Numerical Aspects of π-line reconstruction algorithms in tomography

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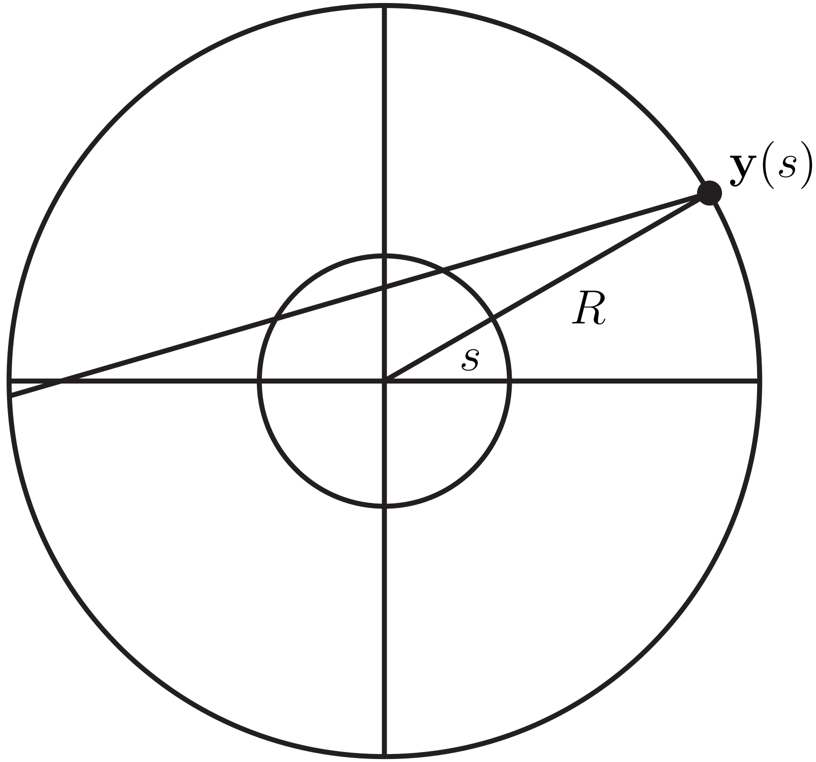
Tomography with sources on a curve

Data: Measurements of the divergent beam transform

$$\mathcal{D}f(\mathbf{y}, \boldsymbol{\theta}) = \int_0^\infty f(\mathbf{y} + t\boldsymbol{\theta})\, dt.$$

$$\mathbf{y}(s) \;=\; \text{source curve.}$$

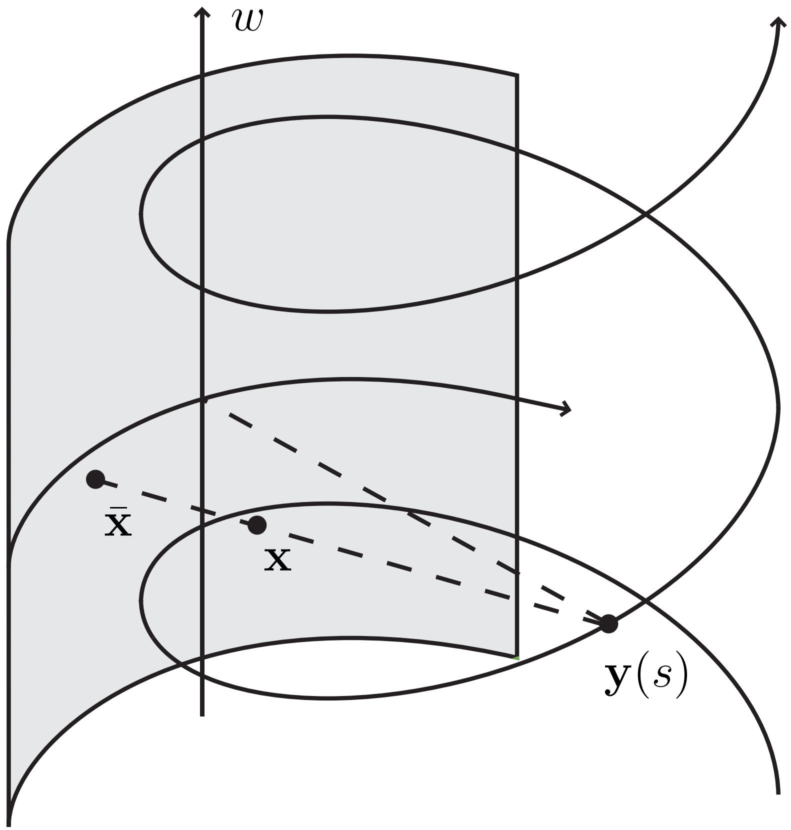
Example 1: 2D fan-beam tomography



$$\mathbf{y}(s) = R(\cos(s), \sin(s))$$

Let S denote the interior of the source circle.

Example 2: 3D Helical Tomography



Source Curve: $\mathbf{y}(s) = \left[R\cos(s),\ R\sin(s),\ \dfrac{P}{2\pi}s \right]$

Let S denote the interior of the helix cylinder. supp$(f) \subset S$.

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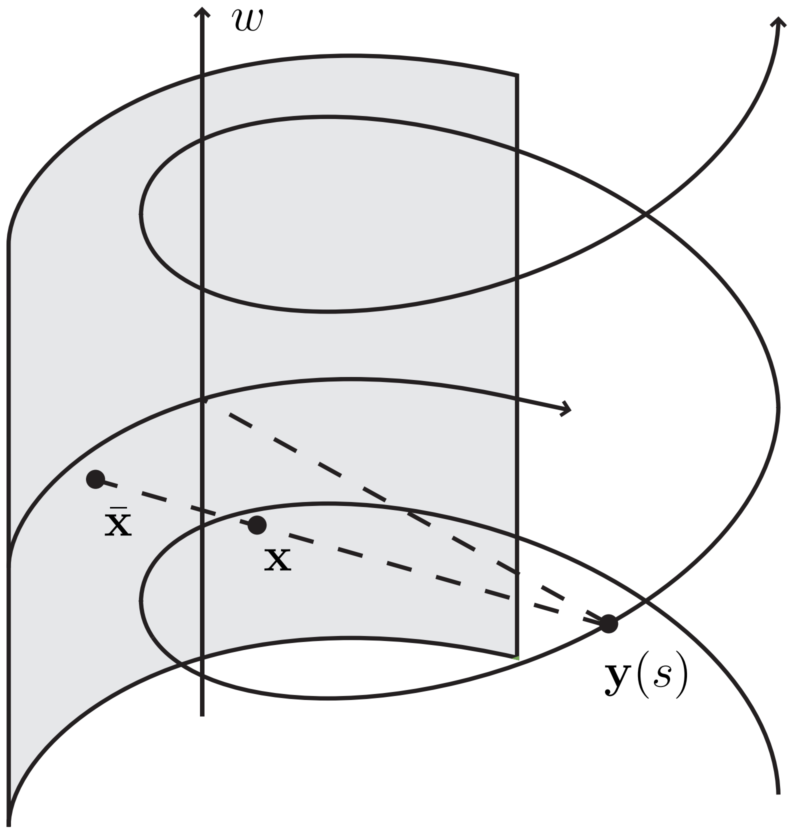
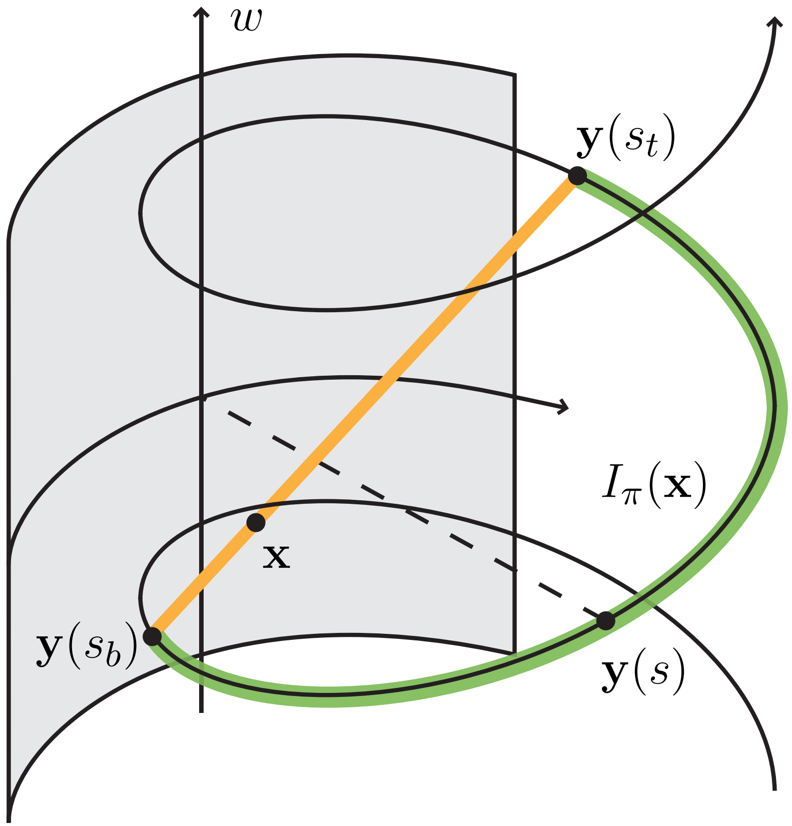


