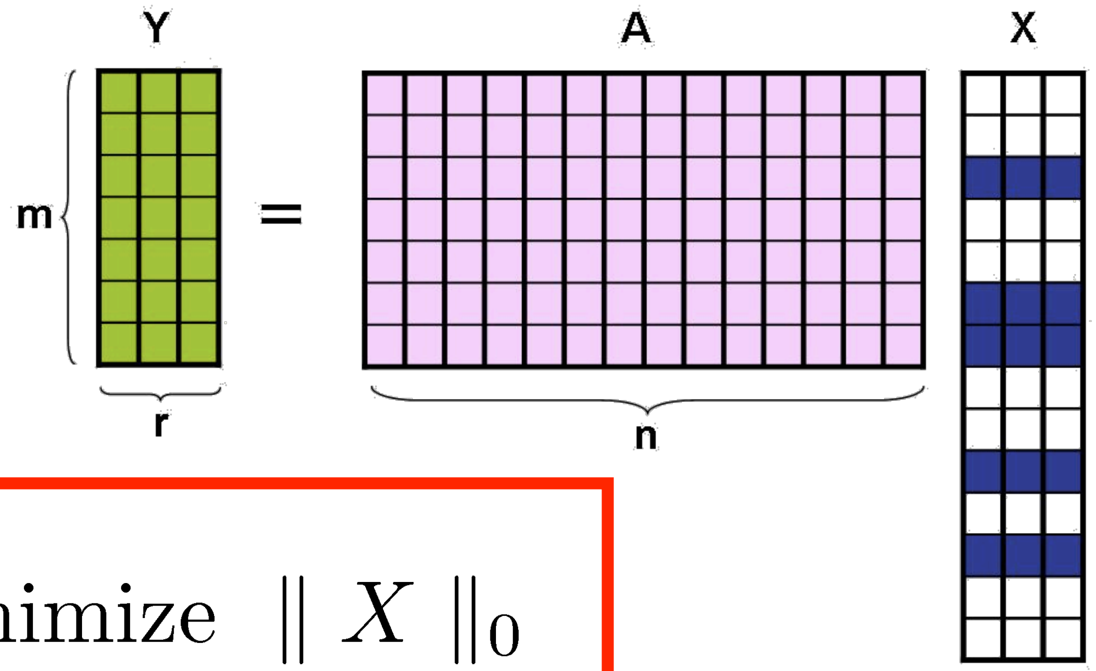
Joint Sparse Model in DOT

$$v(\mathbf{x}; L) = \int g_0(\mathbf{x}, \mathbf{x}') u(\mathbf{x}'; L) \Delta\mu_a(\mathbf{x}') d\mathbf{x}'$$

unknown value



Joint Sparse Recovery Model for DOT
(Ye et al, IEEE TMI, 2011)



$$\text{minimize} \quad \| X \|_0$$
$$\text{subject to } Y = AX$$

r : number of snapshots
m: number of sensor elements

$\| X \|_0$ denotes the # of nonzero rows

Exact & Non-iterative Reconstruction

(Lee, Ye, 2008, Ye,Bresler, Lee, 2008)

- **1st step : estimate the active index set** Λ

$$\Lambda = \{j \in \{1, 2, \cdots, n\} : \Delta\mu_a(\mathbf{x}_j) \neq 0\}$$

- **2nd step : reconstruct the** $\Delta\mu_a(\mathbf{x})$

$$\tilde{X} = A_{\Lambda}^{\dagger} Y \quad \Rightarrow \quad \Delta\tilde{\mu}_a(\mathbf{x}_{(j)}) = \frac{\sum_{L=1}^{r} \left(\tilde{u}(\mathbf{x}_{(j)};L)\right)^* \tilde{X}(j,L)}{\sum_{L=1}^{r} |\tilde{u}(\mathbf{x}_{(j)};L)|^2}$$

J.C. Ye et al, Proceedings of the IEEE ISBI 2008.

Foldy-Lax equation

$$u(\mathbf{x}_{(j)};l) = u_0(\mathbf{x}_{(j)};l) - \sum_{i\neq j} g_0(\mathbf{x}_{(j)}, \mathbf{x}_{(i)}) u(\mathbf{x}_{(i)};l) \Delta\mu_a(\mathbf{x}_{(i)})$$

$$i, j = 1, \cdots, k, \quad l = 1, 2, \cdots, r$$

l_0 Uniqueness Result of MMV

Definition

Given a matrix A, let spark(A) denote the smallest number of linearly dependent columns of A.

Theorem (Chen, Huo (2006))

If a matrix X satisfies $AX = B$ and

$$\|X\|_0 < \frac{spark(A) + rank(B) - 1}{2},$$

then X is the unique solution to the problem [P0].

With increasing number of snapshots, more non-zero el ements can be recovered

Conventional MMV Algorithms

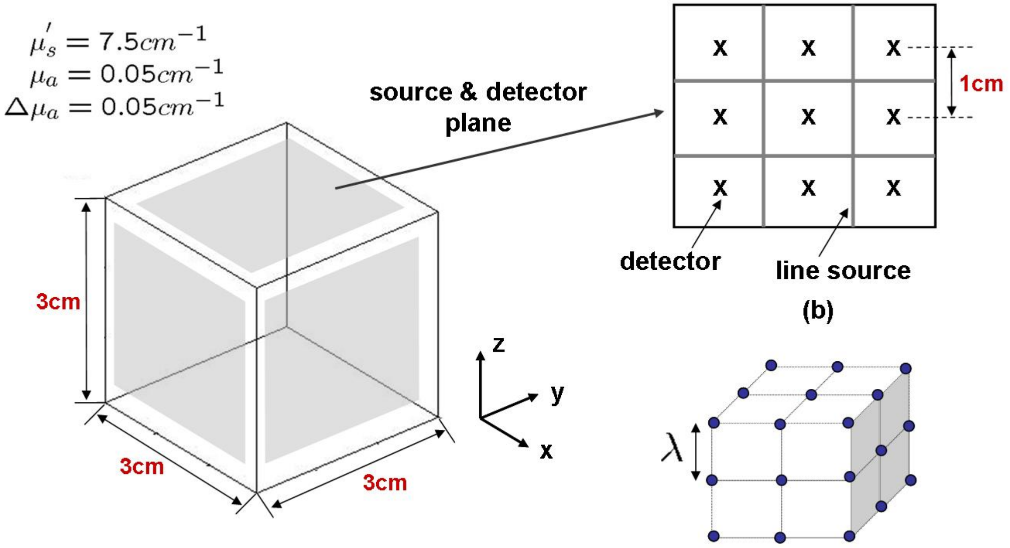
- ***Compressive sensing approaches***
  - ***p-thresholding***
  - ***S-OMP***
  - ***Convex relaxation with mixed norm***
  - ***ReMBo (Reduce Mmv and Boost)***
  - ***Model based CS using block sparsity***
  - ***M-FOCUSS***
  - ***M-SBL***
  - ***etc***

Probabilistic guarantee

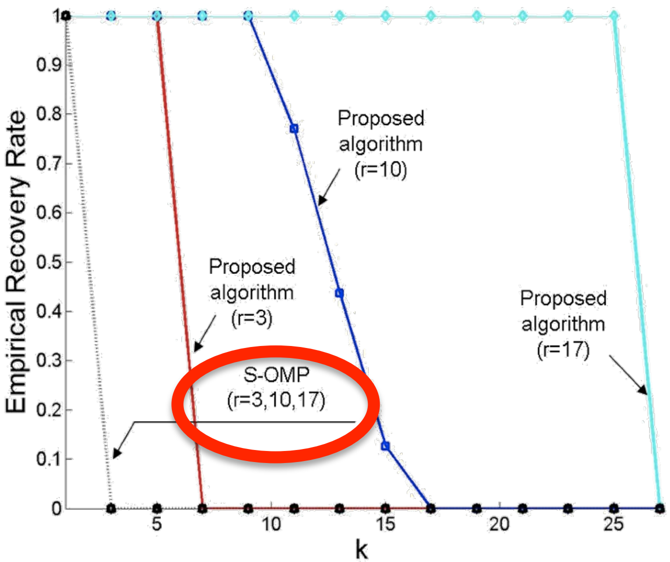
- ***Array signal processing approaches***
  - ***MUSIC***
  - ***ESPRIT***
  - ***IQML***
  - ***Maximum likelihod***
  - ***etc***

