Reconstructing conductivities with boundary corrected D-bar method

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The aim of electrical impedance tomography (EIT) is to form an image of the conductivity distribution inside an unknown body using electric boundary measurements.

Applications in medical imaging, nondestructive testing, subsurface monitoring:

- monitoring heart and lungs of unconscious patients,
- detecting pulmonary edema,
- breast cancer detection,
- detecting cracks in concrete structures,
- environmental applications...

The *inverse conductivity problem* introduced by Calderón is to find the conductivity σ when the *Dirichlet-to-Neumann* -map (DN-map) Λ_σ is known.

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0 & \text{in} \quad \Omega_1, \\ u = f & \text{on} \quad \partial\Omega_1. \end{cases} \tag{1}$$

$$\Lambda_\sigma f = \sigma \frac{\partial u}{\partial \nu}|_{\partial\Omega_1} \tag{2}$$

The problem of reconstructing σ from Λ_σ is nonlinear and ill-posed.

In the two-dimensional case, unique reconstruction can be obtained:

- For $\sigma \in W^{2,p}(\Omega_1), p > 1$, Nachman 1996 (the *D-bar method*), Knudsen-Lassas-Mueller-Siltanen 2009
- For $\sigma \in W^{1,p}(\Omega_1), p > 2$, Brown-Uhlmann 1997
- and for $\sigma \in L^\infty(\Omega_1)$ Astala-Päivärinta 2003.

Nachman's method reduces the reconstruction problem to the case $\sigma = 1$ near the boundary. This procedure is tested numerically in this joint work with S. Siltanen.

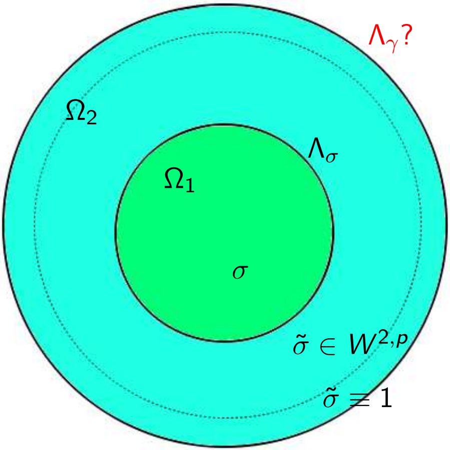
We transform the original conductivity equation to Schrödinger equation: by writing $u = \sigma^{-1/2}\tilde{u}$ we get

$$(-\triangle - q)\tilde{u} = 0 \quad \text{in } \Omega_1, \tag{3}$$

where $q = \sigma^{-1/2}\triangle(\sigma^{1/2})$. This means we have to have $\sigma \geq c_0 > 0$. The DN-map becomes

$$\Lambda_q = \sigma^{-1/2}(\Lambda_\sigma + \frac{1}{2}\frac{\partial\sigma}{\partial\nu})\sigma^{-1/2}. \tag{4}$$

In order for this transformation to be useful we need to have $\sigma \equiv 1$ near $\partial\Omega_1$ so that $\Lambda_q = \Lambda_\sigma$.



We extend σ:

$$\gamma(x) = \begin{cases} \sigma(x), & x \in \Omega_1, \\ \tilde{\sigma}(x), & x \in \Omega_2 \setminus \overline{\Omega_1}, \end{cases}$$

$$\tilde{\sigma}|_{\partial\Omega_1} = \sigma|_{\partial\Omega_1} \tag{5}$$

$$\frac{\partial \sigma}{\partial \nu}|_{\partial\Omega_1} = \frac{\partial \tilde{\sigma}}{\partial \nu}|_{\partial\Omega_1} \tag{6}$$

giving us $\gamma \in W^{2,p}(\Omega_2)$ and $\Lambda_q = \Lambda_\gamma$.