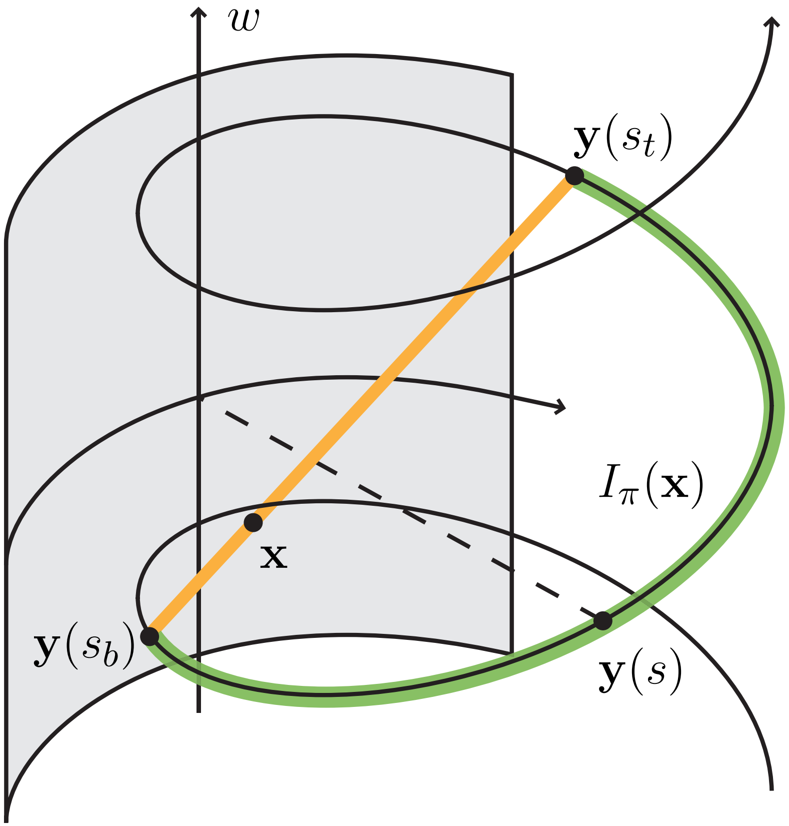
Which source positions are needed for reconstruction at a point $\mathbf{x}$?

π-line and π-interval



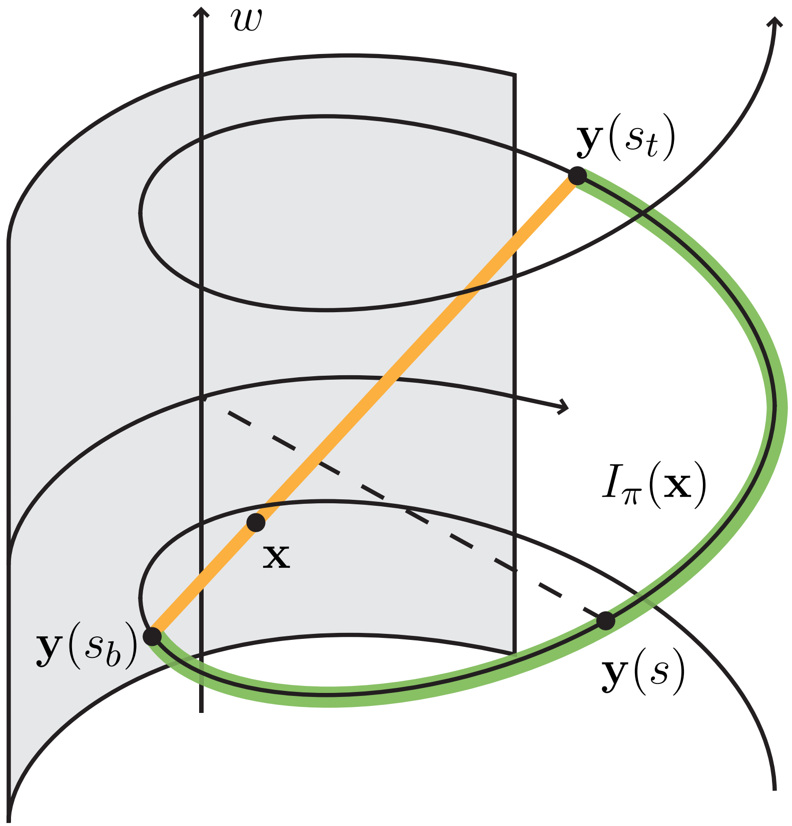
A so-called π-line through $\mathbf{x}$ intersects the source curve twice within one turn.

π-line and π-interval



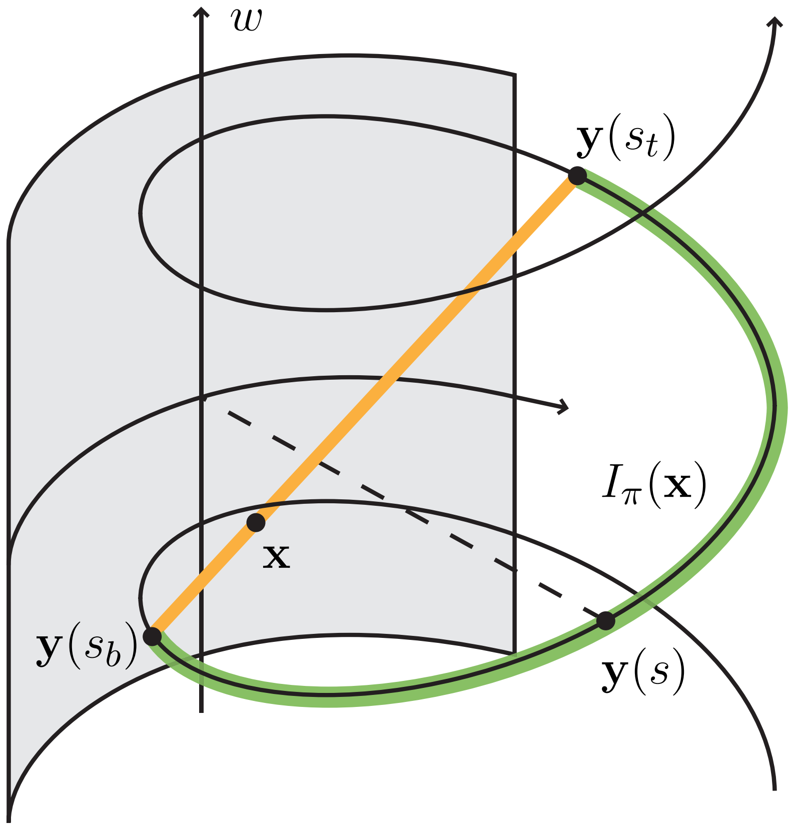
A so-called π-line through $\mathbf{x}$ intersects the source curve twice within one turn.
For the helix there is a unique π-line through $\mathbf{x}$.