Deterministc guarantee

Counter Example



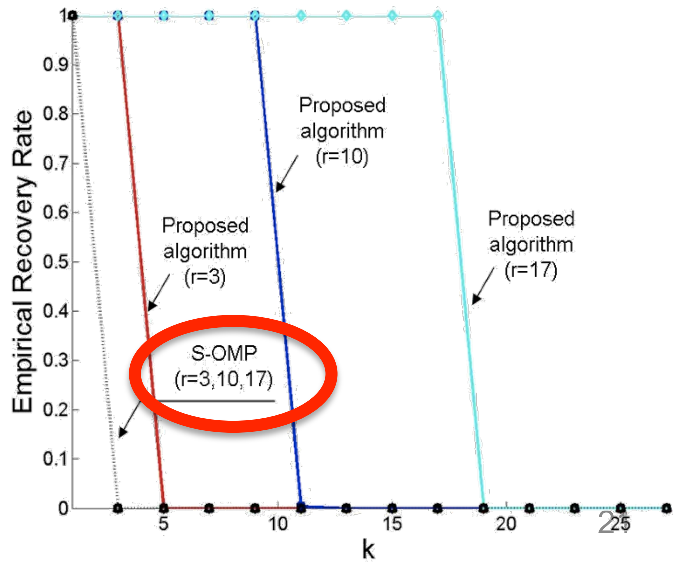
(a) (b) (c)



$SNR = 40dB$

← $\lambda = 7mm$

$\lambda = 5mm$ →



Why new MMV algorithm is necessary ?

- ***S-OMP, Convex relaxation using mixed norm***
  - ***Worst case analysis***
    - ***no improvement over SMV***

$$\max_{j \in \text{supp} X} \|A_S^{\dagger} \mathbf{a}_j\|_1 < 1$$

$$\|X\|_0 < \frac{1}{2}\left(\frac{1}{\mu} + 1\right)$$

- ***Average case analysis***
  - ***improvement with increasing number of snapshot***
  - ***Simulation results show saturation effects***

Why new MMV algorithm is necessary ?

- ***ReMBo (Reduce MMV and Boost)***

Theorem (Mishali, Eldar (2008))

Let $\overline{X}$ be the unique k-sparse solution matrix of $AX = B$ with $k < spark(A)/2$. In addition, let $\mathbf{a} \in \mathbb{R}^r$ be a random vector with an absolutely continuous distribution and define $\mathbf{b} = B\mathbf{a}$ and $\bar{\mathbf{x}} = \overline{X}\mathbf{a}$. Then for a random SMV system $A\mathbf{x} = \mathbf{b}$, we have

- *For every $\mathbf{a}$, the vector $\bar{\mathbf{x}}$ is the unique k-sparse solution.*
- $\mathcal{P}(supp(\bar{\mathbf{x}}) = supp(\overline{X})) = 1$.

The performance of ReMBo is dependent on randomly chosen input vectors so that it takes long time to reproduce the exact solution and its $L0$-performance is same as the SMV problem.

Why new MMV algorithm is necessary ?

- ***MUSIC Algorithm***
  - ***If rank(B)=k, the following MUSIC criterion holds***

$$
\begin{aligned}
Q^* \mathbf{a}_j &= 0 \\
j &\in \mathrm{supp} X
\end{aligned}
$$

- ***Dichotomy:***
  - ***Achieves l0 bound when rank(B) = k***

$$\|X\|_0 < \mathrm{spark}(A) - 1$$

  - ***Fails when rank(B) < k***
    - ***→coherent source problem***

Research Goal

- ***The Best of Both Worlds***

  •*At rank(B)=k, it should be red uced to MUSIC*
  •*At rank(B)→1, it should be re duced to CS*
  •*At all rank(B), it should be su perior to all existing methods*

Generalized MUSIC Criterion (Kim, Lee, Ye, 2010, Lee, Bresler, 2010)

Theorem

Assume that $A \in \mathbb{R}^{m \times n}$ *satisfies* $0 < \delta_{2k-r+1}(A) < 1$. *If* $I_{k-r} \subset \mathrm{supp}X$ *and* $A_{I_{k-r}}$ *is a matrix which consists of columns whose indexes are in* I_{k-r}. *Then for any* $j \in \{1, \cdots, N\} \setminus I_{k-r}$,

$$\mathrm{rank}(Q^*[A_{I_{k-r}}, a_j]) = k - r$$

if and only if

$$j \in \mathrm{supp}X$$

where $\mathrm{supp}X = \{i : X^i \neq 0\}$.

For the canonical form MMV, $A \in \mathbb{R}^{m \times n}$ *satisfies RIP with* $0 \leq \delta^L_{2k-r+1} < 1$ *if and only if*

$$k < \frac{\mathrm{spark}(A) + \mathrm{rank}(B) - 1}{2}.$$

Kim et al, "Compressive MUSIC: Missing link between compressive sensing and array signal processing", Arxiv preprint arXiv:1004.4398, 2010

Lee et al, 'Subspace augumented MUSIC for sparse recovery", Arxiv preprint arXiv:1004.3071, 2010

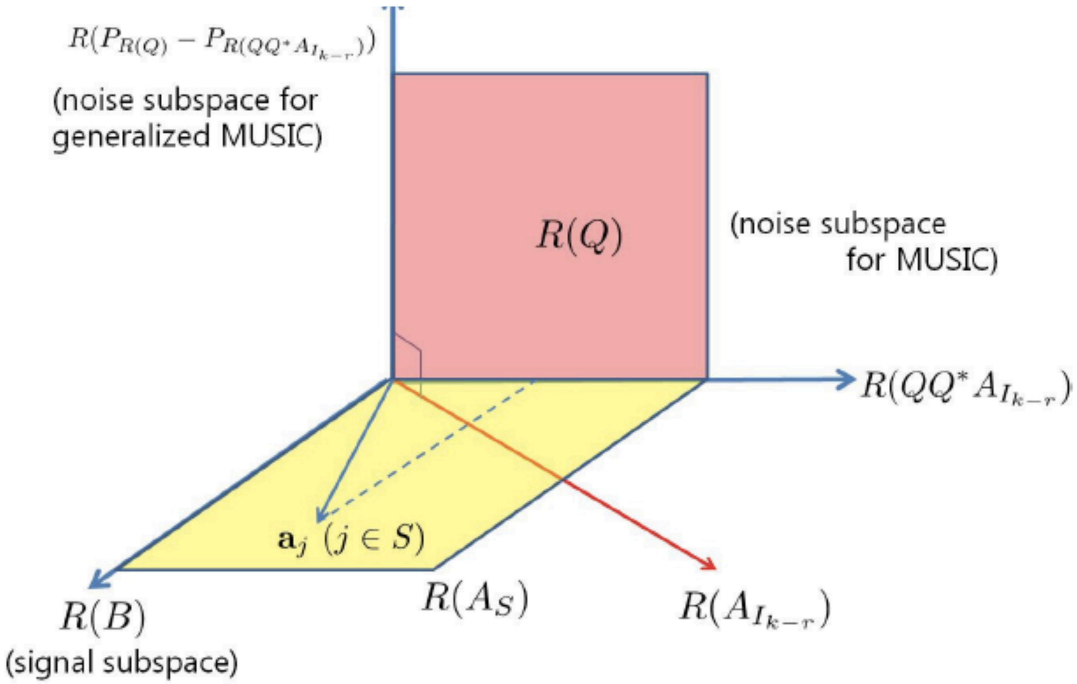
Geometry of Generalized MUSIC Criterion

Corollary

Assume that $A \in \mathbb{R}^{m \times n}$, $X \in \mathbb{R}^{n \times r}$, $B \in \mathbb{R}^{m \times r}$, $I_{k-r} \subset \mathrm{supp} X$. *Then,*

$$\mathbf{a}_j^* \left[P_{R(Q)} - P_{R(P_{R(Q)} A_{I_{k-r}})} \right] \mathbf{a}_j = 0$$

if and only if $j \in \mathrm{supp} X$.



