Newton regularizations for EIT: Complete Electrode Model and convergence by local injectivity

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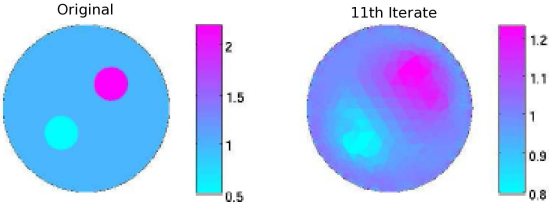
Motivation: Newton-like Methods for EIT



Outline

Newton Regularization

- $F : \mathcal{D}(F) \subset X \to Y$ nonlinear, Frechét-differentiable
- Aim: Stable approximation to x^+ solving $F(x^+) = y^+$, when given y^δ s.th. $\|y^\delta - y^+\| \leq \delta$
- Approximate x^+ by $x_{n+1} = x_n + s_n$, s_n close to $s_n^e = x^+ - x_n$
- s_n^e solves $F'(x_n)s_n^e = y^+ - F(x_n) - E(x^+, x_n)$
- Linearization error:
  $E(x^+, x_n) = F(x^+) - F(x_n) - F'(x_n)(x^+ - x_n)$
- Set $b_n^\delta = y^\delta - F(x_n)$ and solve $F'(x_n)s_n = b_n^\delta$
  by, e.g., Landweber iteration, Showalter method, (iterated) Tikhonov regularization, ν-methods, conjugate gradients

Inexact Newton Scheme to solve $F(x) = y^\delta$

Initial guess x_0
while $\|b_n^\delta\| \geq R\delta$:
repeat:
Approximate $F'(x_n)s_{n,m} = b_n^\delta$ in a stable way
m=m+1
until $\|F'(x_n)s_{n,m} - b_n^\delta\| < \mu_n\|b_n^\delta\|$
$x_{n+1} = x_n + s_{n,m}$
n=n+1

Inexact Newton Scheme to solve $F(x) = y^\delta$

Initial guess x_0
while $\|b_n^\delta\| \geq R\delta$:
  repeat:
    Approximate $F'(x_n)s_{n,m} = b_n^\delta$ Inner Iteration
    m=m+1
    until $\|F'(x_n)s_{n,m} - b_n^\delta\| < \mu_n \|b_n^\delta\|$
  $x_{n+1} = x_n + s_{n,m}$ Outer Iteration
  n=n+1

- L. & Rieder '10: Norm convergence if F satisfies tangential cone condition
- Convergence orders under source conditions require further assumptions on F, *à la* $F'(v) = Q(v, w)F'(w)$
- Newton-type methods are a vast field – see books of Kaltenbacher, Neubauer & Scherzer '08; Rieder '03; Engl, Hanke & Neubauer '96

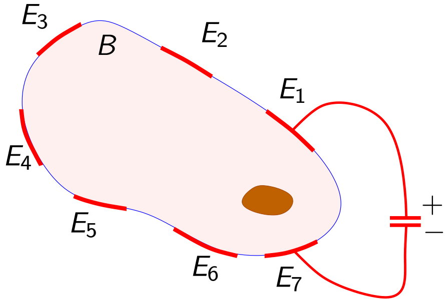
Tangential cone condition (TCC)

- Scherzer '93: Linearization error bounded by non-linear residual:

$$\|F(v) - F(w) - F'(w)(v - w)\| \leq L\|F(v) - F(w)\|$$

  for all $v, w \in B(x^+, \rho) \subset \mathcal{D}(F)$ with $L < 1$
- Assume that initial guess $x_0 \in B(x^+, \rho)$ with ρ small enough
- Then there is $N(\delta)$: all iterates $x_1, \ldots, x_{N(\delta)}$ stay in $B(x^+, \rho)$ and only final iterate satisfies discrepancy principle
- Monotonously decreasing error: $\|x_n - x^+\|_X < \|x_{n-1} - x^+\|$
- If x^+ is the only solution to $F(x) = y^+$, then $\|x_{N(\delta)} - x^+\| \to 0$ as $\delta \to 0$

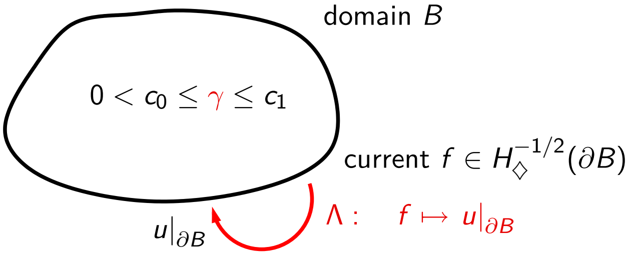
Impedance Tomography



- Body B with conductivity distribution γ
- Apply currents through electrodes and measure resulting voltages
- Seek for information about the conductivity distribution γ inside B!

