Random walks for deformable image registration

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with
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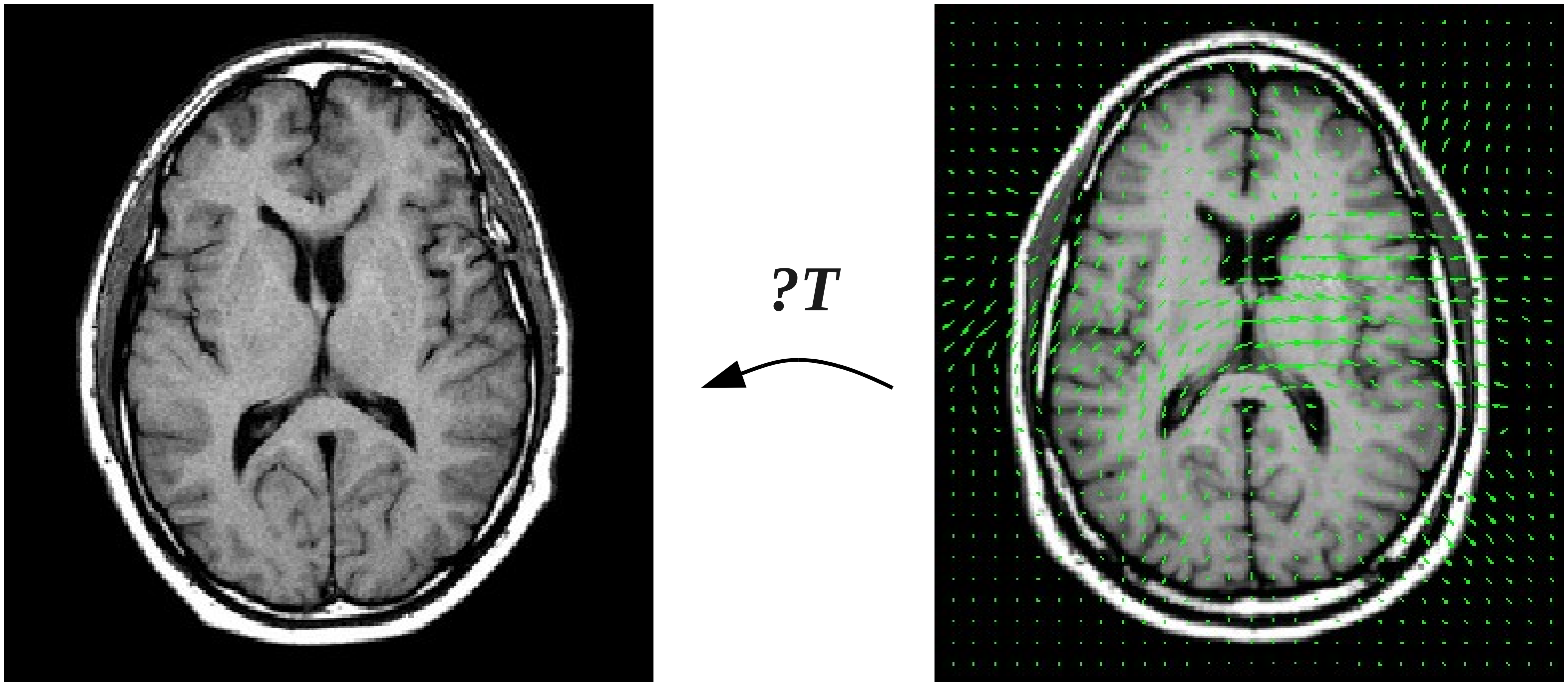
Deformable image registration



I **target** ***J*** **source**

Outline:

- Continuous formulation of deformable registration
- Discrete formulation and related work
- Diffusion and regularization – from continuous to discrete
- Random walker for deformable registration
- Examples and discussion

Deformable image registration

I **target**

$?T$ ←

J **source**

$$I, J : \Omega \rightarrow \mathbb{R}, \Omega \in \mathbb{R}^d$$

transformation $T : \Omega \rightarrow \mathbb{R}^d, T = Id + u$

$\mathbf{u} = (u_x, u_y)$
displacement field

$$E(u) = E_D(I, J \circ T(u)) + \alpha E_R(u)$$

Data similarity — regularization

$$u = argmin_u E(u)$$

Variational image registration

$$\bar{u} = argmin_u \int_{x \in \Omega} \Phi(I(\mathbf{x}), J(x + u(\mathbf{x})))\, d\,\mathbf{x} \quad +$$

Data similarity (point-wise score)

$$\alpha \int_{x \in \Omega} \Psi(\nabla J(\mathbf{x}), \nabla u(\mathbf{x}))\, d\,\mathbf{x}$$

Adaptable regularization

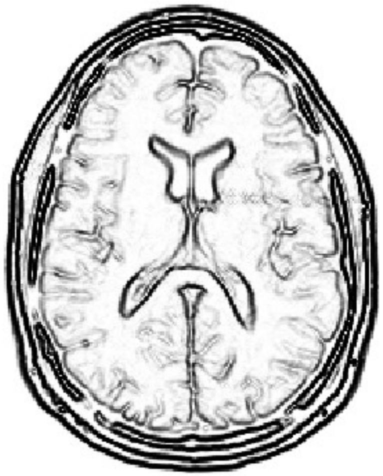
<u>**EXAMPLE:**</u>

SSD

$$\Phi(I(\mathbf{x}), J(\mathbf{x} + u(\mathbf{x}))) = (I(\mathbf{x}) - J(\mathbf{x} + u(\mathbf{x})))^2$$

Linear isotropic regularization

$$\Psi(\nabla J(\mathbf{x}), \nabla u(\mathbf{x})) = w(|\nabla J(\mathbf{x})|^2)(|\nabla u_x|^2 + |\nabla u_y|^2)$$



scalar valued diffusivity

Euler-Lagrange equations

$$min_u \ E_D(I, J \circ T) + \alpha \int_{\boldsymbol{x} \in \Omega} w(.)(|\nabla u_x|^2 + |\nabla u_y|^2)\, d\boldsymbol{x}$$

$$\frac{\partial E_D}{\partial u_x} - \alpha \nabla \cdot (w(.) \nabla u_x) = 0$$

$$\frac{\partial E_D}{\partial u_y} - \alpha \nabla \cdot (w(.) \nabla u_y) = 0$$

Linear isotropic diffusion

Continuous solution to deformable registration:

- Nonlinear optimization (gradient descent, Gauss-Newton ...)
- Semi-implicit scheme to solve the EL PDEs
- Regularization can be decoupled from the data term update – *demon's algorithm*