π-line and π-interval



A π-line through $\mathbf{x}$ gives rise to the π-interval $I_\pi(\mathbf{x}) = [s_b(\mathbf{x}),\ s_t(\mathbf{x})]$.

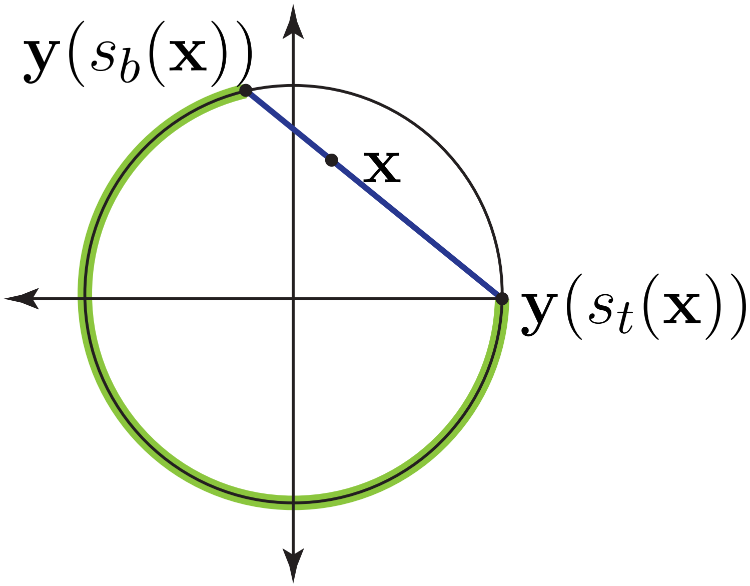
π-line and π-interval



A π-line through $\mathbf{x}$ gives rise to the π-interval
$I_\pi(\mathbf{x}) = [s_b(\mathbf{x}),\ s_t(\mathbf{x})]$.
Sources $\mathbf{y}(s)$ with $s \in I_\pi(\mathbf{x})$ lie on the green arc.

Non-uniqueness of π-lines in 2D

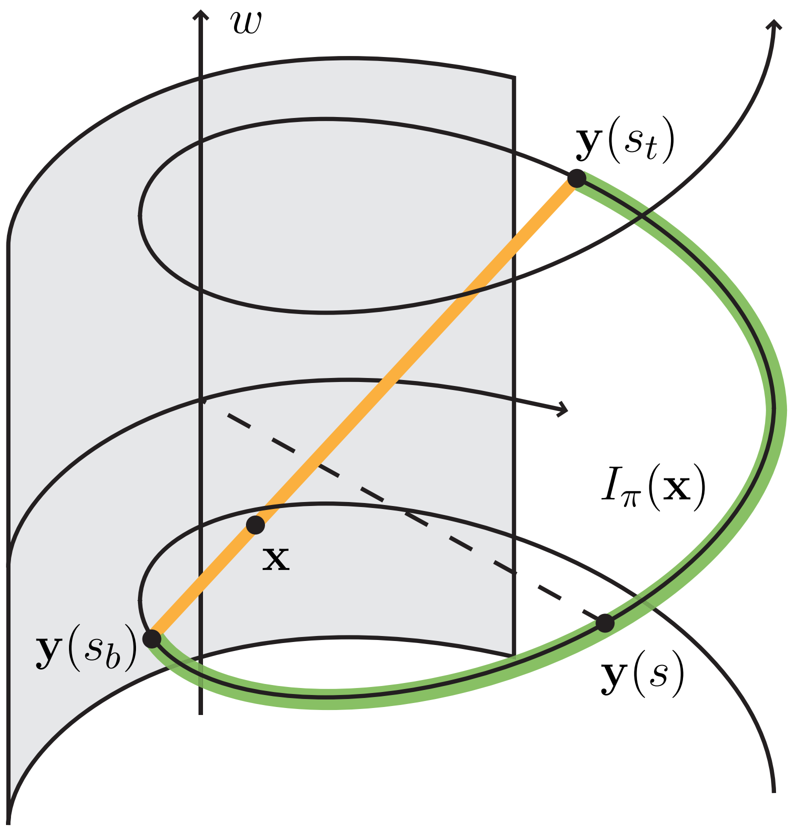
For $\mathbf{y}(s) = R(\cos s, \sin s)$, we lack uniqueness of π-lines. Any line through $\mathbf{x}$ may be chosen as the π-line of $\mathbf{x}$, denoted by $L_\pi(\mathbf{x})$.



$I_\pi(\mathbf{x})$ may be chosen to correspond to either of the two arcs.

π-line reconstruction formulas

Definition 1 *A π-line reconstruction formula uses for reconstruction at a point $\mathbf{x}$ only data from sources within the π-interval of $\mathbf{x}$.*



Example: Backprojection-filtration

Define the Hilbert transform of f in direction $\boldsymbol{\theta} \in S^{n-1}$ as

$$H_{\boldsymbol{\theta}} f(\mathbf{x}) = \frac{1}{\pi} \int_{\mathbf{R}} \frac{f(\mathbf{x} - t\boldsymbol{\theta})}{t} \, dt.$$

Then

$$\frac{-1}{2\pi} \int_{I_\pi(\mathbf{x})} \frac{1}{|\mathbf{x} - \mathbf{y}(s)|} \frac{\partial}{\partial q} \mathcal{D} f(\mathbf{y}(q), \boldsymbol{\beta}(s, \mathbf{x}))\Big|_{q=s} ds \; = \; H_{\boldsymbol{\beta}(s_b(\mathbf{x}),\mathbf{x})} f(\mathbf{x})$$

$\boldsymbol{\beta}(s, \mathbf{x}) =$ unit vector pointing from $\mathbf{y}(s)$ to $\mathbf{x}$.

Right-hand side is Hilbert transform along the π-line of $\mathbf{x}$.

Originally due to Gel’fand and Graev (1991). Basis for backprojection-filtration algorithm (Zou and Pan (2004)).

Example: Filtered backprojection

$$f(\mathbf{x}) = \frac{-1}{2\pi^2} \int_{I_\pi(\mathbf{x})} \frac{1}{|\mathbf{x}-\mathbf{y}(s)|} \int_0^{2\pi} \frac{\partial}{\partial q} \mathcal{D}f(\mathbf{y}(q), \boldsymbol{\Theta}(s,\mathbf{x},\gamma))\Big|_{q=s} \frac{d\gamma\, ds}{\sin\gamma}$$

$$\boldsymbol{\Theta}(s,\mathbf{x},\gamma) = \cos(\gamma)\boldsymbol{\beta}(s,\mathbf{x}) + \sin(\gamma)\boldsymbol{\beta}^{\perp}(s,\mathbf{x}).$$

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(Katsevich 02, 04, Katsevich & Kapralov 07)