In this work,

$$\begin{aligned}
\Omega_1 &= D(\mathbf{0}, r_1), \quad r_1 = 1.0 \\
\Omega_2 &= D(\mathbf{0}, r_2), \quad r_2 = 1.2 \\
\sigma &\in L^\infty(\Omega_1) \\
\tilde{\sigma} &\in W^{2,p}(\Omega_2 \setminus \overline{\Omega_1}) \\
\gamma &\in L^\infty(\Omega_2),
\end{aligned}$$

Define two Dirichlet problems, for $j = 1, 2$,

$$\begin{cases} \nabla \cdot (\tilde{\sigma} \nabla u_j) = 0 & \text{in} \quad \Omega_2 \setminus \overline{\Omega_1} \\ u_j = f_j & \text{on} \quad \partial\Omega_j \\ u_j = 0 & \text{on} \quad \partial\Omega_i, \quad i = 1, 2, \quad i \neq j. \end{cases} \tag{7}$$

Four new DN maps in $\Omega_2 \setminus \overline{\Omega_1}$ can be characterized by

$$\Lambda^{ij} f_j = \tilde{\sigma} \frac{\partial u_j}{\partial \nu}|_{\partial\Omega_i}, \quad i, j = 1, 2 \tag{8}$$

so $\lambda^{ij} : H^{1/2}(\partial\Omega_j) \to H^{-1/2}(\partial\Omega_i)$.

Proposition (6.1 in Nachman 1996)

Let $\gamma \in W^{2,p}(\Omega_2), p > 1$, *then*

$$\Lambda_\gamma = \Lambda^{22} + \Lambda^{21}(\Lambda_\sigma - \Lambda^{11})^{-1}\Lambda^{12}. \tag{9}$$

The proof consists of showing that the operator $\Lambda_\sigma - \Lambda^{11}$ is invertible, and the identities

$$(\Lambda_\gamma - \Lambda^{22})f_2 = \Lambda^{21}(u|_{\partial\Omega_1}) \tag{10}$$

$$(\Lambda_\sigma - \Lambda^{11})(u|_{\partial\Omega_1}) = \Lambda^{12}f_2 \tag{11}$$

for any u that solves the conductivity equation in Ω_2 with $u|_{\partial\Omega_2} = f_2$.

The D-bar method:

- based on Nachman 1996
- robust algorithm was given by Siltanen-Mueller-Isaacson
- The method has been successfully tested on a chest phantom and on *in vivo* human chest data (Isaacson-Mueller-Newell-Siltanen).

$$\Lambda_\gamma \to \mathbf{t}(k) \to \gamma$$

The CGO-solution (Complex Geometric Optics) on $\partial\Omega_2$ can be solved from

$$\psi(\cdot, k)|_{\partial\Omega_2} = e^{ikx} - S_k(\Lambda_\gamma - \Lambda_1)\psi(\cdot, k)|_{\partial\Omega_2}, \qquad (12)$$

in the Sobolev space $H^{1/2}(\partial\Omega_2)$ for all $k \in \mathbb{C} \setminus \{0\}$. Here S_k is a single-layer operator

$$(S_k\phi)(x) := \int_{\partial\Omega_2} G_k(x-y)\phi(y)ds,$$

and G_k is Faddeev's Green function defined by

$$G_k(x) := e^{ikx}g_k(x), \quad g_k(x) := \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{e^{ix\cdot\xi}}{|\xi|^2 + 2k(\xi_1 + i\xi_2)}d\xi.$$

Using the CGO-solution we can define *the scattering transform*:

$$\mathbf{t}(k) = \int_{\partial\Omega_2} e^{i\bar{k}\bar{x}}(\Lambda_\gamma - \Lambda_1)\psi(\,\cdot\,, k)ds. \tag{13}$$

For each fixed $x \in \Omega_2$, we would solve the following integral formulation of the D-bar equation:

$$\mu(x,k) = 1 + \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{\mathbf{t}(k')}{(k-k')\bar{k}'} e^{i(k'x+\overline{k'x})}\overline{\mu(x,k')}dk_1'dk_2', \tag{14}$$

where $\mu(x,k) = \exp(-ik(x_1 + ix_2))\psi(x,k)$. Then the conductivity would be perfectly reconstructed as $\gamma(x) = \mu(x,0)^2$.

In this work we omit the requirement of continuity of the derivative $\frac{\partial\gamma}{\partial\nu}$ and use the method of Nakamura-Tanuma-Siltanen-Wang to calculate $g(\theta) \approx \sigma|_{\partial\Omega_1}$.