Main limitation – local minima
> need for better frameworks

Discrete deformable registration

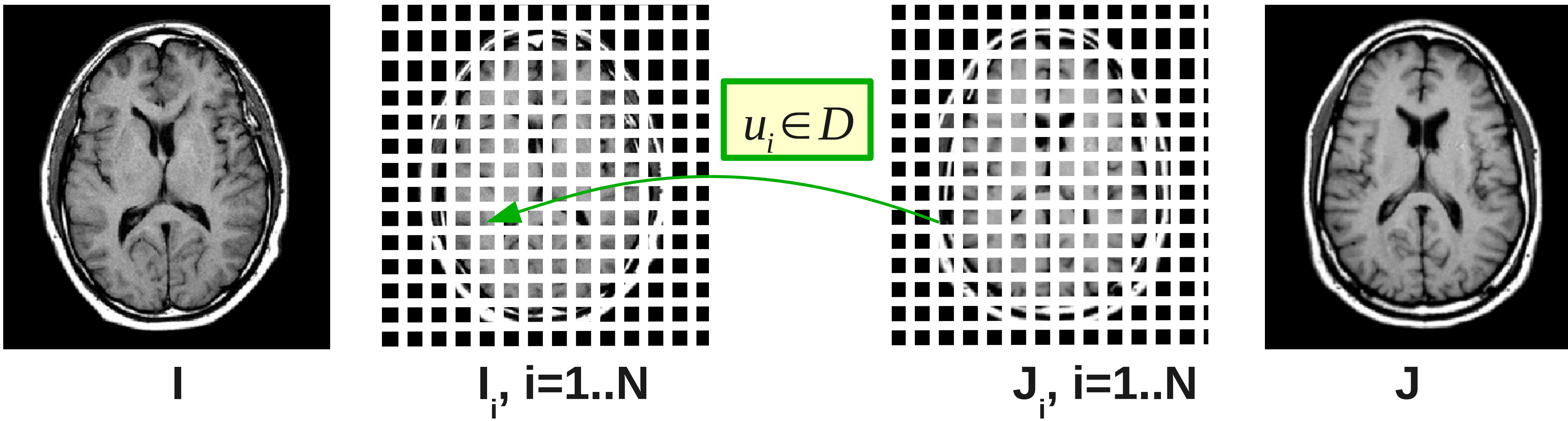
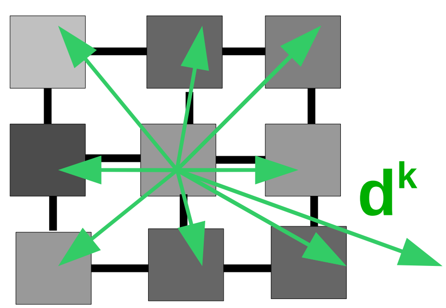


Image = Graph

$G = (N, E)$



Quantized deformations

$D = \{d^1, d^2, \ldots, d^K\}$

Labels

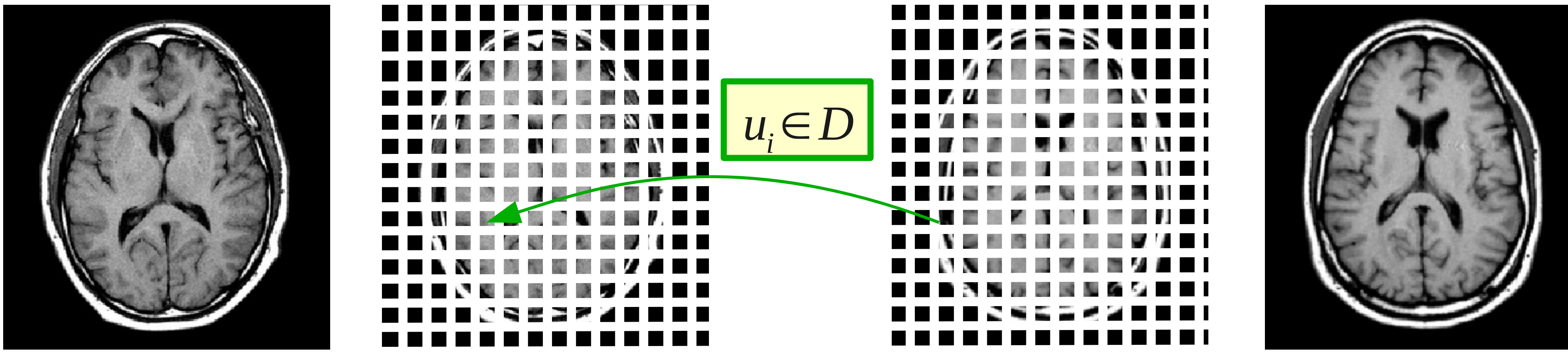
$L = \{u^1, u^2, \ldots, u^K\}$

Note: D can be very large

Ex: deformations [0-20pix]

$41^3 = 68921$ labels

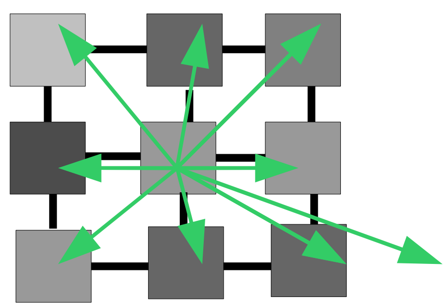
Discrete deformable registration



I | **I_i, i=1..N** | **J_i, i=1..N** | **J**

Image = Graph

$\mathbf{G} = (N, E)$



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Labels

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Note: D can be very large

Ex: deformations [0-20pix]

$41^3 = 68921$ labels

Registration as labeling : MRF

$$E(\boldsymbol{u}) = \sum_{i \in N} \Phi_i(u_i) + \alpha \sum_{(i,j) \in E} \Psi_{ij}(u_i, u_j)$$

data similarity

Regularization Interaction

Related work

$$E(\boldsymbol{u}) = \sum_{i \in N} \Phi_i(u_i) + \alpha \sum_{(i,j) \in E} \Psi_{ij}(u_i, u_j)$$

Global optimum

- Ψ_{ij} metric – graph cut optimization
  [T.W. Tang et al. MICCAI 2007][L. Tang et al. MICCAI 2010]
- More complex interaction terms
  linear programming method based on primal-dual principle
  [Glocker et al. MIA 2008]

Related work

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<u>We propose :</u>
Gaussian MRF for deformable registration
solved using random walker algorithm **[Grady PAMI 2006, CVPR 2005]**

- Fast algorithm
- Can easily incorporate a large number of displacements
- Robustness to noise
- Unique global minimum (has to be relaxed)

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Here:
Gaussian MRF for deformable registration
solved using random walker algorithm **[Grady PAMI 2006, CVPR 2005]**

- Fast algorithm
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NEXT:
- diffusion and regularization
- diffusion on graphs
- RW for image registration
- Experiments and discussion

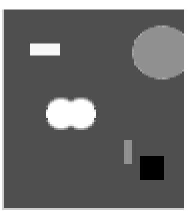
Regularization → **diffusion**

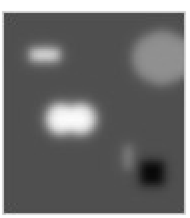
$$E_R(u) = \int_\Omega |\nabla u|^2 \, dxdy \quad \Rightarrow \quad \Delta u = 0$$

Diffusion

Uniform

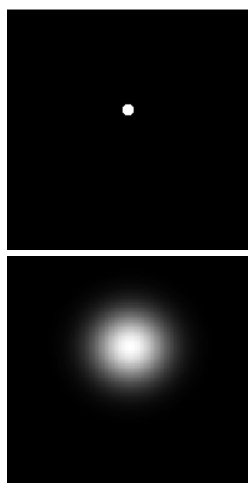
$$\frac{\partial u}{\partial t} = div(\nabla u) = \Delta u$$
$$u(0) = u_0$$


u(0)


u(t)

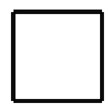
...