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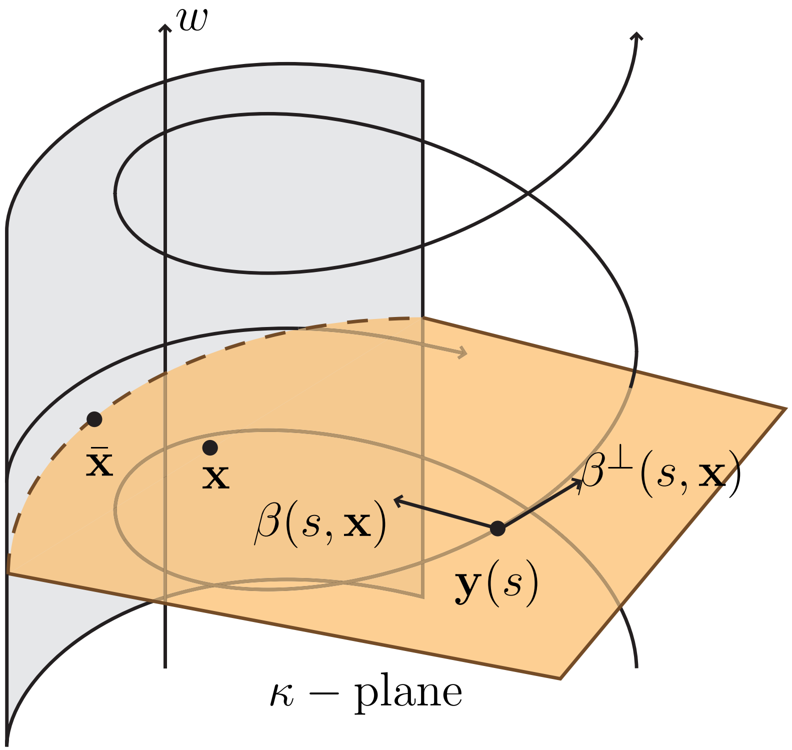
(Katsevich 02, 04, Katsevich & Kapralov 07)

Both formulas hold in dimensions 2 and 3 for a large family of source curves.

In dimension 3, $\boldsymbol{\beta}^\perp$ has to be carefully chosen (Katsevich 02, 04).

κ-Plane and Katsevich's formula



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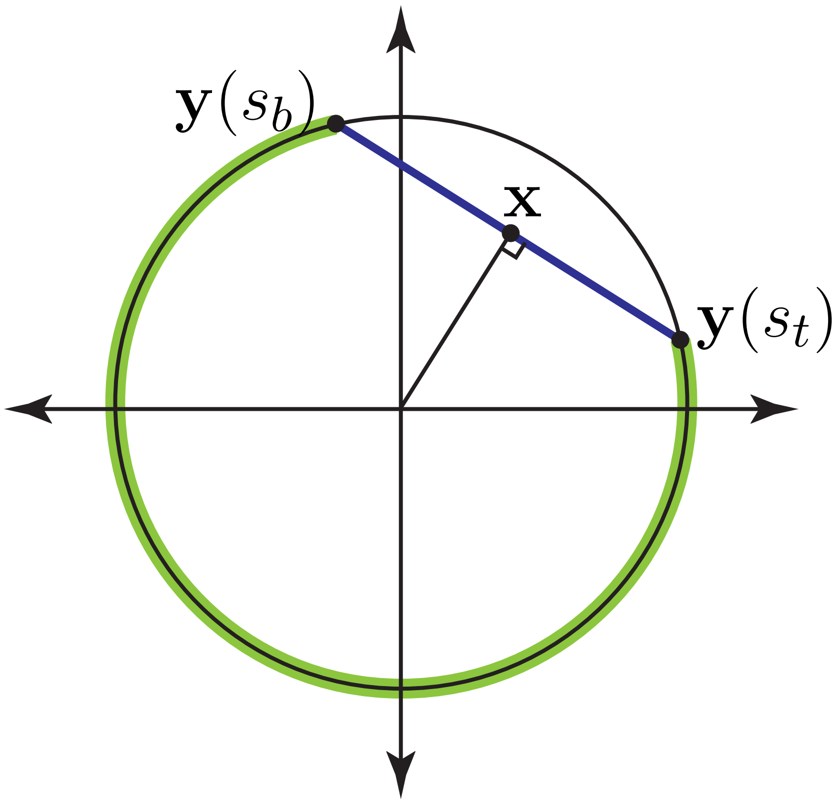
Characteristics of π-line formulas

- Flexibility in choosing π-lines in 2D.

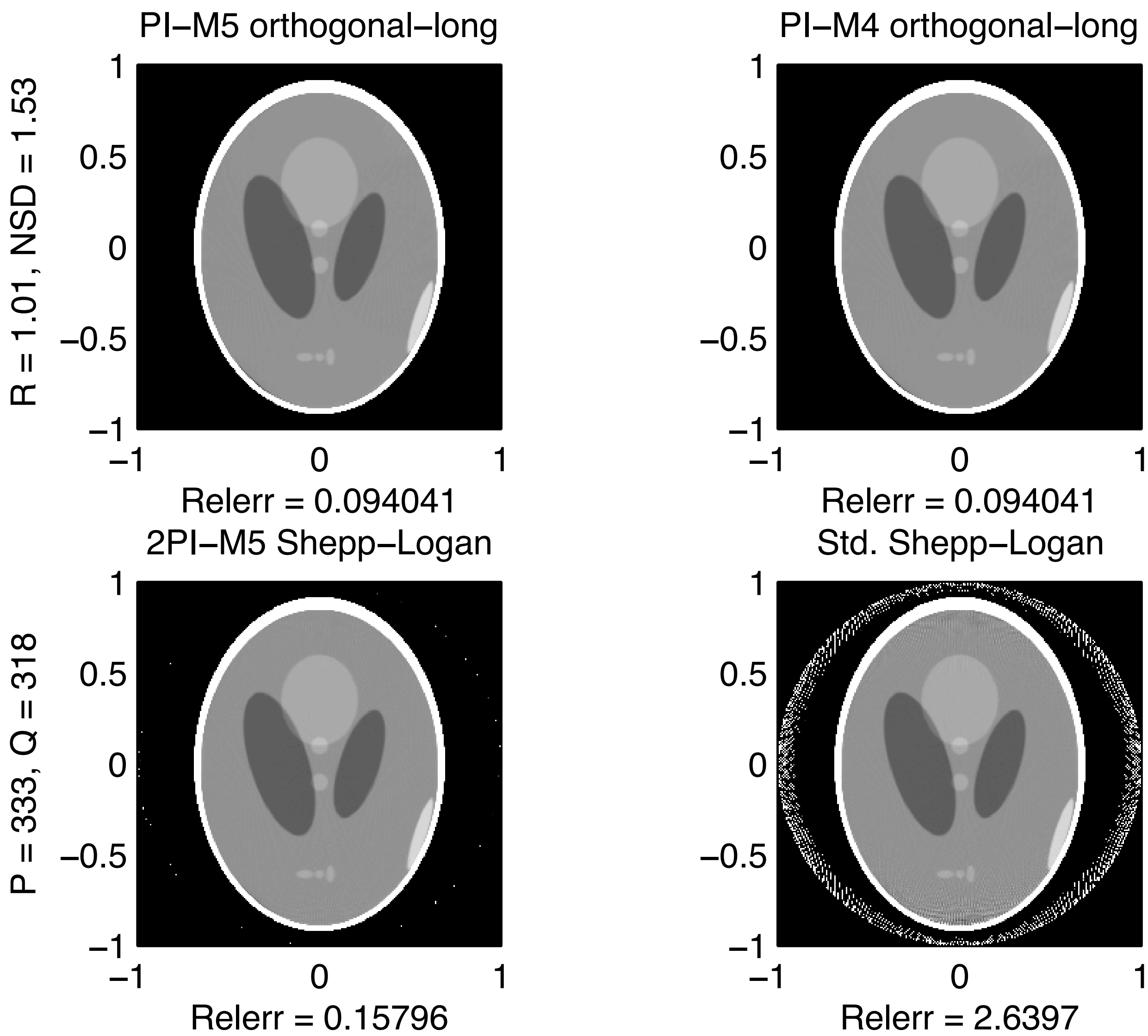
Example: Orthogonal-long π-lines

$L_\pi(\mathbf{x})$ is orthogonal to $\mathbf{x}$ and $I_\pi(\mathbf{x}) = [s_b(\mathbf{x}), s_t(\mathbf{x})]$ corresponds to the longer arc.