We extend σ to γ using the following extension in polar coordinates:

$$\gamma(\rho,\theta) = \begin{cases} \sigma(\rho,\theta), & \rho \le r_1, \\ (g(\theta)-1)f_m(\rho)+1, & r_1 < \rho \le r_e, \\ 1, & r_e < \rho \le r_2, \end{cases} \tag{15}$$

where $r_e = 1.175$ and $f_m(\rho) \ge 0$ is a suitable third-degree polynomial satisfying $f_m(r_1) = 1$ and $f_m(r_e) = 0$.

Any linear operator $\Lambda : H^{1/2}(\partial\Omega_i) \to H^{-1/2}(\partial\Omega_j)$ can be represented by a matrix in the following way. Define a truncated orthonormal basis at the boundary $\partial\Omega_k$:

$$\phi_k^{(n)}(\theta) = \frac{1}{\sqrt{2\pi r_k}} e^{in\theta}, \quad n = -N, ..., N, \quad k = 1, 2. \tag{16}$$

Write any function $f : \partial\Omega_i \to \mathbb{C}$ as a vector

$$\vec{f} = [\hat{f}(-N), \hat{f}(-N+1), \ldots, \hat{f}(N-1), \hat{f}(N)]^T, \quad \hat{f}(n) = \int_{\partial\Omega_i} f \overline{\phi_i^{(n)}} ds.$$

Then the operator Λ is approximated by the matrix $L = [\widehat{u}(\ell, n)]$, where

$$\widehat{u}(\ell, n) = \int_{\partial\Omega_j} (\Lambda\phi_i^{(n)}) \overline{\phi_j^{(\ell)}} ds. \tag{17}$$

In practice in EIT we measure the *Neumann-to-Dirichlet* -map approximated by the matrix R_σ. We simulate measurement noise by using the matrix

$$R_\sigma^\varepsilon := R_\sigma + cE, \tag{18}$$

where E is a matrix with random entries independently distributed according to the Gaussian normal density. Then, the noisy DN-matrix L_σ^ε is roughly speaking the inverse of R_σ^ε, and the boundary correction procedure gives us

$$L_\gamma^\varepsilon = L^{22} + L^{21}(L_\sigma^\varepsilon - L^{11})^{-1}L^{12}, \tag{19}$$

provided that the matrix $L_\sigma^\varepsilon - L^{11}$ is invertible.

The D-bar method numerically: expand $e^{ikx}|_{\partial\Omega_2}$ as a vector $\vec{g}$ in our finite trigonometric basis, solve the CGO-solution

$$\vec{\psi}_k := [I + \mathbf{S}_k(L_\gamma^\varepsilon - L_1)]^{-1}\vec{g}. \tag{20}$$

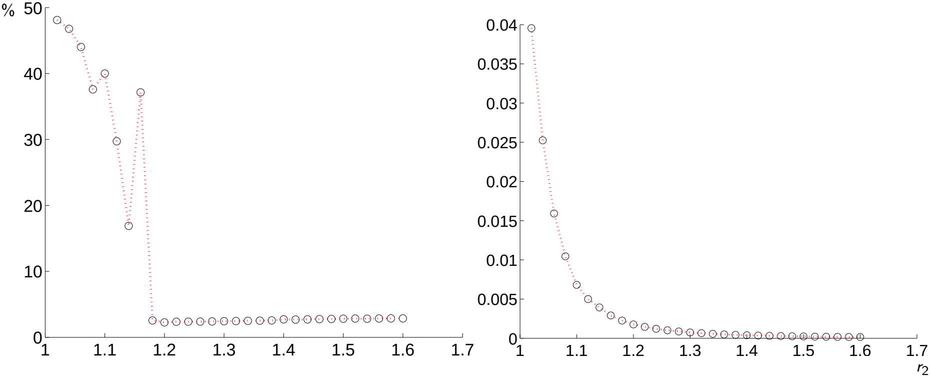
for k ranging in a grid inside the disc $|k| < R$ (the truncation radius $R > 0$). Define the truncated scattering transform by

$$\mathbf{t}_R(k) = \begin{cases} \int_{\partial\Omega_2} e^{i\bar{k}\bar{x}} \mathcal{F}^{-1}((L_\gamma^\varepsilon - L_1)\vec{\psi}_k)(x)ds & \text{for } |k| < R, \\ 0, & \text{otherwise}, \end{cases} \tag{21}$$