Neumann-to-Dirichlet Operator Λ

- $u \in H^1_\diamondsuit(B)$ solves $\begin{cases} \nabla(\gamma \nabla u) = 0 & (B) \\ (\gamma \nabla u).\nu = f & (\partial B) \end{cases}$



- Λ bounded $H_\diamondsuit^{-1/2}(\partial B) \to H_\diamondsuit^{1/2}(\partial B)$
- Reconstruction of γ from Λ? Sylvester & Uhlmann '86, Nachman '96, Astala & Päivärinta '05

Complete Electrode Model (CEM)

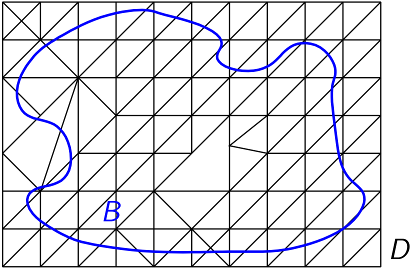
- Continuum model is far from “real-world” EIT
- Complete Electrode Model: Electrodes $\{E_j\}_{j=1}^{p}$ are perfect conductors; impedance b.c. at the electrode/object interface
- $\mathcal{E}_p := \mathrm{span}\{\chi_{E_1}, \dots, \chi_{E_p}\} \cap L^2_\diamondsuit(\partial B) \subset L^2_\diamondsuit(\partial B)$
- For a current “vector” $I \in \mathcal{E}_p$, seek $(u, U) \in H^1(B) \oplus \mathcal{E}_p$:

$$
\begin{aligned}
&-\nabla.(\gamma\nabla u) = 0 \quad (B), && u + z_j(\gamma\nabla u).\nu = U \quad (\cup_j E_j), \\
&(\gamma\nabla u).\nu \equiv 0 \quad (\partial B \setminus \cup_j \overline{E_j}), && \int_{E_j} (\gamma\nabla u).\nu \mathrm{d}s = I|_{E_j}.
\end{aligned}
$$

- Current-to-voltage map $\Lambda_p : I \mapsto U$ (on $\mathcal{E}_p$)

What can we reconstruct from Λ_p ?

- Λ_p has only $p(p-1)/2$ degrees of freedom
- Can one reconstruct γ in finite-dimensional space from Λ_p ?
- Consider γ in $V_{\mathcal{T}}^{+} = V_{\mathcal{T}} \cap L_{+}^{\infty}(B)$ where $V_{\mathcal{T}}$ are piecewise polynomials



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- Consider γ in $V_{\mathcal{T}}^{+} = V_{\mathcal{T}} \cap L_{+}^{\infty}(B)$ where $V_{\mathcal{T}}$ are piecewise polynomials
- Show: Forward operator $F_p : V_{\mathcal{T}}^{+} \subset V_{\mathcal{T}} \to \mathcal{L}(\mathcal{E}_p)$, $F_p(\gamma) = \Lambda_p$, has injective Frechét derivative: $F_p'(\gamma)[h] \neq 0$ if $h \neq 0$. Then finite-dimensionality implies TCC and local convergence theory for inexact Newton methods
- 2 steps:
  - First, show that $F : V_{\mathcal{T}}^{+} \subset V_{\mathcal{T}} \to \mathcal{L}(L_{\diamondsuit}^2(\partial B), L_{\diamondsuit}^2(\partial B))$, $F(\gamma) = \Lambda$, has injective Frechét derivative
  - Second, use approximation results for electrode models to show that $F_p'(\gamma)$ is also injective

Continuum Model - Injectivity of Frechet Derivative I

- Recall: $F(\gamma) = \Lambda$, and $F : V_{\mathcal{T}}^{+} \subset V_{\mathcal{T}} \to \mathcal{L}(L^2_\diamondsuit(\partial B), L^2_\diamondsuit(\partial B))$
- F is Frechét differentiable with derivative $F'(\gamma)[h]f = u'|_{\partial B}$,

$$\int_B \gamma \nabla u' \cdot \nabla \varphi \, \mathrm{d}x = -\int_B h \nabla u(f) \cdot \nabla \varphi \, \mathrm{d}x \quad \text{for all } \varphi \in H^1_\diamondsuit(B)$$