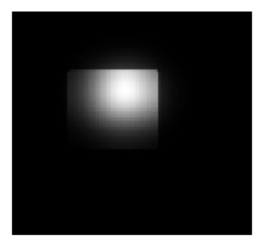

Steady state

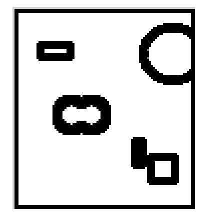


Isotropic

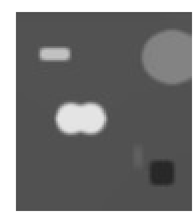
$$\frac{\partial u}{\partial t} = div(w \nabla u)$$
$$u(0) = u_0$$







$$w(s^2) = \frac{1}{\sqrt{s^2 + \varepsilon}}$$
$$s = \nabla u_0$$


u(t)

Scalar-valued diffusivity

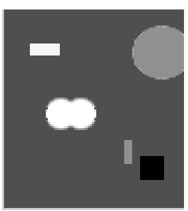
<u>Changes the metric of the space</u>

Slows down/blocks diffusion at edges

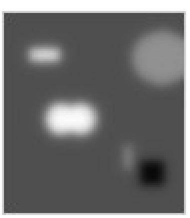
Diffusion

Uniform

$$\frac{\partial u}{\partial t} = div(\nabla u) = \Delta u$$
$$u(0) = u_0$$



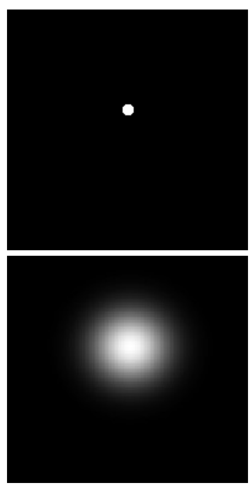
u(0)



u(t)

...

Steady state



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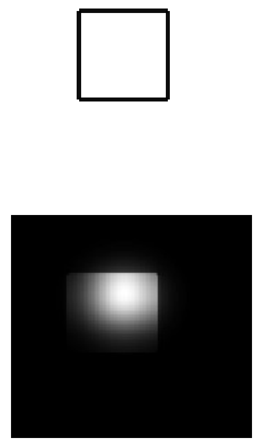

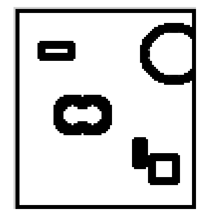

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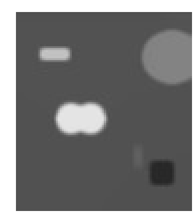
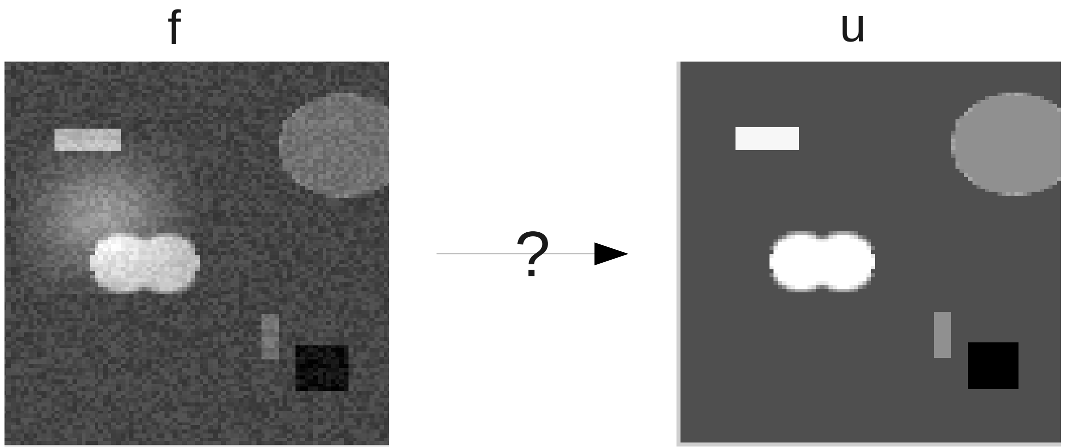


u(t)

Scalar-valued diffusivity
<u>Changes the metric of the space</u>
Slows down/blocks diffusion at edges

<u>Numerical solution:</u>
Parabolic PDEs
- fully implicit time discretization –> elliptic PDEs
- solve with implicit or semi-implicit FD schemes

Regularization and diffusion



data — regularization

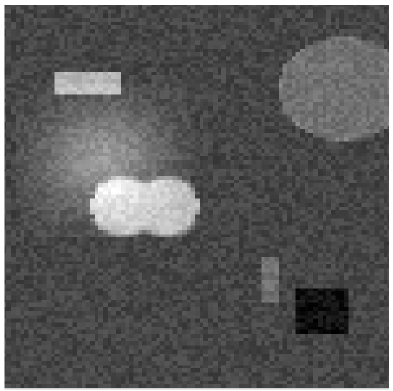
Energy functional : $E_u(u) = \frac{1}{2}\int_{\Omega} \left(|f-u|^2 + \alpha |\nabla u|^2 \right) dxdy$

$$u = argmin_u E_u(u)$$

Euler Lagrange Equation : $\frac{u-f}{\alpha} = \Delta u$

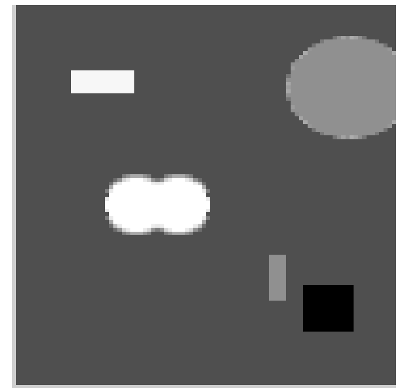
Regularization and diffusion

f



?

u



data regularization

Energy functional : $E_u(u) = \frac{1}{2} \int_\Omega \left(|f-u|^2 + \alpha |\nabla u|^2 \right) dxdy$

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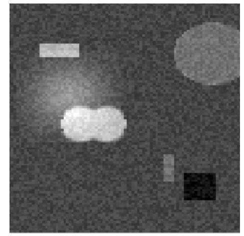
Euler Lagrange Equation : $\frac{u-f}{\alpha} = \Delta u$

Uniform diffusion

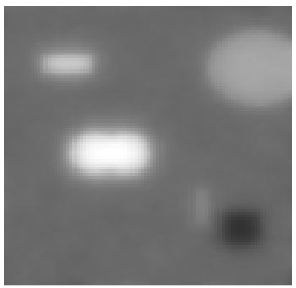
$$\frac{\partial u}{\partial t} = \Delta u$$