Geometry of Generalized MUSIC Criterion

Theorem

Let $U \in \mathbb{R}^{m\times r}$ and $Q \in \mathbb{R}^{m\times(m-r)}$ consist of orthonormal columns such that $R(U) = R(B)$ and $R(Q)^{\perp} = R(B)$. Then the following properties hold :

(a) $UU^* + P_{R(QQ^*A_{I_{k-r}})}$ *is equal to the orthogonal projection onto* $R(B) + R(QQ^*A_{I_{k-r}})$.

(b) $QQ^* - P_{R(QQ^*A_{I_{k-r}})}$ *is equal to the orthogonal projection onto* $R(Q) \cap R(QQ^*A_{I_{k-r}})^{\perp}$.

(c) $QQ^* - P_{R(QQ^*A_{I_{k-r}})}$ *is equal to the orthogonal complement of* $R([U\ A_{I_{k-r}}])$ *or* $R([B\ A_{I_{k-r}}])$.

Compressive MUSIC Algorithm

- **1st Step: Compressive sensing step**

  - Find I_{k-r} with existing MMV-algorithm such as 2-thresholding and SOMP and let $S = I_{k-r}$.

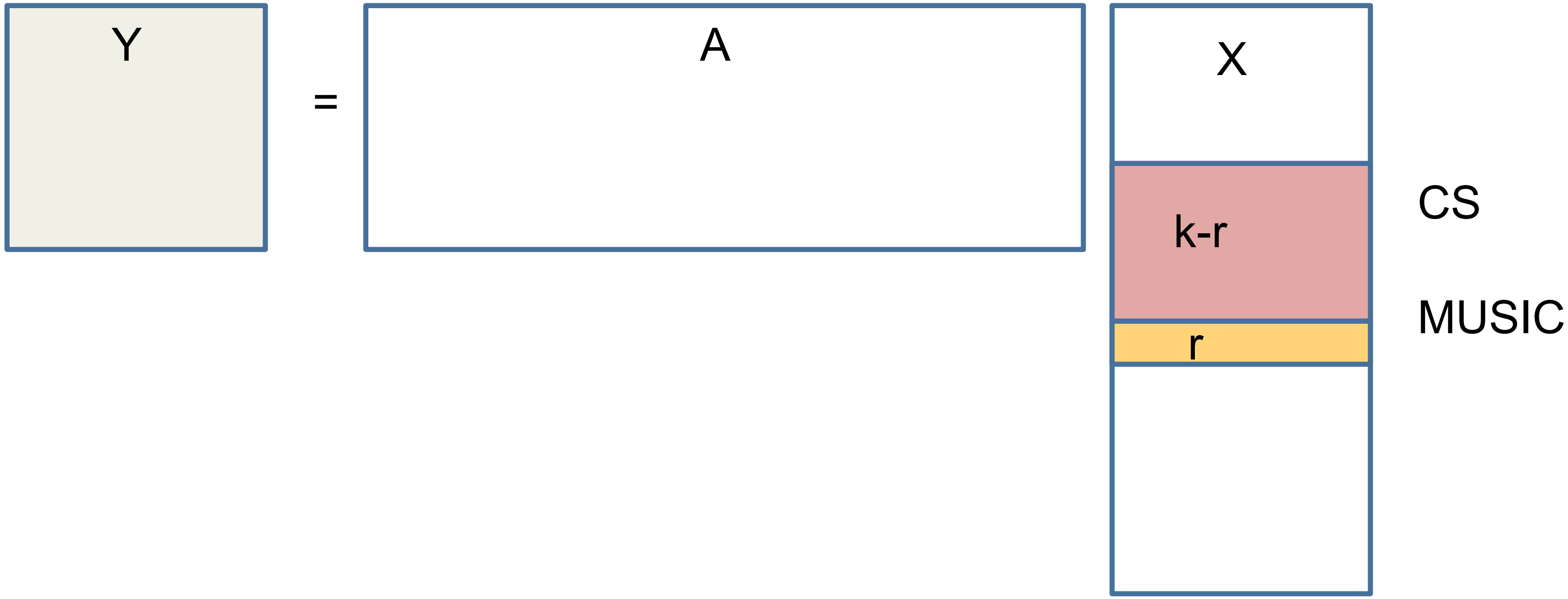
  Probabilistic performance guarantee

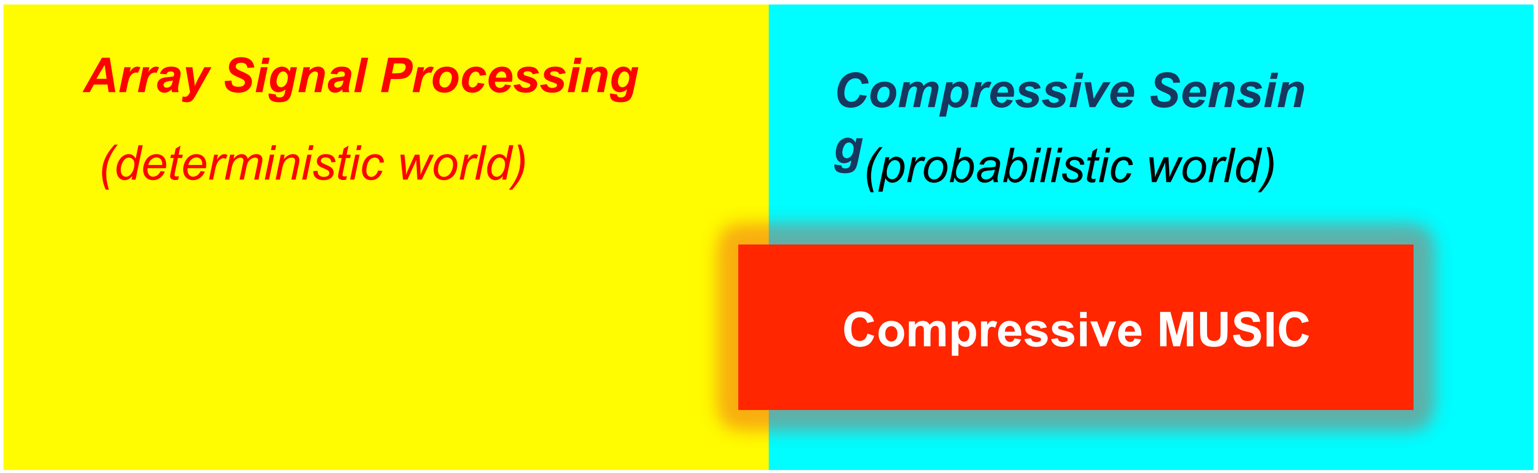
- **2nd Step: generalized MUSIC step**

  - For $j \in \{1, \cdots, N\} \setminus I_{k-r}$, calculate $\mathbf{g}_j^* P_{G_{I_{k-r}}}^{\perp} \mathbf{g}_j = 0$:
    - If $\mathbf{g}_j^* P_{G_{I_{k-r}}}^{\perp} \mathbf{g}_j = 0$, then add j into S.