Superior performance for R close to 1!

Comparison for R=1.01



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- Region of Backprojection not equal to S.
  RBP(s) $=$ set of all points where data from source $\mathbf{y}(s)$ is used for reconstruction $= \{\mathbf{x} : s \in I_\pi(\mathbf{x})\}$.
  RBP(s) depends on the family of π-lines.

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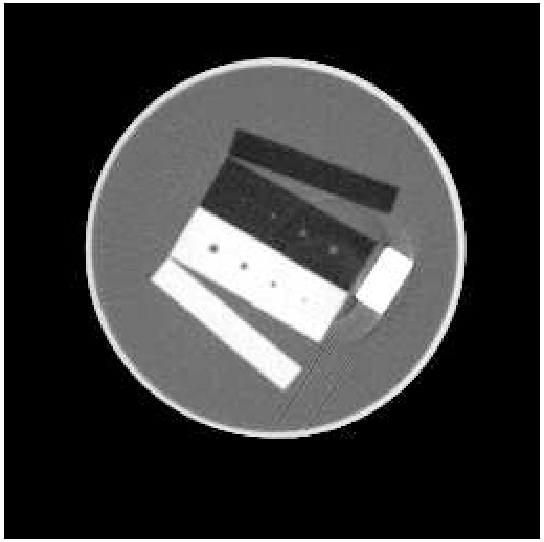
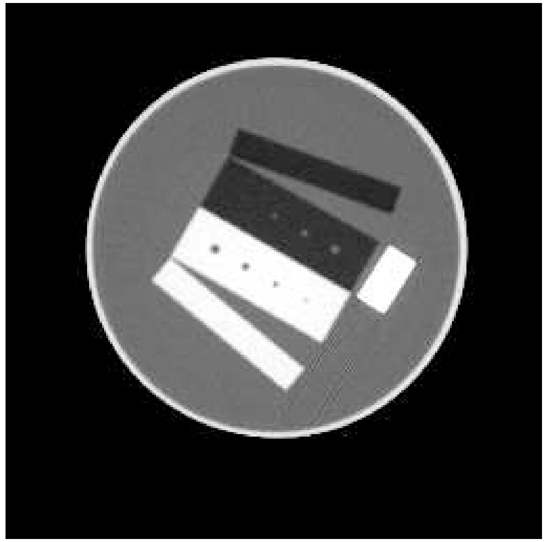
  RBP(s) depends on the family of π-lines.
- Comet tail artifacts.

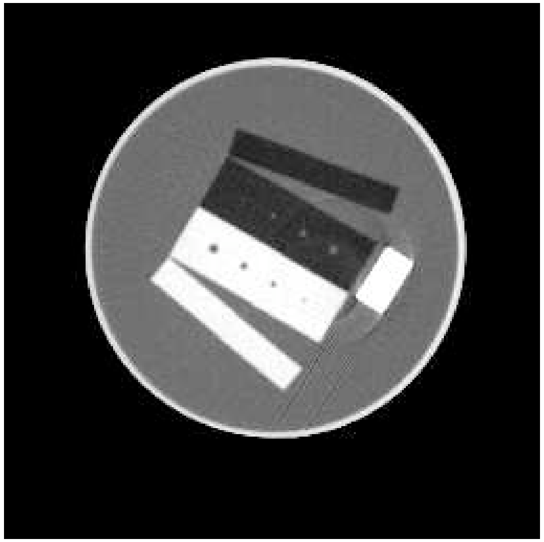
Comet tail artifacts

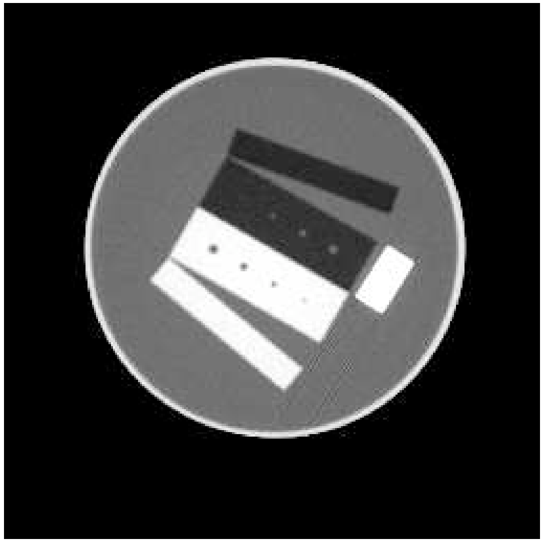




Reconstructions from real data. The reconstruction from the π-line filtered backprojection formula (left) shows a large comet tail artifact that is not present in a standard reconstruction (right).

Comet tail artifacts

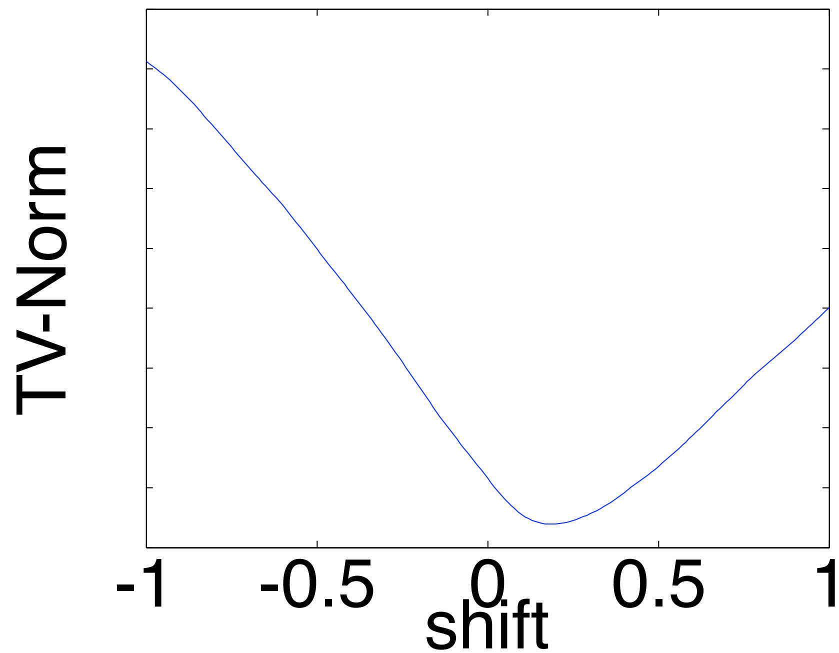




Reconstructions from real data. The reconstruction from the π-line filtered backprojection formula (left) shows a large comet tail artifact that is not present in a standard reconstruction (right).

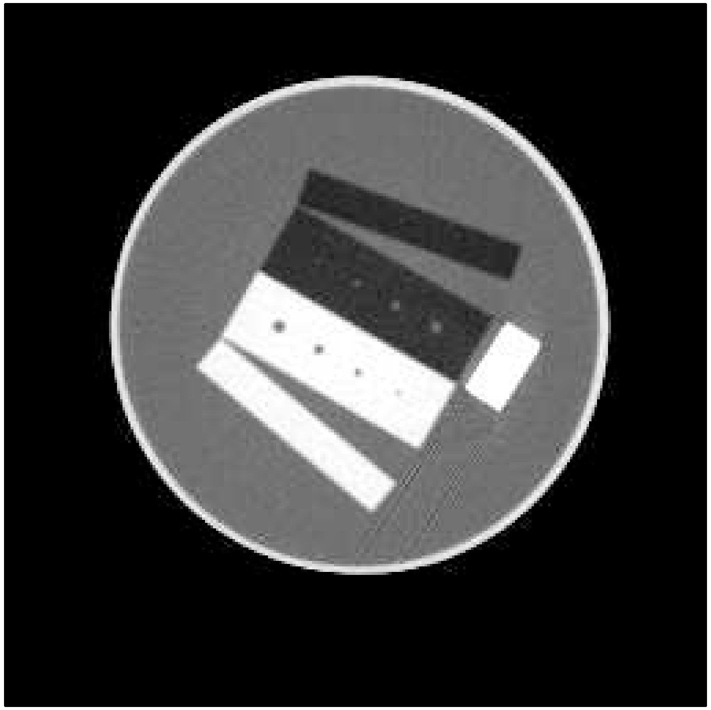
In this case most of the artifact is due to a previously undetected data misalignment in the fan angle. The π-line formula is much more sensitive to such misalignments.