Fully implicit time discretization of the uniform diffusion with initial value f and time step α

Regularization



Uniform



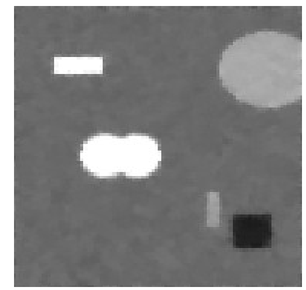
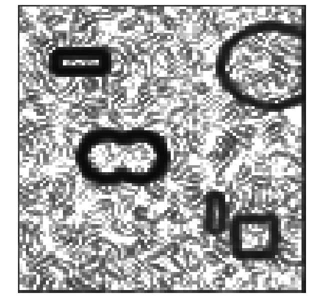
$$E_R(u) = \int_\Omega |\nabla u|^2 \, dxdy$$

Euler
Lagrange

$$\frac{u-f}{\alpha} = \Delta u$$

Isotropic

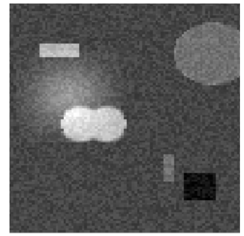
W

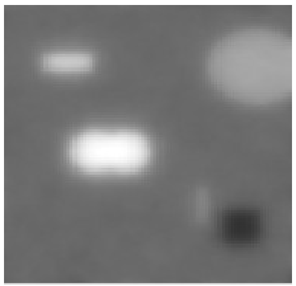
$$E_R(u) = \int_\Omega w(|\nabla f|^2) |\nabla u|^2 \, dxdy$$

$$\frac{u-f}{\alpha} = \nabla \cdot (w(.) \nabla u)$$

Regularization



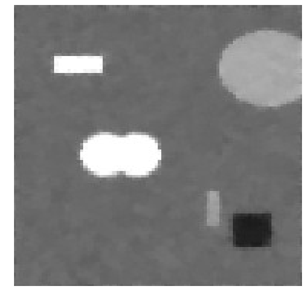
Uniform



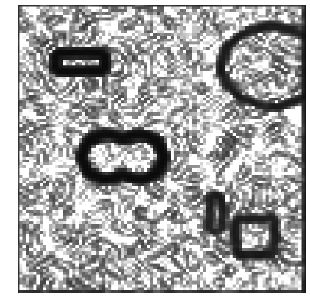
$$E_R(u) = \int_\Omega |\nabla u|^2 \, dxdy$$

Euler Lagrange $$\frac{u-f}{\alpha} = \Delta u$$

Isotropic



w



$$E_R(u) = \int_\Omega w(|\nabla f|^2) |\nabla u|^2 \, dxdy$$

$$\frac{u-f}{\alpha} = \nabla (w(.) \nabla u)$$

Deformable registration

$$min_u \; E_D(I, J \circ T) + \alpha \int_{x \in \Omega} w(.)(|\nabla u_x|^2 + |\nabla u_y|^2) \, d\mathbf{x}$$

$$\frac{\partial E_D}{\partial u_x} - \alpha \nabla (w(.) \nabla u_x) = 0$$

$$\frac{\partial E_D}{\partial u_y} - \alpha \nabla (w(.) \nabla u_y) = 0$$

Discrete diffusion

$$\Delta u = 0$$

$$u(x) = f_B, x \in K_B$$

Boundary conditions

Finite differences + implicit scheme

$\Delta u:$

	1	
1	-4	1
	1	

•

$u_{i-1,j-1}$	$u_{i-1,j}$	$u_{i-1,j+1}$
$u_{i,j-1}$	$u_{i,j}$	$u_{i,j+1}$
$u_{i+1,j-1}$	$u_{i+1,j}$	$u_{i+1,j+1}$

Pixel location

u Nx1
L NxN

$$Lu = 0$$

$$\begin{bmatrix} L_u & B \\ B & L_B \end{bmatrix} \begin{bmatrix} u_u \\ f_B \end{bmatrix} = 0$$

$$L_u u_u = -B^T f_B$$

Discrete diffusion

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Generalize L – isotropic diffusion

Combinatorial Laplacian

$$L_{ij} = \begin{cases} d_i = \sum_k w_{ik} & if\ i = j \\ -w_{ij} & if\ (i, j) \in E \\ 0 & otherwise \end{cases}$$

Positive semi-definite (PSD)

Discrete diffusion

$$\Delta u = 0$$

$$u(x) = f_B,\ x \in K_B$$ Boundary conditions

Finite differences + implicit scheme

$\Delta u:$

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•

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Pixel location

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$$\begin{bmatrix} L_u & B \\ B & L_B \end{bmatrix} \begin{bmatrix} u_u \\ f_B \end{bmatrix} = 0$$

$$L_u u_u = -B^T f_B$$

Combinatorial Dirichlet integral

$$D[u] = \int_\Omega |\nabla u|^2 dx$$

$$D[\mathbf{u}] = \mathbf{u}^t A^T A \mathbf{u} = \mathbf{u}^t L \mathbf{u}$$

$A_{(ij)k}$ incidence matrix (combinatorial gradient)

Critical points = minima [L-PSD]

$$L\mathbf{x} = 0$$

Generalize L – isotropic diffusion

Combinatorial Laplacian

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Positive semi-definite (PSD)

Random walker algorithm

[Grady PAMI 2006]

$$Lu = 0$$

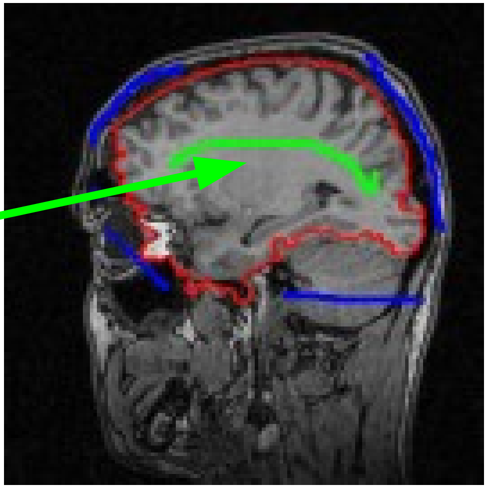
$$\begin{bmatrix} L_u & B \\ B & L_B \end{bmatrix} \begin{bmatrix} u_u \\ f_B \end{bmatrix} = 0$$

$$L_u u_u = -B^T f_B$$

Combinatorial Laplacian

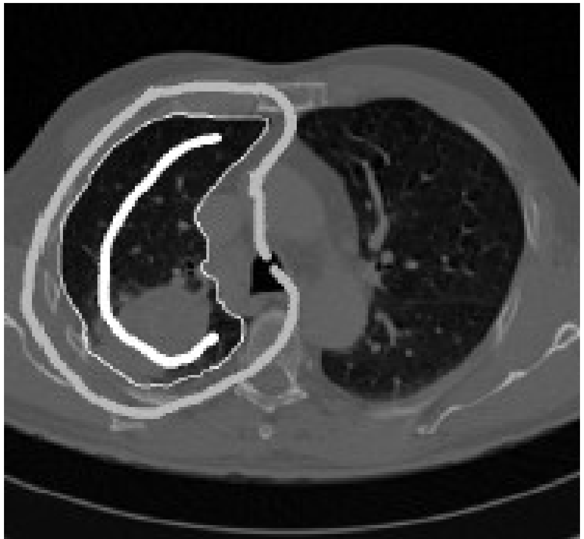
$$L_{ij} = \begin{cases} d_i = \sum_k w_{ik} & \text{if } i = j \\ -w_{ij} & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$$