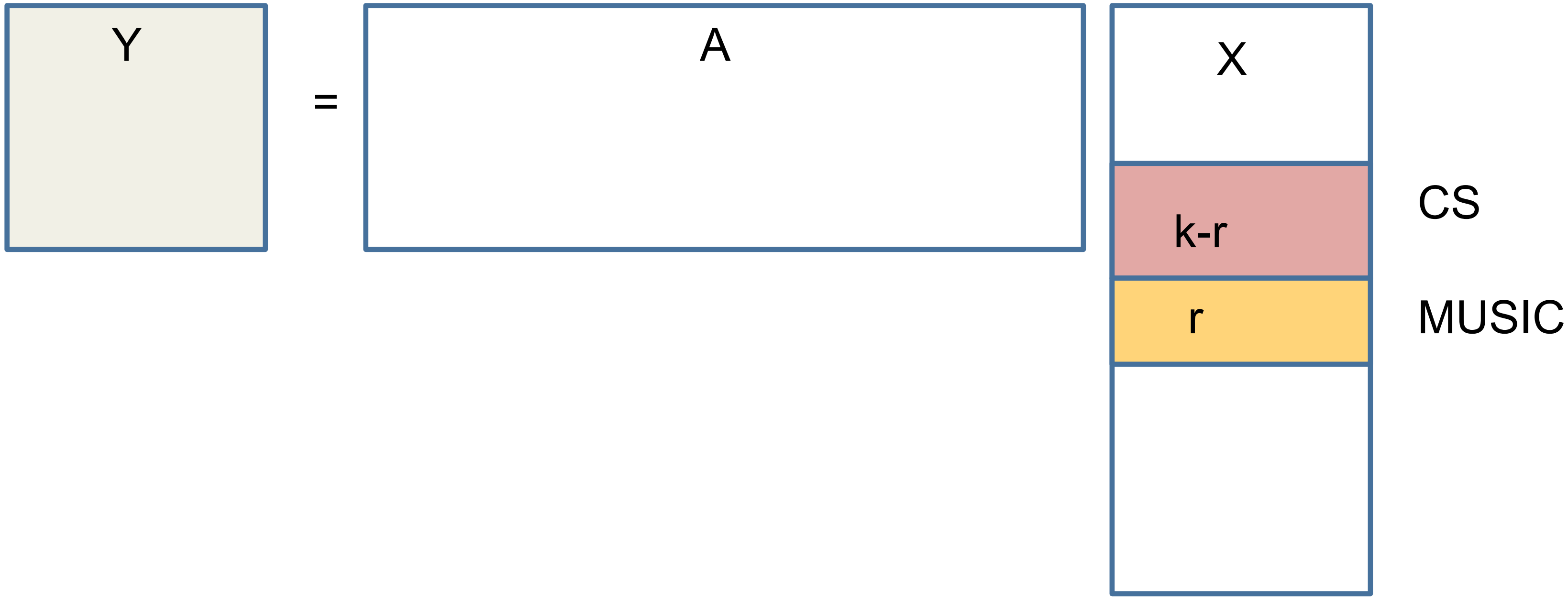
  Deterministic performance guarantee

Compressive MUSIC (r=1)





Compressive MUSIC (r=k/2)



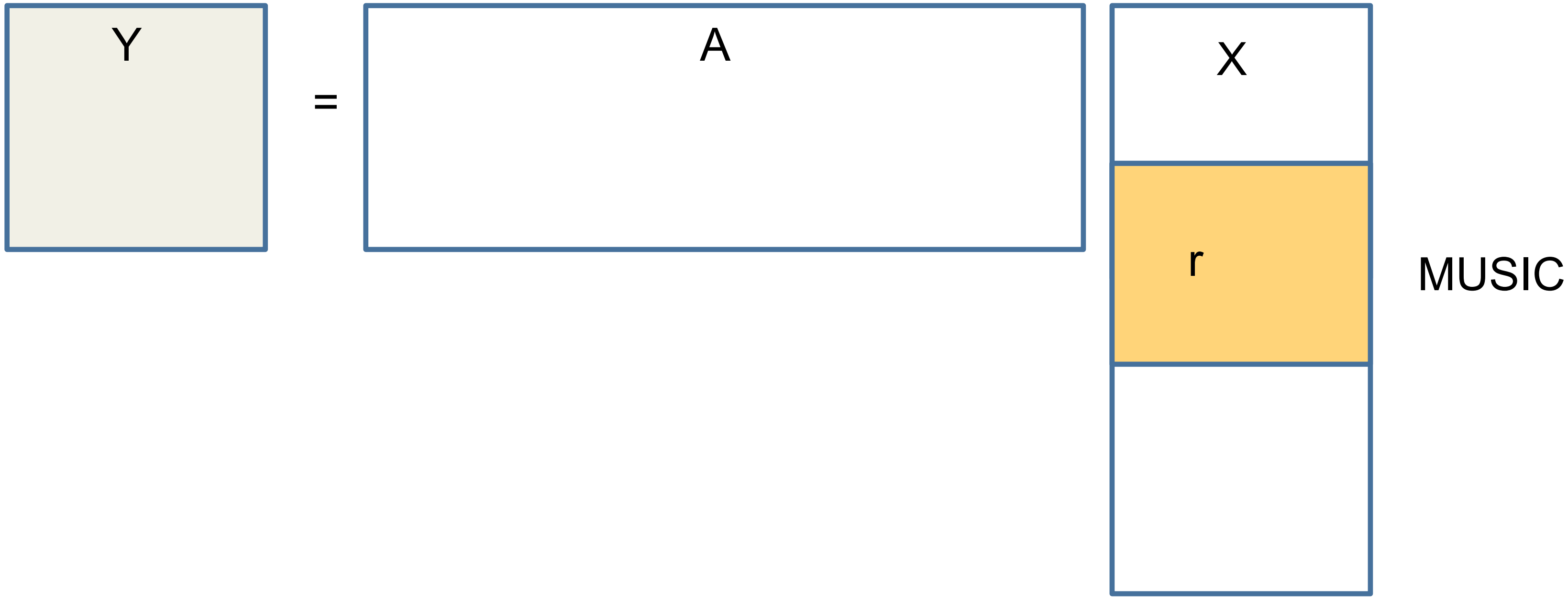
Array Signal Processing

(deterministic world)

Compressive Sensin g *(probabilistic world)*

Compressive MUSIC

Compressive MUSIC (r=k)



Array Signal Processing

(deterministic world)

Compressive MUSIC

Compressive Sensing

(probabilistic world)

Number of Sensor Elements

- ***Partial Support Recovery using SS-OMP***

Theorem

(a) r is a fixed finite number.
(b) Let $\mathrm{SNR}_{\min}(Y)$ satisfy

$$\mathrm{SNR}_{\min}(Y) > 1 + \frac{4k}{r}(\kappa(B) + 1).$$

If we have

$$m > k(1+\delta)\left[1 - \frac{4k}{r}\frac{(\kappa(B)+1)}{\mathrm{SNR}_{\min}(Y) - 1}\right]^{-1}\frac{2\log(n-k)}{r},$$

then we can find $k - r$ correct indices of $\mathrm{supp}X$ by applying subspace S-OMP.

Number of Sensor Elements

- ***Partial Support recovery using SS-OMP***

Theorem

(a) r is proportionally increasing with respect to k so that $\alpha := \lim_{n\to\infty} r(n)/k(n) > 0$ exist.
(b) Let $\mathrm{SNR}_{\min}(Y)$ satisfy

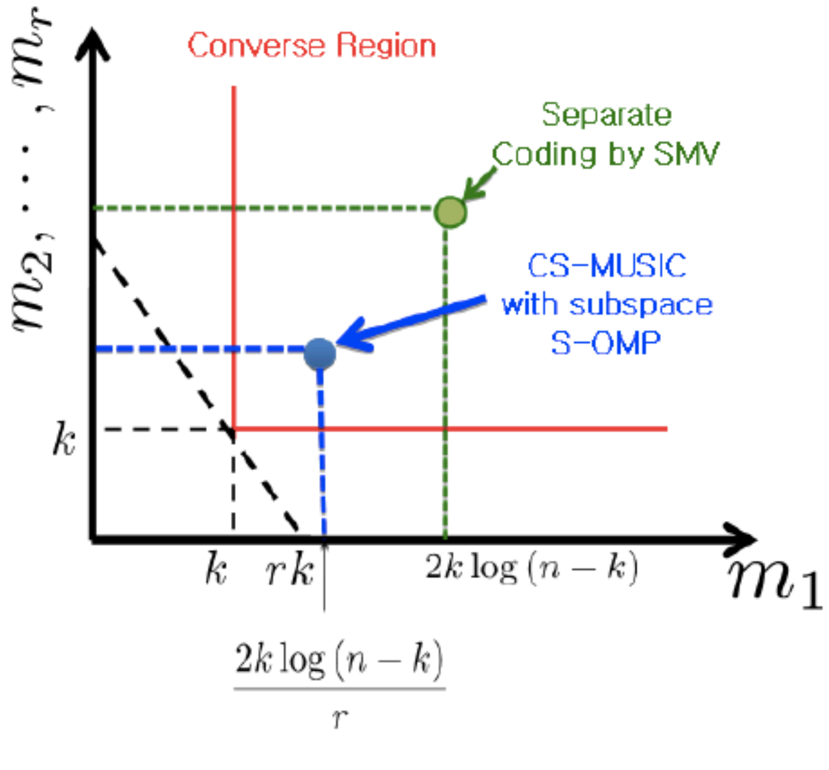
$$\mathrm{SNR}_{\min}(Y) > 1 + \frac{4}{\alpha}(\kappa(B) + 1).$$