Finding the correct alignment



The correct alignment (about 0.19 detector widths) corresponds here to a minimum of the total variation $TV(f) = \int |\nabla f(\mathbf{x})| \, d\mathbf{x}$ (here of a subregion of the image).

Reconstruction with corrected alignment

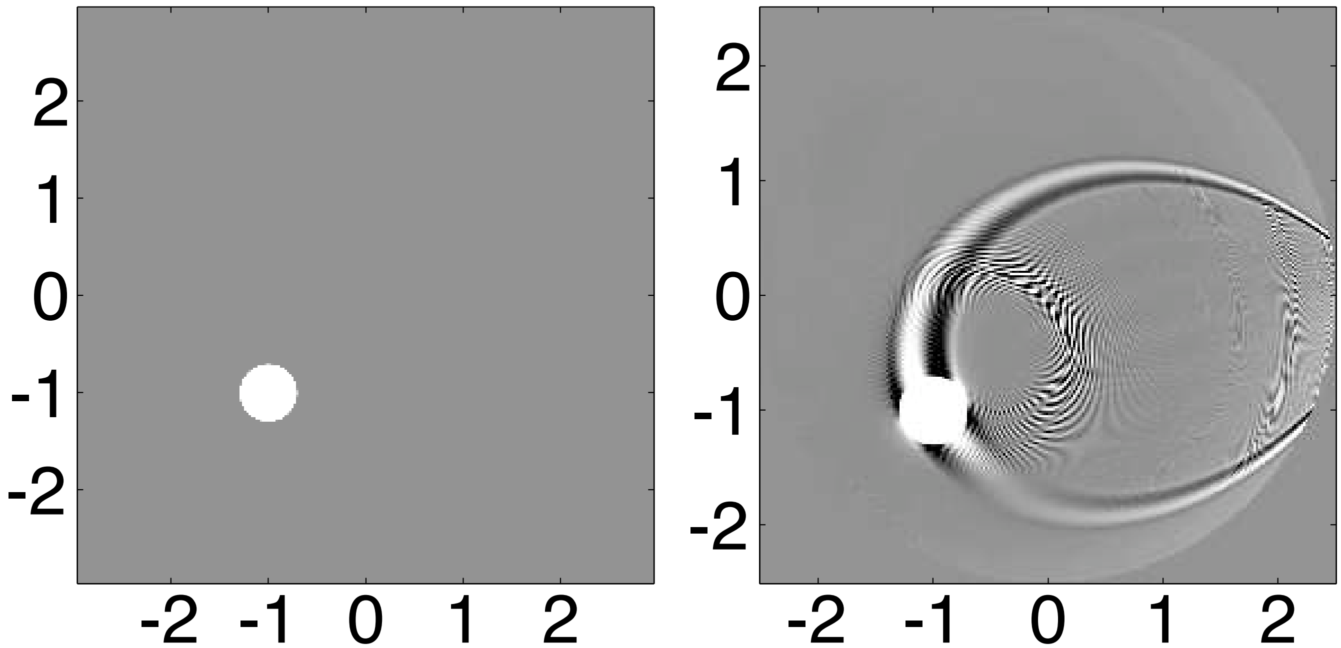


The comet tail artifact is much reduced.

However ...

... a small comet tail artifact may remain even with well-aligned data.

Exact smooth function $x_3 = 0$



Original and 3D reconstruction displayed in plane $x_3 = 0$. Display window is [-1.e-4, 1.e-4] while the image maximum is 1.

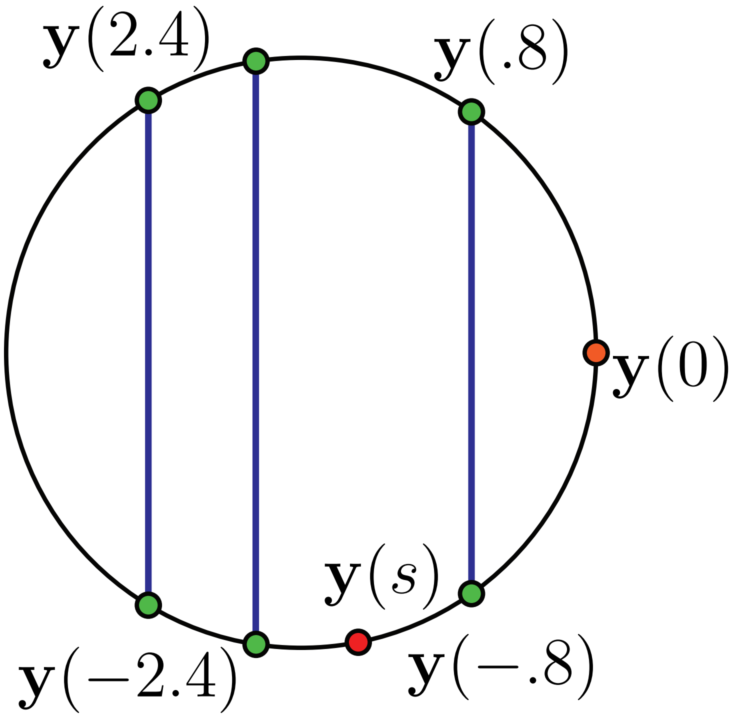
A first heuristic principle

The comet tail artifact is related to the boundary of the region of backprojection.

This motivated further study of RBP(s) ...

We begin with additional examples of families of π-lines in 2D.

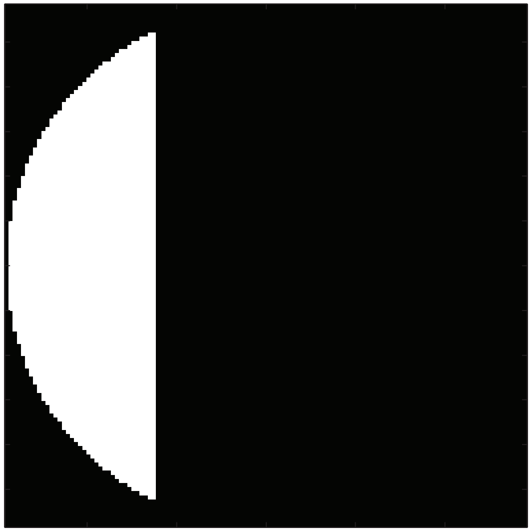
Parallel π-lines



The π-line of a point $\mathbf{x}$ is the vertical line through $\mathbf{x}$.
The π-interval corresponds to the right arc.
Points to the left of $\mathbf{y}(s)$ are in $RBP(s)$.