$$w_{ij} = \exp(-\beta (I_i - I_j)^2)$$

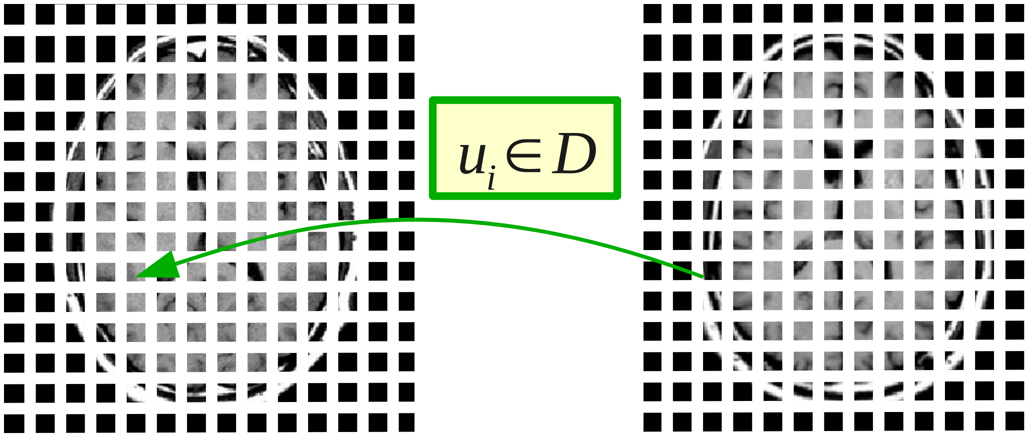


$\mathbf{f_B}$ seeds ;

$\mathbf{u_i}$ prob that a random walker from node i first reaches a seed.



RW for deformable registration



I_i, i=1..N $\quad$ J_i, i=1..N

Quantized deformations

$D = \{\mathbf{d}^1, \mathbf{d}^2, \ldots, \mathbf{d}^K\}$

Labels

$L = \{u^1, u^2, \ldots, u^K\}$

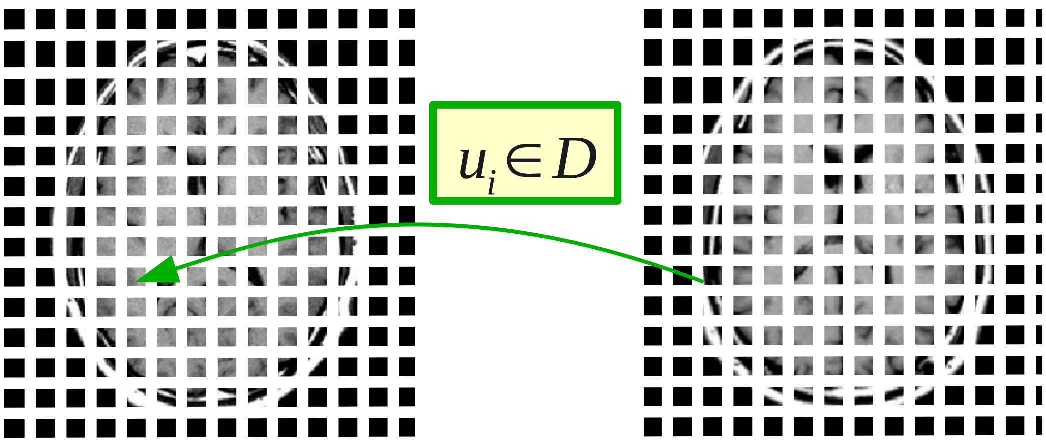
Registration as labeling : MRF

$$E(\boldsymbol{u}) = \sum_{i \in N} \Phi_i(u_i) + \alpha \sum_{(i,j) \in E} \Psi_{ij}(u_i, u_j)$$

data similarity potential

Regularization Interaction

RW for deformable registration



I_i, i=1..N J_i, i=1..N

Quantized deformations

D = {$\mathbf{d}^1$, $\mathbf{d}^2$, ... ,$\mathbf{d}^K$}

Labels

L = {u^1, u^2 , ... ,u^K}

Registration as labeling : MRF

$$E(\mathbf{u}) = \sum_{i \in N} \Phi_i(u_i) + \alpha \sum_{(i,j) \in E} \Psi_{ij}(u_i, u_j)$$

data similarity potential

Regularization Interaction

Gaussian MRF :

$$E^k(u^k) = \sum_{i \in N} \left(\sum_{l=1:K, l \neq k} \lambda_i^l (u_i^k)^2 + \lambda_i^k (1 - u_i^k)^2 \right)$$
$$+ \alpha \sum_{ij \in E} w_{ij} (u_i^k - u_j^k)^2$$

Modifications :

1. u_i^k = prob of node i having label u^k
2. Gaussian MRF with interaction
$$\Psi(u_i^k, u_j^k) = w_{ij}(u_i^k - u_j^k)^2$$
3. 'priors' for probability of a displacement
$$\lambda_i^k = \exp(-\gamma (I_i - J_{i+\mathbf{d}^k})^2)$$

Gaussian MRF
for deformable registration

Displacement probability

$$\lambda_i^k = \exp\left(-\gamma\left(I_i - J_{i+\mathbf{d}^k}\right)^2\right)$$

1 – if displacement $\mathbf{d}^k$ fits perfectly the similarity score
Small - otherwise

Data potential

$$\Phi_i(u_i^k) = \left(\sum_{l=1:K, l \neq k} \lambda_i^l (u_i^k)^2 + \lambda_i^k (1 - u_i^k)^2\right)$$

Displacement **l≠k** has **high prob.** > **low prob.** for label **k**

Displacement **k** has **high prob.** > **high prob.** for label **k**

Gaussian MRF for deformable registration

Displacement probability

$$\lambda_i^k = \exp(-\gamma(I_i - J_{i+\mathbf{d}^k})^2)$$

1 – if displacement **d**^{k} fits perfectly the similarity score
Small - otherwise

Data potential

$$\Phi_i(u_i^k) = \left(\sum_{l=1:K, l\neq k} \lambda_i^l (u_i^k)^2 + \lambda_i^k (1-u_i^k)^2\right)$$

Displacement **l≠k** has **high prob.** > **low prob.** for label **k**

Displacement **k** has **high prob.** > **high prob.** for label **k**

Regularization Interaction

$$\Psi(u_i^k, u_j^k) = w_{ij}(u_i^k - u_j^k)^2$$

$$w_{ij} = \exp(-\beta(J_i - J_j)^2)$$

⇔

Recall : regularization term (Dirichlet integral)

$$D[u] = \int_\Omega w(|\nabla J|^2)|\nabla u|^2 dx$$

$$D[\mathbf{u}] = \mathbf{u}^t L \mathbf{u}$$

Encourages two neighbors to have similar labels if the image intensity is similar.