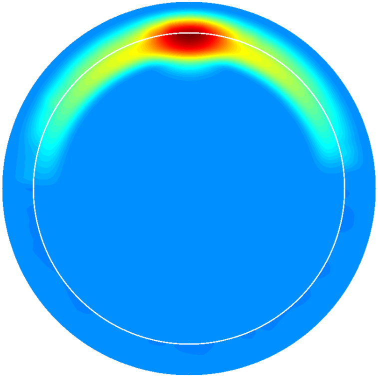
Finally solve the equation (14) with the numerical algorithm of Knudsen-Mueller-Siltanen using $\mathbf{t}_R$ and denote the solution by $\mu_R(x,k)$. Then $\gamma(x) \approx \mu_R(x,0)^2$.

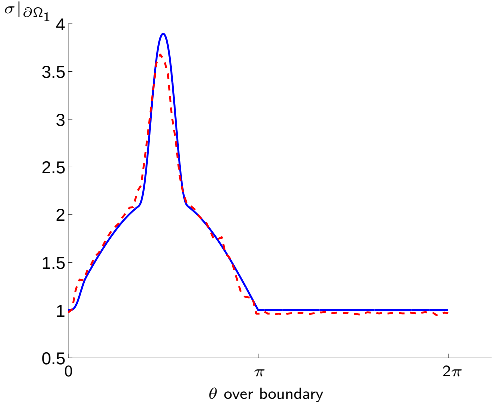
- Solutions to the Dirichlet problems are calculated using FEM, with 1048576 triangles in the disk Ω_1 and 425984 triangles in the annulus $\Omega_2 \setminus \overline{\Omega_1}$.
- $\|R_1^\epsilon - R_1\|/\|R_1\| = 0.0001$ (the $\mathrm{ACT3}$ impedance tomography imager of Rensselaer Polytechnic Institute has SNR of 95.5 dB $\approx$ noise of 0.0017%)
- $\epsilon_{\mathrm{fem}} = \|R_1^{\mathrm{th}} - R_1\|/\|R_1^{\mathrm{th}}\| = 0.0000173$,
- the error $\|R_\sigma^\epsilon - R_\sigma\|/\|R_\sigma\|$ ranges between 0.00011 and 0.00076,
- The condition number of the matrix $L_\sigma^\varepsilon - L^{11}$ was less than 27 in all our test cases.
- The error $\|L_\gamma^\epsilon - L_\gamma^2\|/\|L_\gamma^2\|$, where L_γ^2 is the DN map calculated directly on the boundary $\partial\Omega_2$, was less than 2.2% in all cases.

$$\|L^{\epsilon}_{\gamma=1} - L^{2}_{\gamma=1}\| / \|L^{2}_{\gamma=1}\| \qquad \|L^{\epsilon}_{\gamma=1} - L_{\gamma=1}\| / \|L_{\gamma=1}\|$$