Then if we have

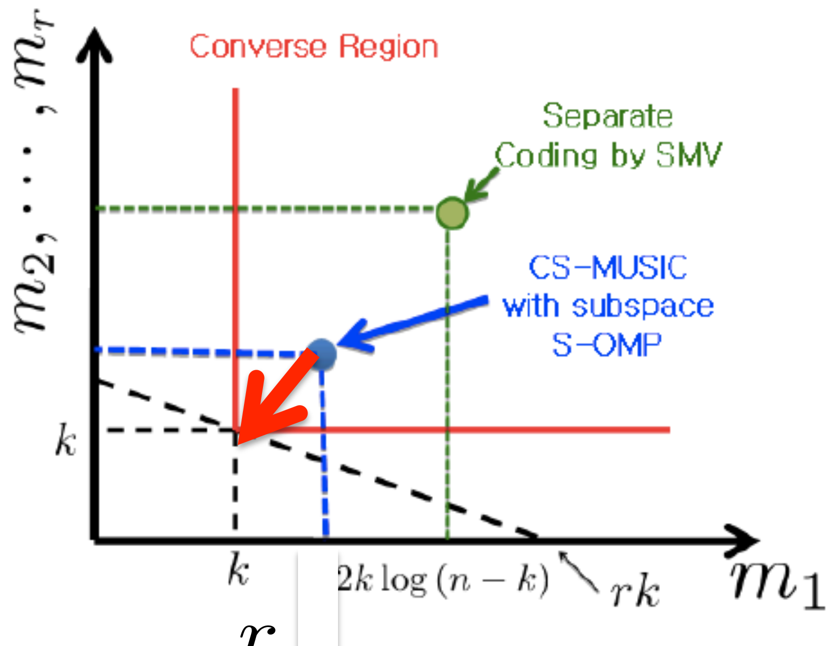
$$m > k(1+\delta)^2 \frac{1}{\left[1 - \frac{4}{\alpha}\frac{\kappa(B)+1}{\mathrm{SNR}_{\min}(Y)-1}\right]^2} \left[2 - F(\alpha)\right]^2,$$

for some $\delta > 0$ where $F(\alpha)$ is an increasing function such that $F(1) = 1$ and $\lim_{\alpha\to 0^+} F(\alpha) = 0$. Then we can find $k - r$ correct indices of $\mathrm{supp}X$ by applying subspace S-OMP.

MMV Coding Region



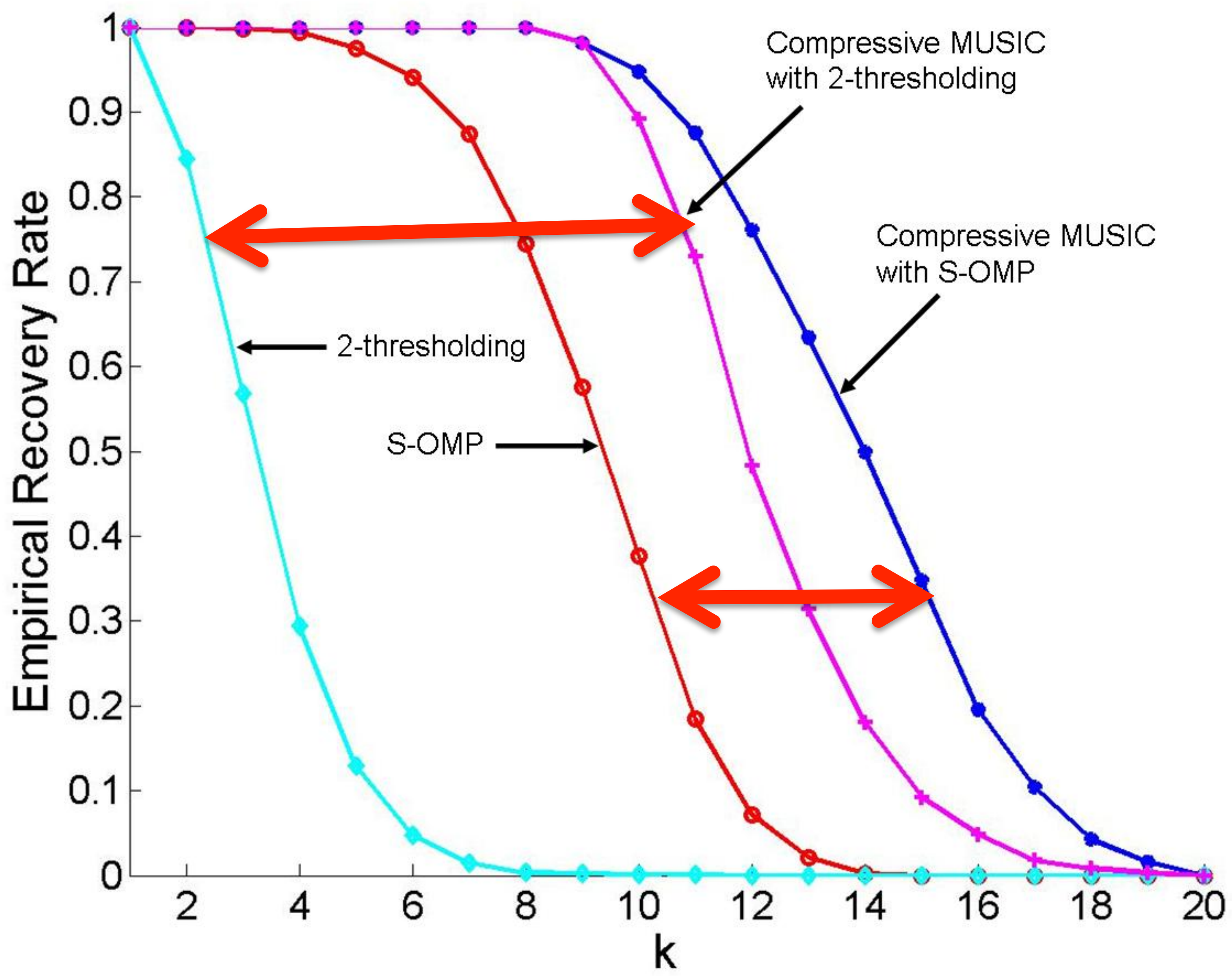
(a)