RW with priors

[following Grady CVPR 2005]

$$E^k(u^k) = \sum_{i \in N} \left(\sum_{l=1:K, l \neq k} \lambda_i^l (u_i^k)^2 + \lambda_i^k (1-u_i^k)^2 \right) + \alpha \sum_{ij \in E} w_{ij} (u_i^k - u_j^k)^2$$

Combinatorial Laplacian L

$$L_{ij} = \begin{cases} d_i = \sum_k w_{ik} & if\ i=j \\ -w_{ij} & if\ (i,j) \in E \\ 0 & otherwise \end{cases}$$

$$\Lambda^k = diag(\lambda^k)$$

RW with priors

[following Grady CVPR 2005]

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$$\Lambda^k = diag(\lambda^k)$$

$$E^k(\mathbf{u}^k) = \sum_{l=1:K, l \neq k} \mathbf{u}^{kT} \Lambda^l \mathbf{u}^k + (1 - \mathbf{u}^k)^T \Lambda^k (1 - \mathbf{u}^k) + \alpha \mathbf{u}^{kT} L \mathbf{u}^k$$

Minimum when

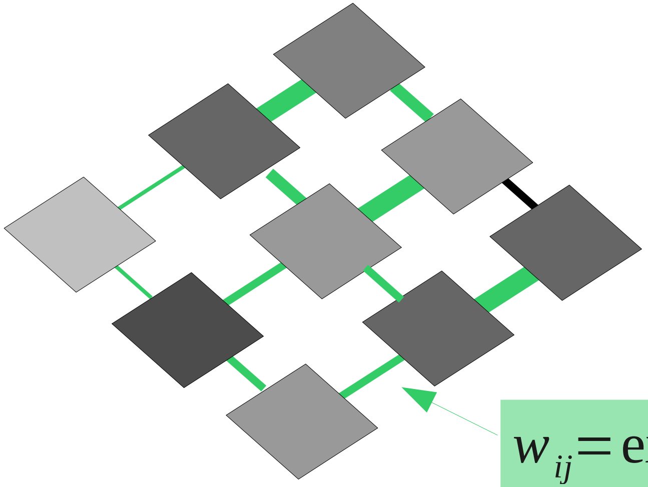
$$\left(\alpha L + \sum_{l=1}^{K} \Lambda^l \right) \mathbf{u}^k = \lambda^k$$

Note:
- The combined matrix still positive semi-definite
- One equation system per label !!

Augmented graph

$$\left(\alpha L + \sum_{l=1}^{K} \Lambda^{l}\right) \boldsymbol{u}^{k} = \lambda^{k}$$

Combinatorial Laplace equation for an augmented graph



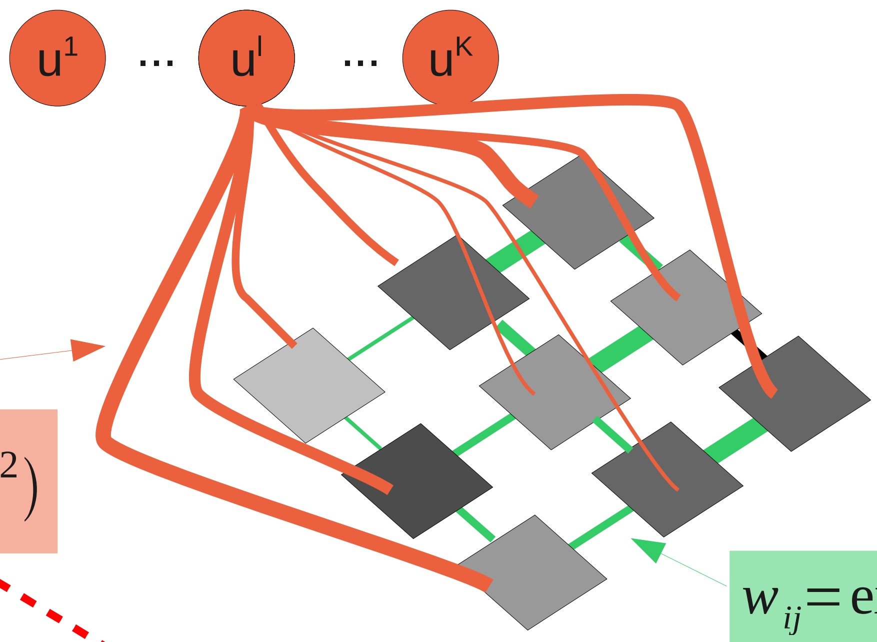
$$w_{ij} = \exp\left(-\beta (J_i - J_j)^2\right)$$

Large (1) if neighbors have similar color (allows diffusion)

Augmented graph

$$\left(\alpha L + \sum_{l=1}^{K} \Lambda^{l}\right) \boldsymbol{u}^{\boldsymbol{k}} = \lambda^{k}$$

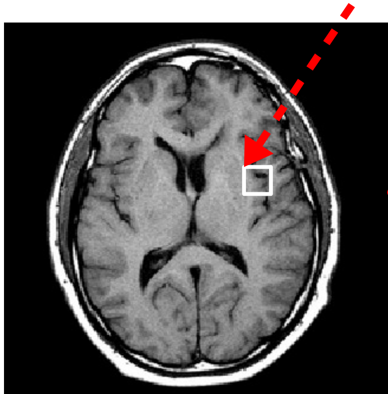
Combinatorial Laplace equation for an augmented graph



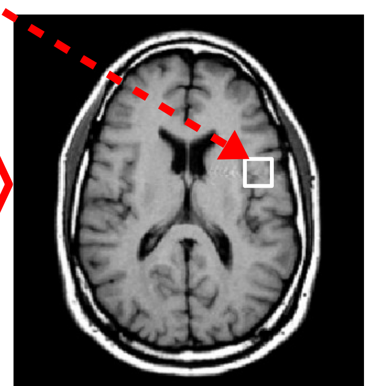
$$\lambda_i^l = \exp\left(-\gamma\left(I_i - J_{i+\boldsymbol{d}^l}\right)^2\right)$$

$$w_{ij} = \exp\left(-\beta\left(J_i - J_j\right)^2\right)$$

Large (1) if neighbors have similar color (allows diffusion)



Large (1) if Matching color (SSD)



Implementation

$$\left(\alpha L + \sum_{l=1}^{K} \Lambda^{l}\right)\boldsymbol{u}^{\boldsymbol{k}} = \lambda^{k}$$

Computational cost

- Need to solve for each displacement label (actually K-1)
  $K = (2*maxdispl+1)^3$
  Ex. maxdispl = 10 , K =9261
- Size of $L = (rows*cols*slices)^2$
  Ex. 200*200*40=16E5