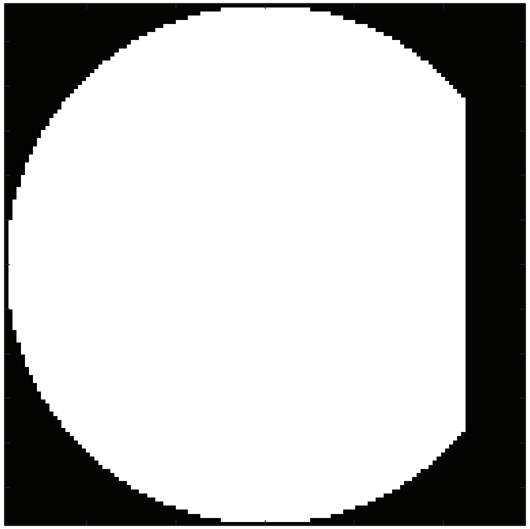
RBP for parallel π-lines

$RBP(-2)$



$\bullet \mathbf{y}(0)$

$RBP(.75)$



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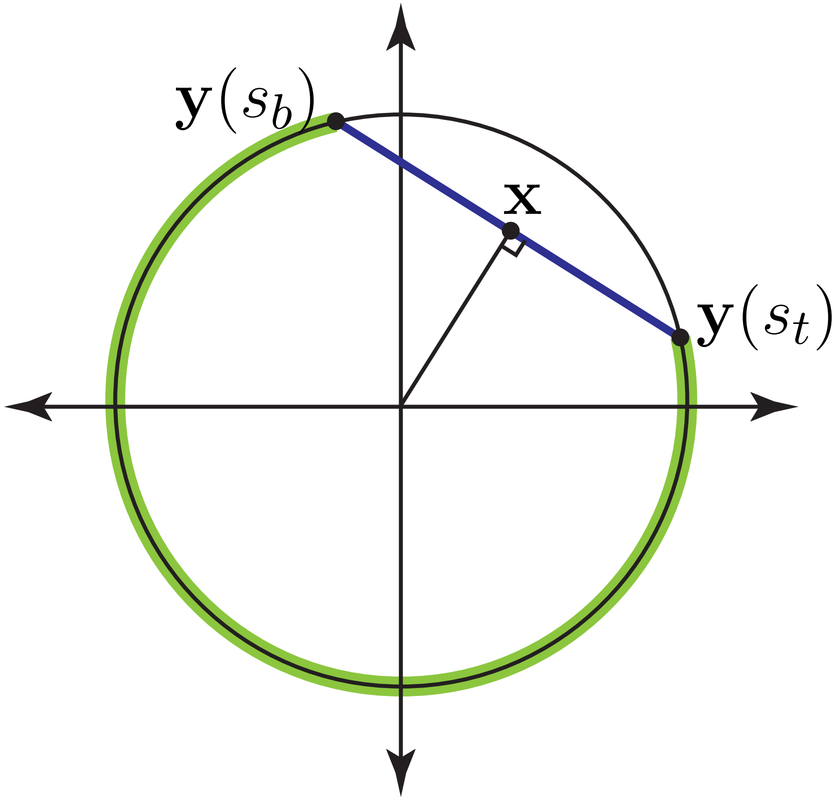
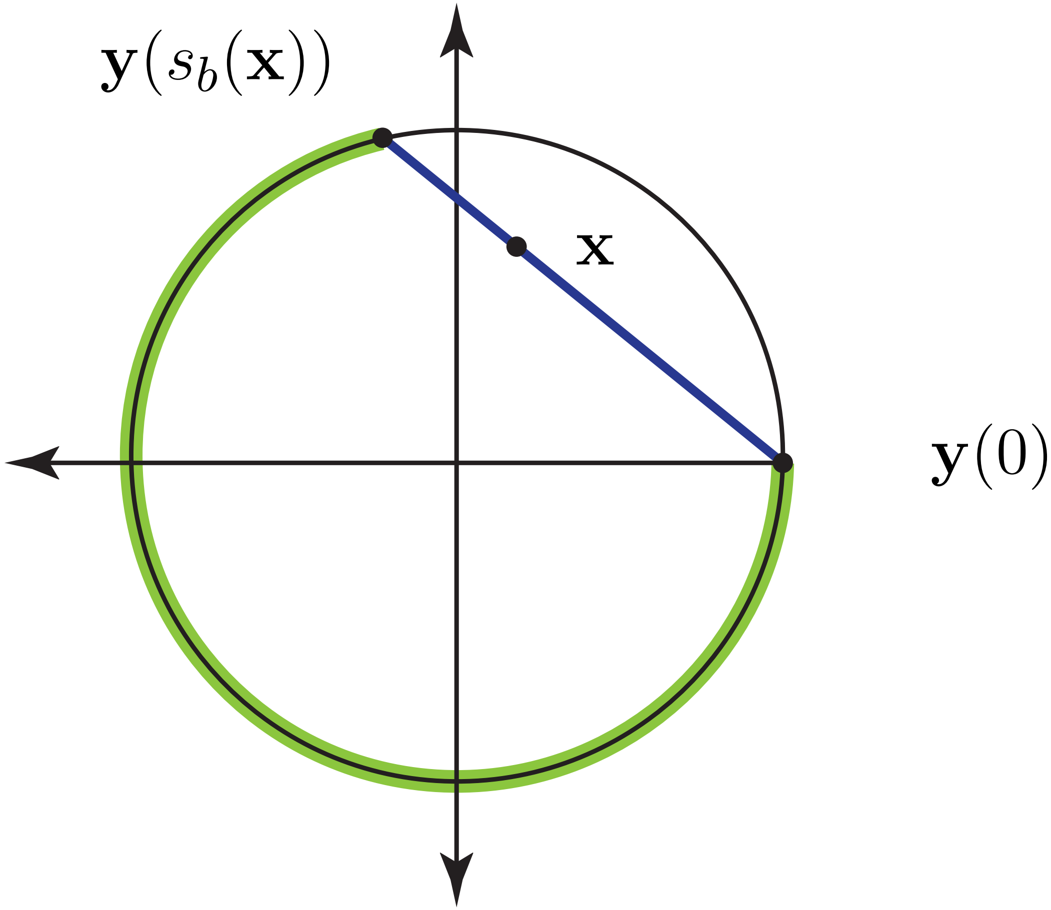
Orthogonal-long π-lines



No two points have the same π-interval. The set $RBP(s)$ and its boundary are not immediately obvious.

Fan-type π-lines



$s_t(\mathbf{x}) = 2\pi$ for all $\mathbf{x} \in S$.

RBP(0) = S, so ∂RBP$(0) \cap S$ is empty.

Some theory for ∂RBP(s)

Let $\mathbf{x} \in S$ and let s_b, s_t be continuous functions of $\mathbf{x}$.

- $\mathbf{x} \in \partial\text{RBP}(s) \;\Rightarrow\; s \in \{s_b(\mathbf{x}), s_t(\mathbf{x})\} \;\Leftrightarrow\; \mathbf{y}(s) \in L_\pi(\mathbf{x})$
- $\mathbf{x} \in \partial\text{RBP}(s_b(\mathbf{x})) \cup \partial\text{RBP}(s_t(\mathbf{x}))$ for all $\mathbf{x} \in S$.

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Property A and B hold for the parallel π-lines and the helix.
Only Property A holds for orthogonal-long π-lines.
Only Property B holds for fan-type π-lines.