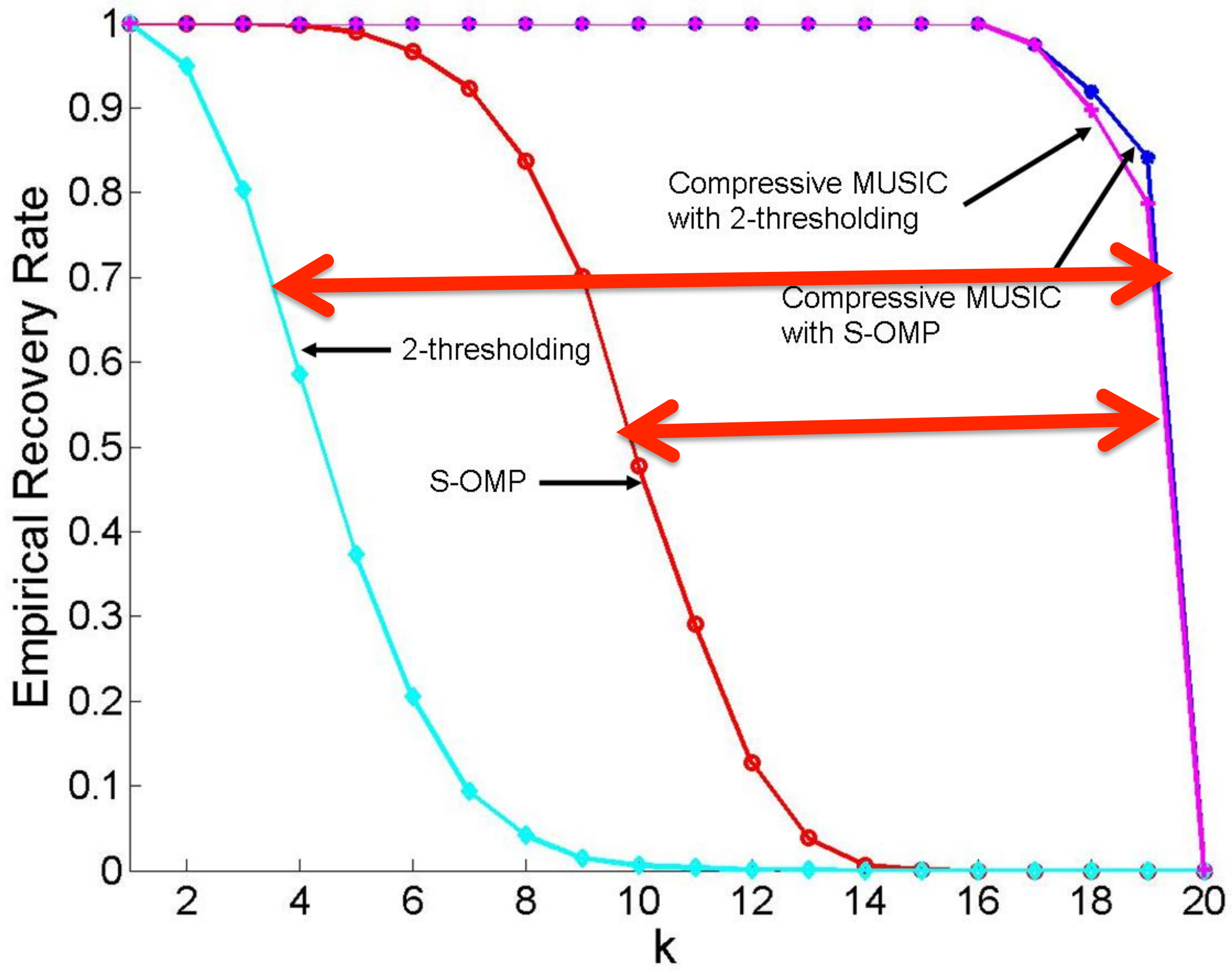
$$\alpha = \frac{r}{k} \to 1$$

(b)

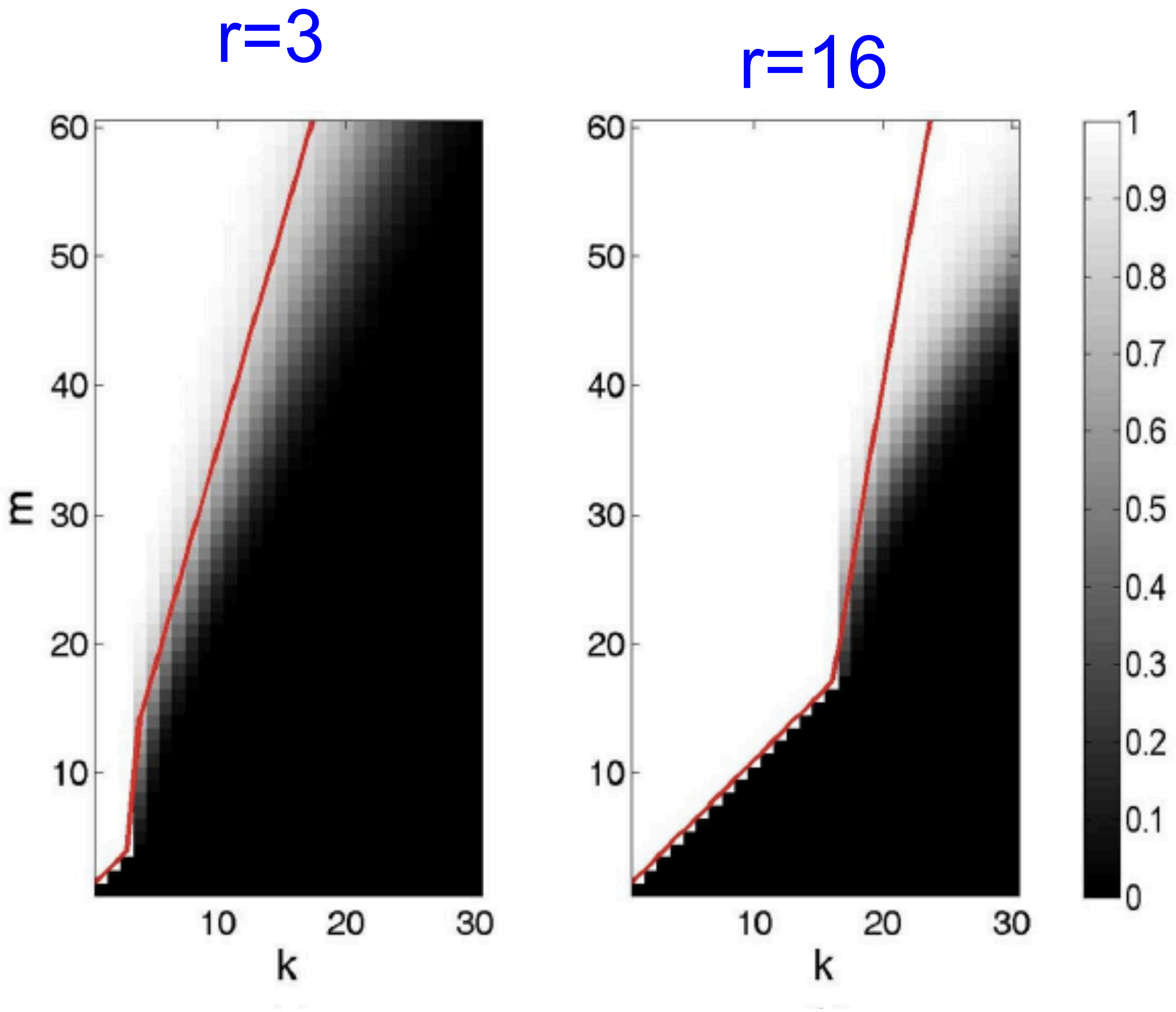
Simulation (Noiseless) **n=100, m=20, r=8**