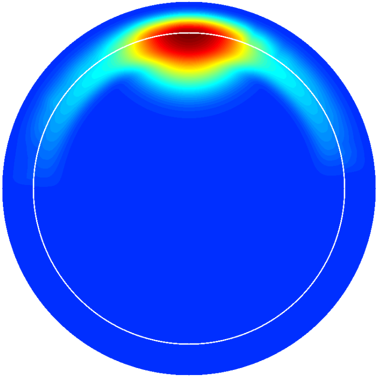


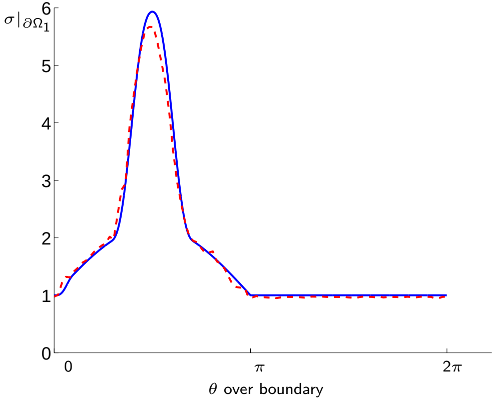
Example 1



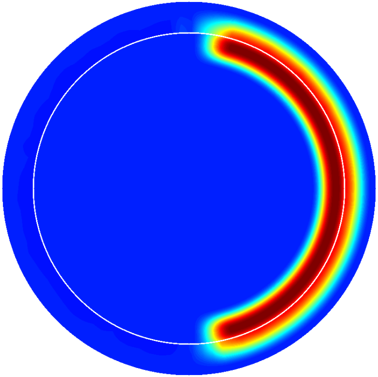

$\sigma|_{\partial\Omega_1}$
4
3.5
3
2.5
2
1.5
1
0.5
0
π
2π
θ over boundary

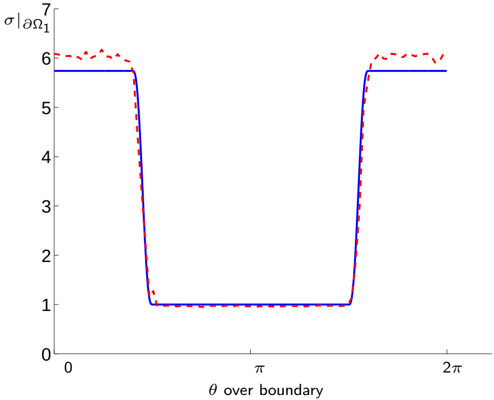
Example 2



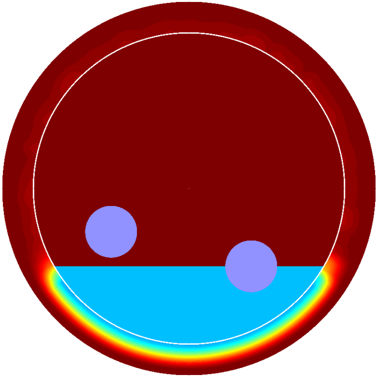


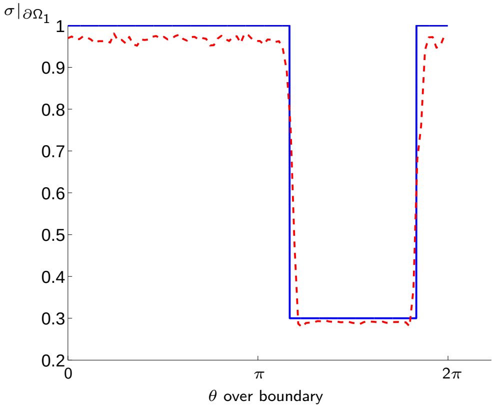
Example 3

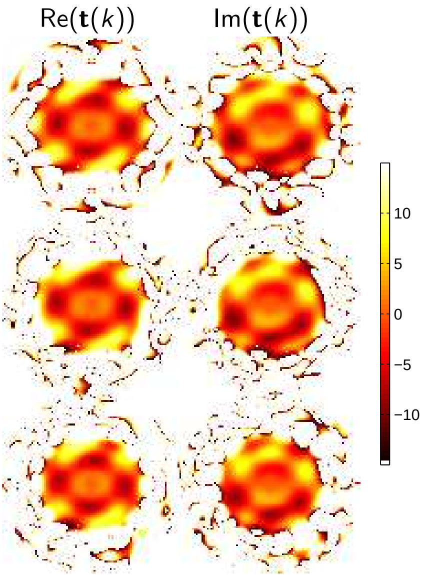




Example 4




$\sigma|_{\partial\Omega_1}$
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0
π
2π
θ over boundary