➢ Multi-resolution pyramid
➢ Loose global optimality

displacements

	#	range
¼	60	[15,15]
½	40	[-10,10]
1	30	[-7,7]

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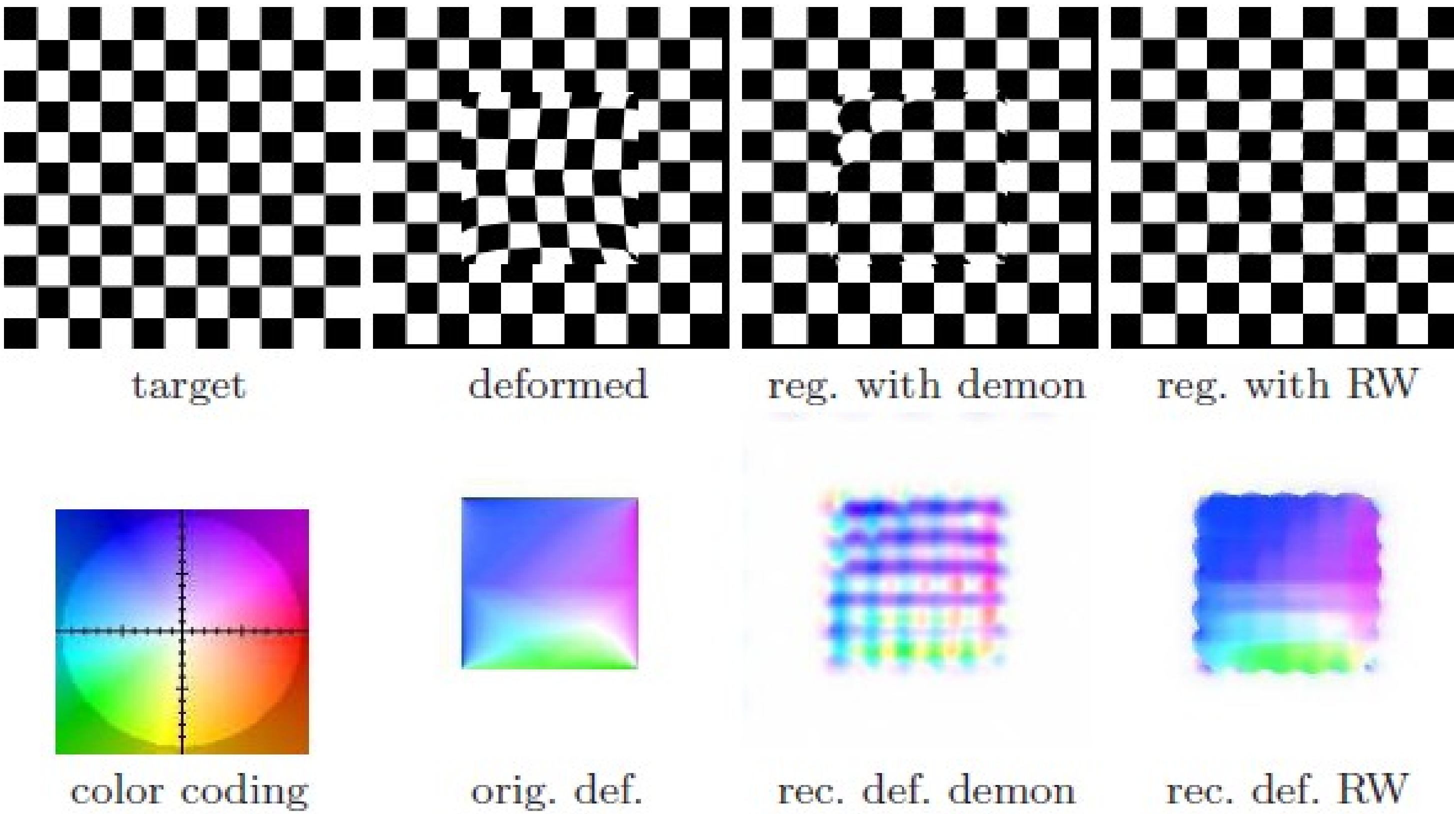
Algorithm 1 Random walker nonlinear registration

generate multi-resolution images $I_1(=I), I_2, I_3$ and J_3, with a factor $r(=2)$
for i=3:1 **do**
  define a set of discrete labels $\{\mathbf{d}_1, \ldots, \mathbf{d}_K\}$
  setup the image graph, compute Laplacian L and priors Λ
  solve for deformations labels $\mathbf{u}^k$ for every k
  assign $u_i = \mathbf{d}^k$ where $k = \text{argmax}(u_i^1, \ldots, u_i^K)$
  if $i > 1$ **then**
    compute source at next level $J_{i-1} = \text{warp}(\text{upsample}(J_i), \text{interp}(ru_i))$
  end if
end for
registered image = $\text{warp}(J_1, u_1)$

Experiments:
Quality of deformations

Synthetic image +	known deformation field	Demon's alg.	RW

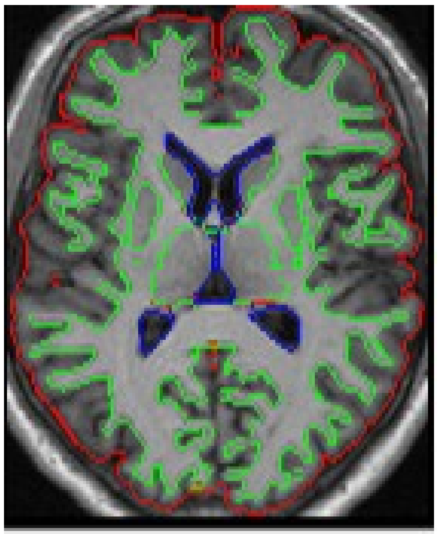


256x256 image
3 levels resol.
160sec (MATLAB)

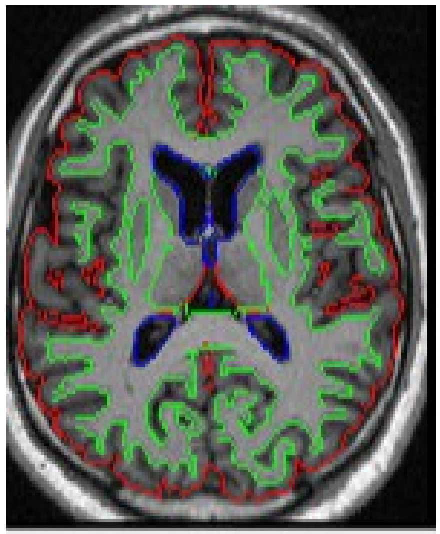
	Demon's alg.	RW
Ang. err. (deg)	1.58 ± 1.20	0.45 ± 0.97
Mag. Err. (deg)	1.12 ± 1.54	1.65 ± 2.89
SSD [0-255]	10.88	5.01