- If $F'(\gamma)[h] = 0$, that is, $F'(\gamma)[h]f = u'|_{\partial B} = 0$ for every $f \in L^2_\diamondsuit(\partial B)$, then

$$0 = \int_{\partial B} f \underbrace{(F'(\gamma)[h]f)}_{=u'|_{\partial B}=0} \mathrm{d}s = \int_B \gamma \nabla u(f) \cdot \nabla u' \, \mathrm{d}x$$

$$= -\int_B h |\nabla u(f)|^2 \, \mathrm{d}x$$

Continuum Model - Injectivity of Frechet Derivative II

- If we find, for all $h \in V_{\mathcal{T}}$, currents f_j such that $\int_B h|\nabla u(f_j)|^2 \,\mathrm{d}x$ blows up, then F' is injective
- Employ the localized potentials of Harrach ('08): For two disjoint subdomains $\Omega_{1,2}$ such that $\overline{B} \setminus (\overline{\Omega_1} \cup \overline{\Omega_2})$ is connected and contains $S \subset \partial B$, there exists $\{f_j\} \in L^2_\diamondsuit(S)$ with

$$\lim_{j\to\infty} \int_{\Omega_1} |\nabla u(f_j)|^2 \,\mathrm{d}x = \infty \text{ and } \lim_{j\to\infty} \int_{\Omega_2} |\nabla u(f_j)|^2 \,\mathrm{d}x = 0$$

- Use that h is piecewise polynomial to construct blow-up
- Due to finite-dimensionality:

$$\min \left\{ \|F'(\gamma)[h]\|_{\mathcal{L}(L^2_\diamondsuit(\partial B))} : h \in V_{\mathcal{T}},\ \|h\|_\infty = 1 \right\} > 0$$

Approximation of CEM by CM

- Consider a sequence of forward operators $\{F_p\}_{p\in\mathbb{N}}$ for electrode configurations $\mathcal{E}_p$
- p corresponds to the number of electrodes $\{E_j^p\}_{j=1}^p$ of $\mathcal{E}_p$
- $P_p : L^2_\diamondsuit(\partial B) \to L^2_\diamondsuit(\partial B)$ the orthogonal projection onto $\mathcal{E}_p$
- $u(f)$ and $u_p(P_p f)$ are electric potentials for continuum model and complete model, respectively
- Under suitable (geometric) assumptions on electrodes: $\lim_{p\to\infty} \sup\left\{\|u(f) - u_p(P_p f)\|_{H^1(B)} : \|f\|_{L^2(\partial B)} = 1\right\} = 0$ (L., Hyvönen, Hakula '08; Hyvönen '09)
- Under the same assumptions on $\mathcal{E}_p$,

$$\lim_{p\to\infty} \sup\left\{\|F'(\gamma)[\cdot] - F'_p(\gamma)[\cdot]P_p\|_{\mathcal{L}(L^\infty(B),\mathcal{L}(L^2_\diamondsuit(\partial B)))}\right\} = 0$$

Complete Model - Injectivity of Frechet Derivative

- Triangle inequality:

$$\begin{aligned}\|F_p'(\gamma)[h]P_p\|_{\mathcal{L}(L^2_\diamondsuit(\partial B))} \geq & \|F'(\gamma)[h]\|_{\mathcal{L}(L^2_\diamondsuit(\partial B))} \\ & - \|F'(\gamma)[h] - F_p'(\gamma)[h]P_p\|_{\mathcal{L}(L^2_\diamondsuit(\partial B))}\end{aligned}$$

- Set $\Gamma(r) := \min \{ \|F'(\gamma)[h]\|_{\mathcal{L}(L^2_\diamondsuit(\partial B))} : \gamma,\ h \in V_\mathcal{T},\ \|h\|_\infty = 1,\ c_0 \leq \gamma \leq r \} > 0$
- If $\gamma \in V_\mathcal{T}^+$ then $F_p'(\gamma) \in \mathcal{L}(V_\mathcal{T}, \mathcal{L}(\mathcal{E}_p))$ is injective for any p such that $\|F'(\gamma)[\cdot] - F_p'(\gamma)[\cdot]P_p\|_{\mathcal{L}(V_\mathcal{T},\mathcal{L}(L^2_\diamond(\partial B)))} < \Gamma(\|\gamma\|_\infty)$
- Necessary requirement for injectivity: $p(p-1) \geq 2\dim(V_\mathcal{T})$

Comments

- TCC allows to prove norm convergence of inexact Newton schemes as the noise level tends to zero (for complete electrode model and piecewise polynomial conductivities)
- Variant: Finite element discretization of variational problems, leading to forward operators $F_{p,\delta} \rightsquigarrow$ TCC for $F_{p,\delta}$ with constants independent of p and $\delta \in (0, \delta_{\mathsf{max}})$
- Result is asymptotic as $p \to \infty$. It would be interesting to know more for specific configurations

Thanks for your attention!