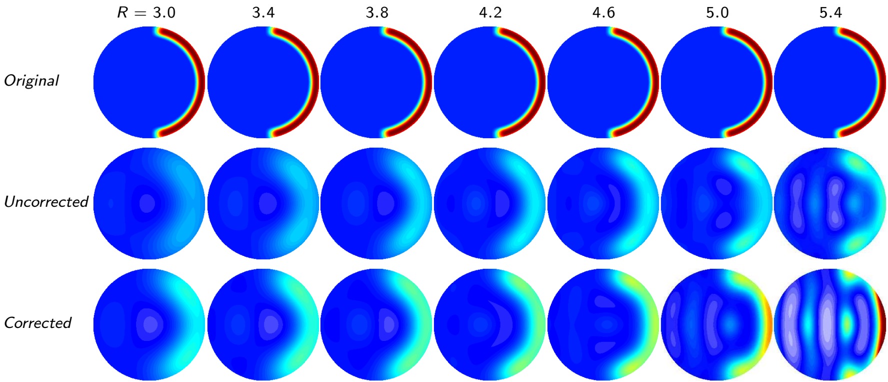


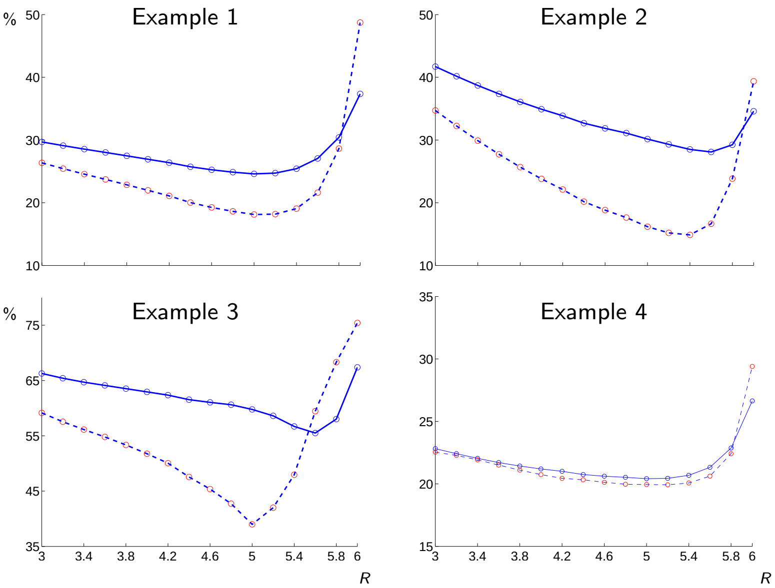
non-noisy
L_σ

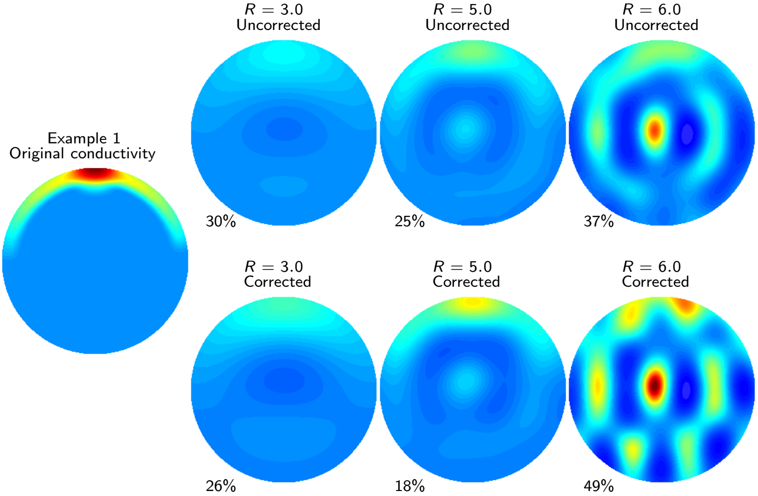
noisy
L_σ^ε

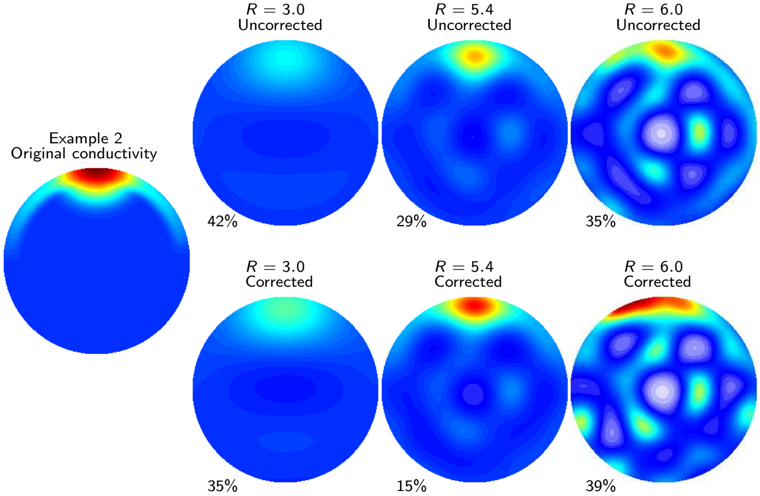
noisy and corrected
L_γ^ε

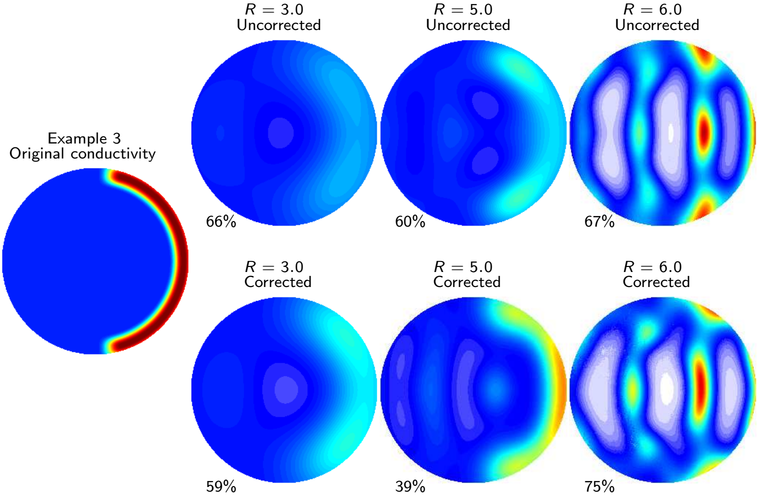
Scattering transform, real and imaginary parts, of Example 4.

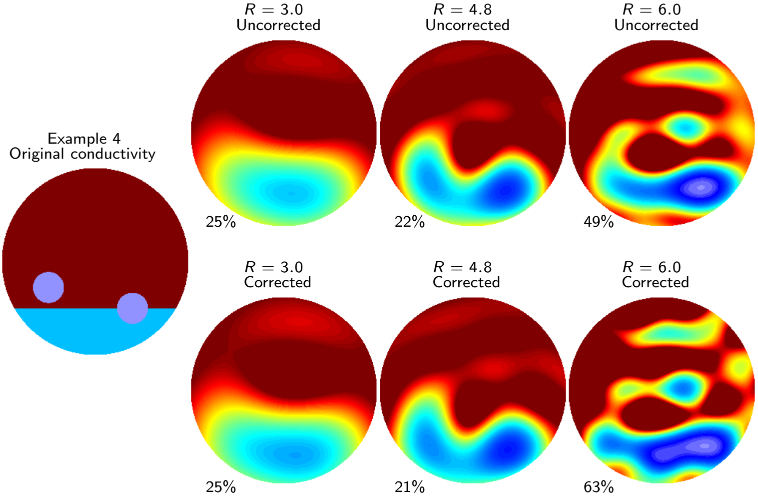

$R = 3.0$
3.4
3.8
4.2
4.6
5.0
5.4
Original
Uncorrected
Corrected


Example 1
Example 2
Example 3
Example 4
%
%
50
40
30
20
10
75
65
55
45
35
35
30
25
20
15
3
3.4
3.8
4.2
4.6
5
5.4
5.8
6
R


Example 1
Original conductivity
$R = 3.0$
Uncorrected
30%
$R = 5.0$
Uncorrected
25%
$R = 6.0$
Uncorrected
37%
$R = 3.0$
Corrected
26%
$R = 5.0$
Corrected
18%
$R = 6.0$
Corrected
49%


Example 2
Original conductivity
$R = 3.0$
Uncorrected
42%
$R = 5.4$
Uncorrected
29%
$R = 6.0$
Uncorrected
35%
$R = 3.0$
Corrected
35%
$R = 5.4$
Corrected
15%
$R = 6.0$
Corrected
39%


Example 3
Original conductivity
$R = 3.0$
Uncorrected
66%
$R = 5.0$
Uncorrected
60%
$R = 6.0$
Uncorrected
67%
$R = 3.0$
Corrected
59%
$R = 5.0$
Corrected
39%
$R = 6.0$
Corrected
75%


Example 4
Original conductivity
R = 3.0
Uncorrected
25%
R = 4.8
Uncorrected
22%
R = 6.0
Uncorrected
49%
R = 3.0
Corrected
25%
R = 4.8
Corrected
21%
R = 6.0
Corrected
63%

Thank you!

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