Brain MRI with WM/GM/CSF segmentation

source

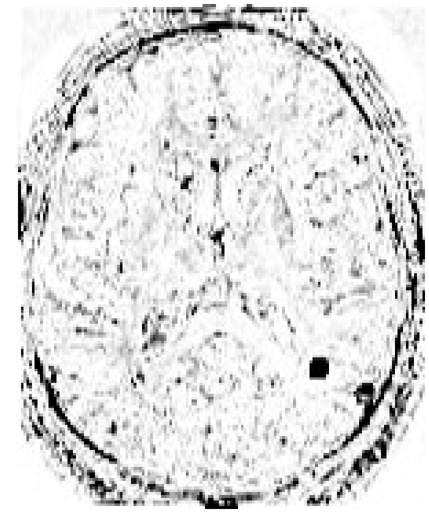


target

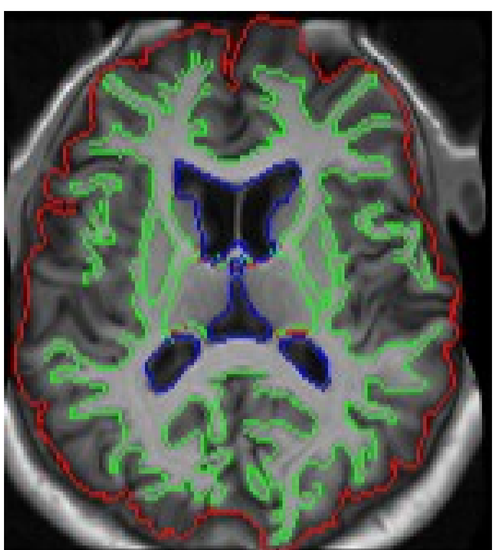


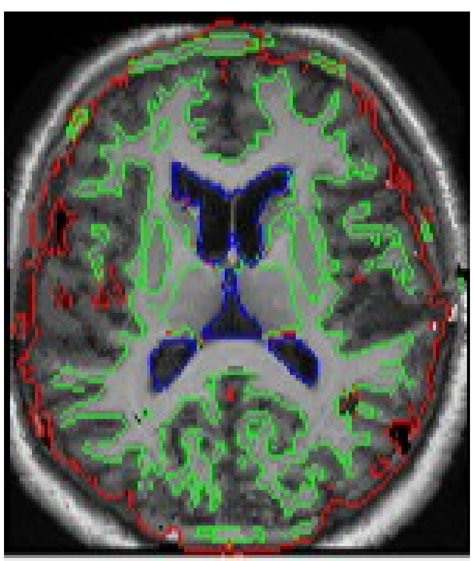
Reg. Demons
SSD=8.82

Reg. RW
SSD=3.38



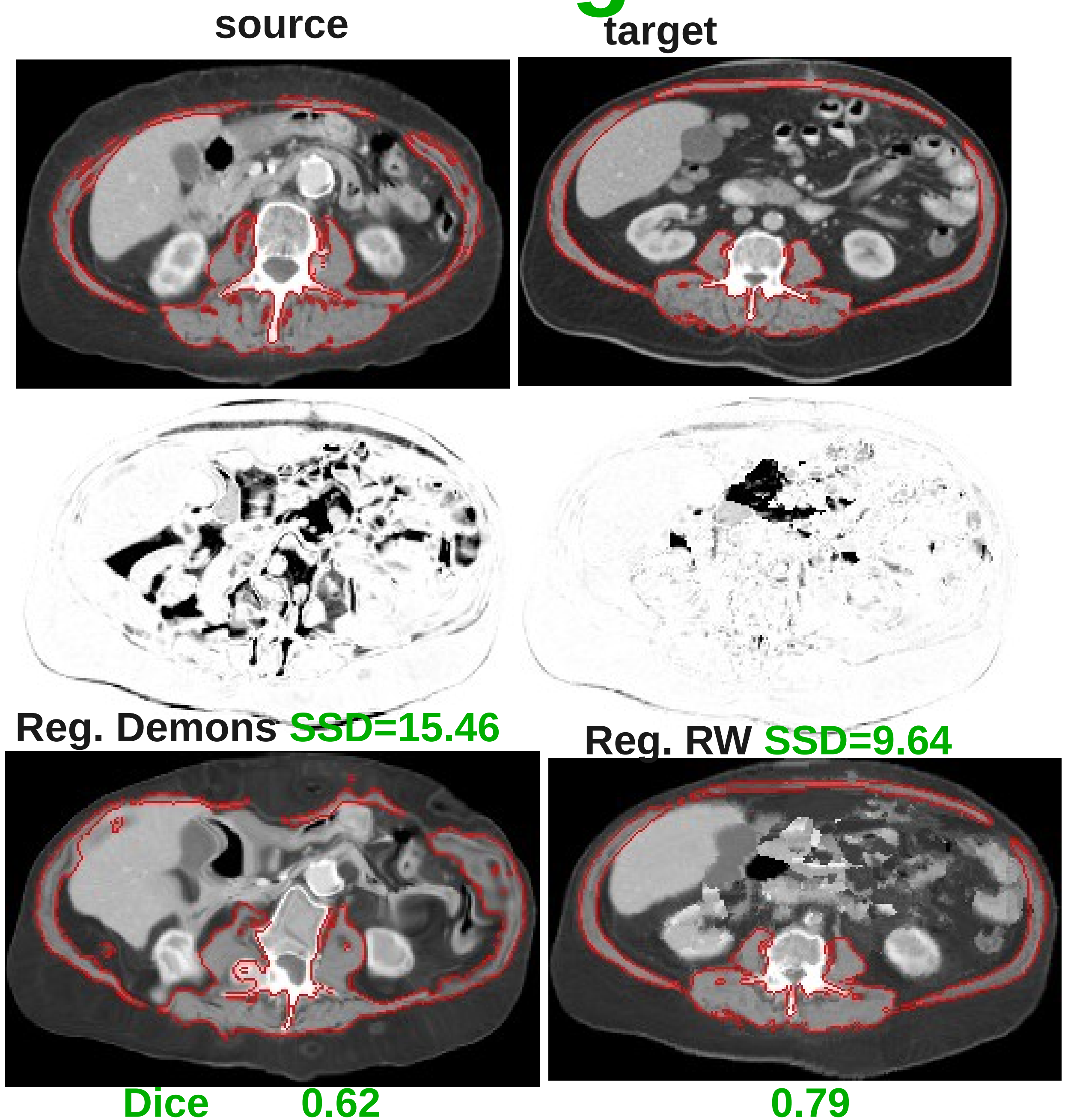
Data from Brain segmentation Repository





Dice 0.82 0.79 0.83 0.84 0.81 0.84

Abdominal CT with muscle segmentation



Data from CCI, Edmonton

Discussion

CONTRIBUTION:

New discrete formulation of deformable image registration based on the RW method

- Image dependent regularization term
- Global solution with respect to discretization

LIMITATIONS AND EXTENSIONS :

Computational cost :

Solve a large equation system for each label

Requirement of a point-wise similarity score

$$E(\boldsymbol{u}) = \sum_{i \in N} \Phi_i(u_i) + \alpha \sum_{(i,j) \in E} \Psi_{ij}(u_i, u_j)$$

Possible solution

Lower dim models :

Compute displacements only at control points

Approximate non-local scores in a neighborhood defined by the control points