Simulation (Noiseless) **n=100, m=20, r=16**

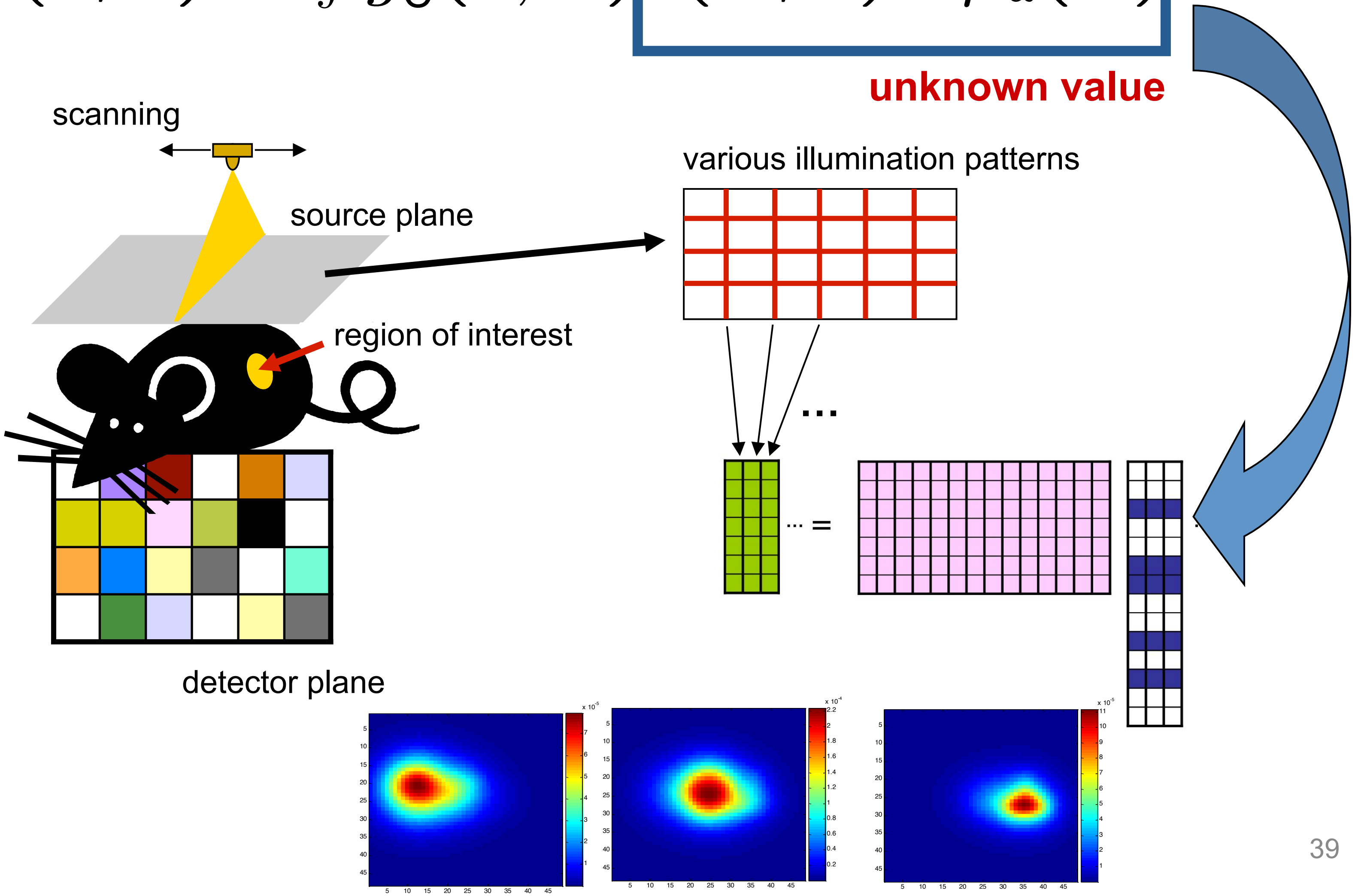


Phase Transition



Joint Sparse Model in DOT

$$v(\mathbf{x};L) = \int g_0(\mathbf{x},\mathbf{x}')\boxed{u(\mathbf{x}';L)\Delta\mu_a(\mathbf{x}')}d\mathbf{x}'$$



Exact & Non-iterative Reconstruction

(Lee, Ye, 2008, Ye,Bresler, Lee, 2008)

- 1st step : estimate the active index set Λ using

  compressed MUSIC

$$\Lambda = \{j \in \{1, 2, \cdots, n\} : \Delta\mu_a(\mathbf{x}_j) \neq 0\}$$

- 2nd step : reconstruct the $\Delta\mu_a(\mathbf{x})$

$$\tilde{X} = A_\Lambda^\dagger Y \quad \Rightarrow \quad \Delta\tilde{\mu}_a(\mathbf{x}_{(j)}) = \frac{\sum_{L=1}^{r} \left(\tilde{u}(\mathbf{x}_{(j)};L)\right)^* \tilde{X}(j,L)}{\sum_{L=1}^{r} |\tilde{u}(\mathbf{x}_{(j)};L)|^2}$$

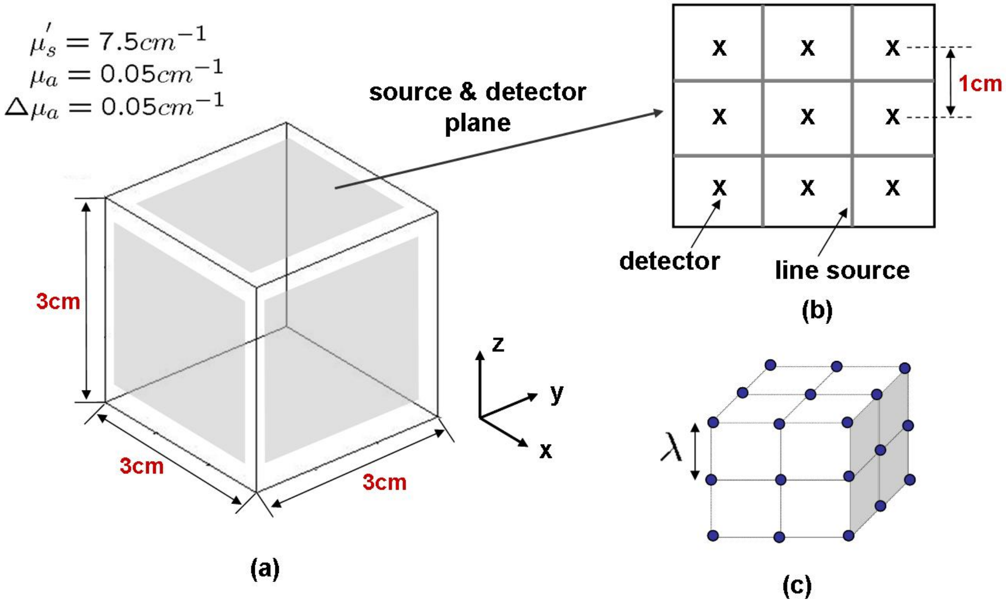
J.C. Ye et al, Proceedings of the IEEE ISBI 2008.

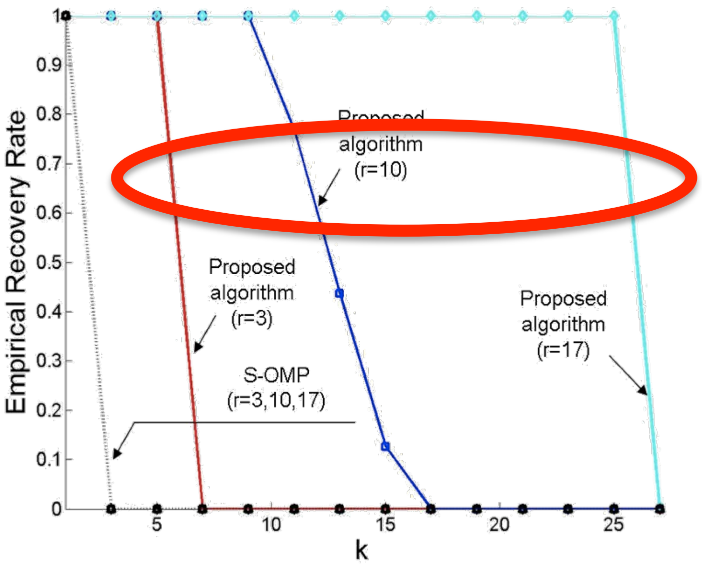
Foldy-Lax equation

$$u(\mathbf{x}_{(j)};l) = u_0(\mathbf{x}_{(j)};l) - \sum_{i\neq j} g_0(\mathbf{x}_{(j)}, \mathbf{x}_{(i)}) u(\mathbf{x}_{(i)};l) \Delta\mu_a(\mathbf{x}_{(i)})$$

$$i, j = 1, \cdots, k, \quad l = 1, 2, \cdots, n$$

Revisit the Counter Example

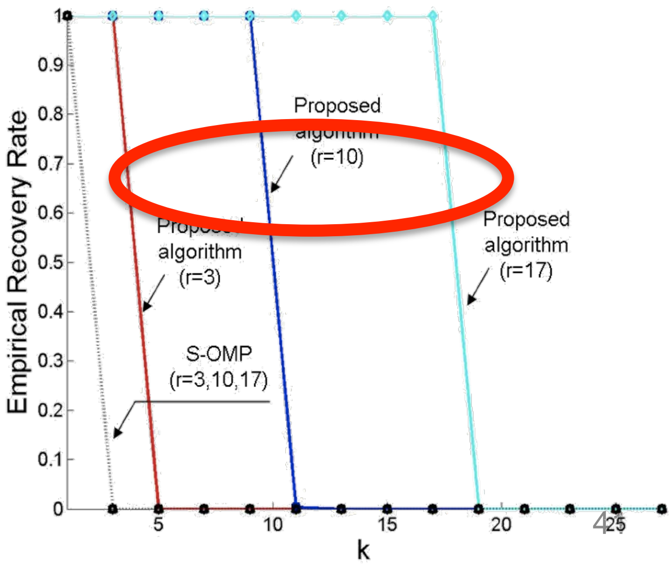




$SNR = 40dB$

← $\lambda = 7mm$

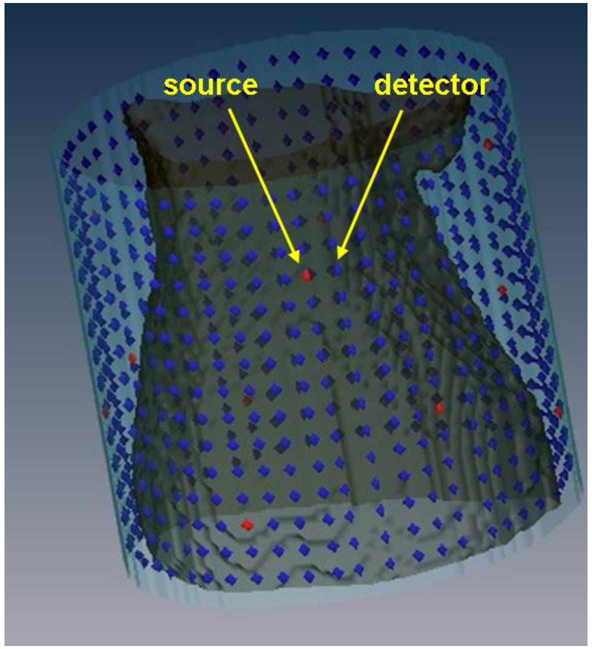
$\lambda = 5mm$ →



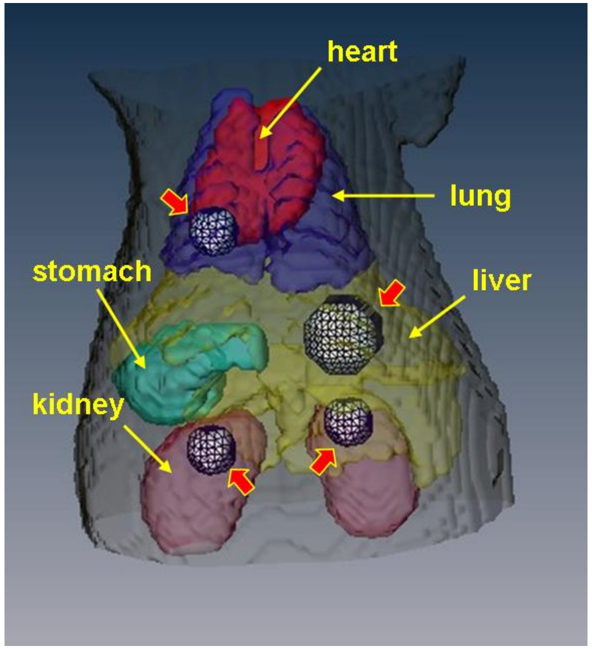
Simulation for Molecular Imaging

$$p(x) = \frac{a(x)^* a(x)}{a(x)^*(P_{R(Q)} - P_{R(P_{R(Q)}A_{I_{k-r}})})a(x)}, \quad x \in \Omega$$

*** Generalized MUSIC spectrum**



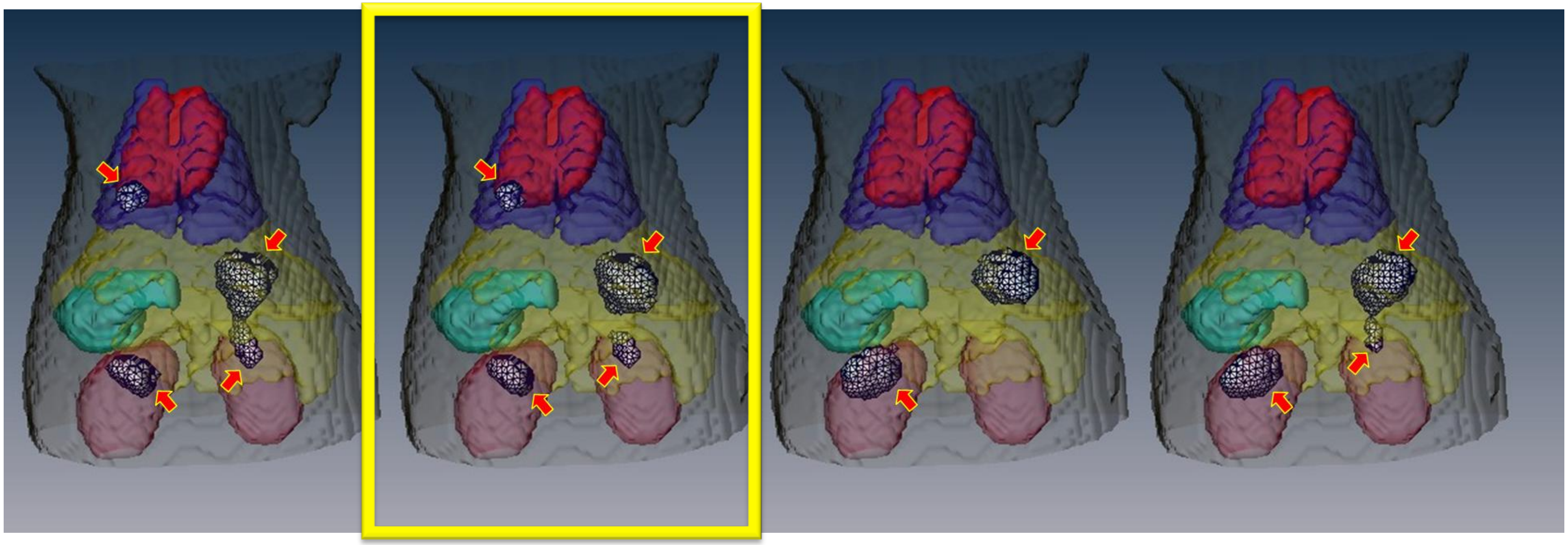
Source and Detector geometry



Original Phantom

	proposed method	Tikhonov regularization	$l1$-penalty regularization
MSE $[10^{-7}]$	5.8696	7.6837	7.5802

Hausdorff distance [mm]	proposed method	Tikhonov regularization	$l1$-penalty regularization
Liver	1.5	2.5	2.1794
Lung (left)	1.118	∞	∞
Kidney (left)	1	2.4495	2.5495
Kidney (right)	1.4142	∞	1.8708



MUSIC | **Compressive MUSIC** | Tikhonov Regularization | L1-penalty Regularization

Conclusion

- Diffuse optical tomography can be formulated joint sparse recovery problem
  - Non-iterative and exact reconstruction algorithm exists !
- Compressive MUSIC outperforms the all existing methods in joint sparse recovery problems
  - Apply compressive MUSIC for DOT
- Due the the simplicity and effectiveness, the proposed method would open a new direction in DOT research

Acknowledgements

국가지정연구실
National Research Lab.

Korea Science and Engineering Foundation (KOSEF) grant funded by the Korea government (MEST) (No.20 10-0000855).

Thank you