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Only Property A holds for orthogonal-long π-lines.
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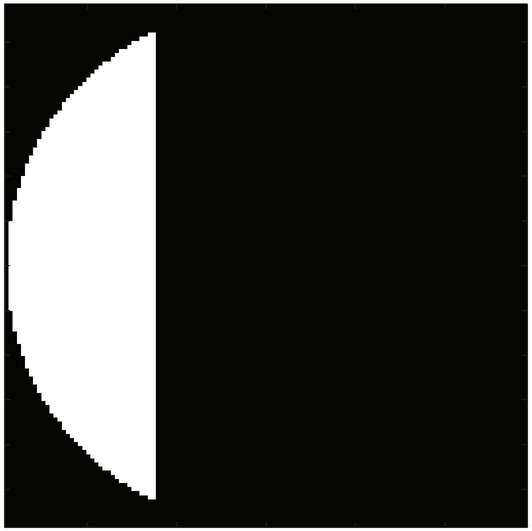
Proposition 1 *If both Property A and B hold, then ∂RBP$(s) \cap S$ equals the intersection of S with the union of all π-lines that contain $\mathbf{y}(s)$.*

RBP for parallel π-lines

$RBP(-2)$



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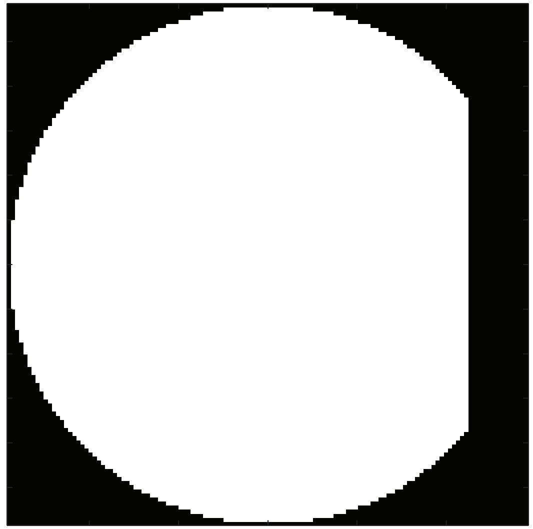


$\bullet \mathbf{y}(0)$

∂RBP(s) for orthog.-long π-lines

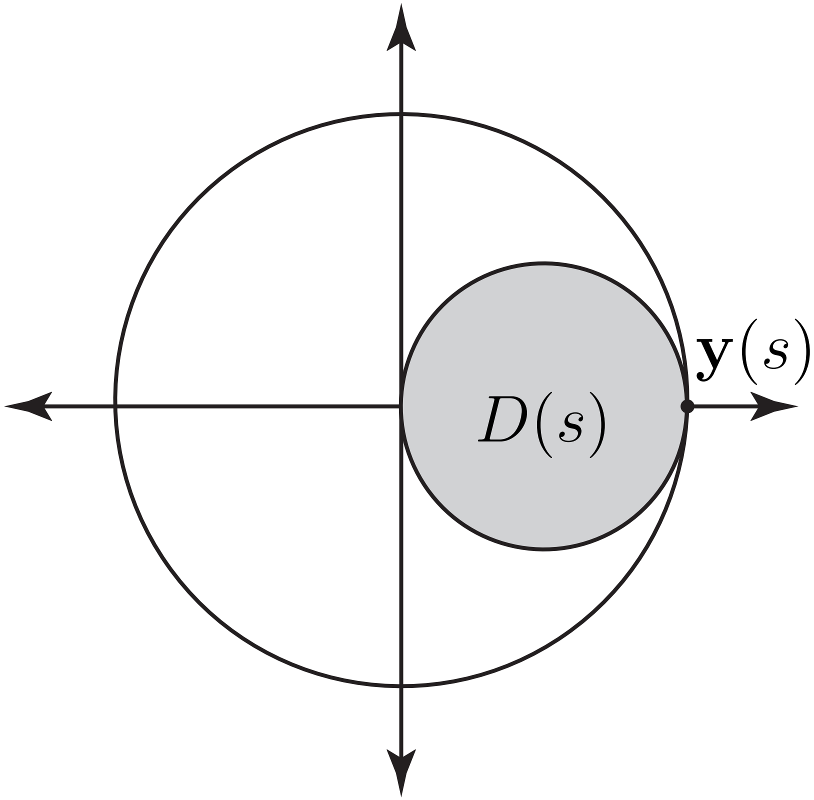
One can show that Property A holds. Hence for $\mathbf{x} \in S$,

$$\begin{aligned} \mathbf{x} \in \partial\text{RBP}(s) \quad &\Leftrightarrow \quad \mathbf{y}(s) \in L_\pi(\mathbf{x}) \\ &\Leftrightarrow \quad \mathbf{x} \perp (\mathbf{x} - \mathbf{y}(s)) \\ &\Leftrightarrow \quad |\mathbf{x} - \mathbf{y}(s)/2| = |\mathbf{y}(s)|/2 \end{aligned}$$

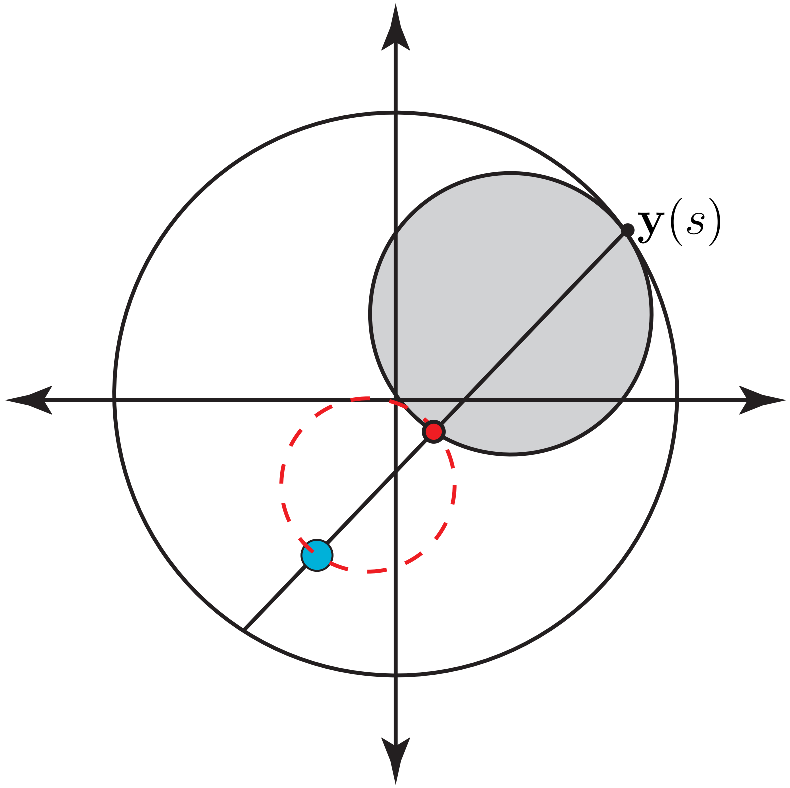
Hence RBP(s) is the circle with center $\mathbf{y}(s)/2$ and radius $|\mathbf{y}(s)|/2$.

RBP for orth.-long π-lines

For orthogonal-long π-lines, $RBP(s)$ contains all points outside the disk $D(s) = \{\mathbf{x} : |\mathbf{x} - \mathbf{y}(s)/2| < |\mathbf{y}(s)/2|\}$.



Location of artifact I



Heuristic principle. A contribution to the artifact will occur at intersections (red) of the boundary of RBP(s) with lines connecting $\mathbf{y}(s)$ and points in the support of the function (blue).

Location of artifact II

For a point $\mathbf{x}_0 \in S$ we define the set $\Gamma_{\mathbf{x}_0}$ by

$$\Gamma_{\mathbf{x}_0} \;=\; \{\mathbf{x} \in S \mid \exists s : \mathbf{x} \in \partial RBP(s) \text{ and } \mathbf{x}_0, \mathbf{x}, \text{ and } \mathbf{y}(s) \text{ are collinear}\}$$

Loosely speaking, $\Gamma_{\mathbf{x}_0}$ gives the support of the artifact that would be caused by $f(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_0)$.

The location of the full artifact is then given by

$$\Gamma \;=\; \bigcup_{\mathbf{x}_0 \in \text{supp}(f)} \Gamma_{\mathbf{x}_0}\,.$$

Equivalent characterization

Recall:

$\Gamma_{\mathbf{x}_0} = \{\mathbf{x} \in S \mid \exists s : \mathbf{x} \in \partial RBP(s)$ and $\mathbf{x}_0$, $\mathbf{x}$, and $\mathbf{y}(s)$ are collinear$\}$.

Proposition 2 *Let s_b, s_t be continuous functions of $\mathbf{x}$. Then*

$$\Gamma_{\mathbf{x}_0} \;=\; \{\mathbf{x} \mid \mathbf{x}_0 \in L_\pi(\mathbf{x})\}.$$

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Proposition 3 *Let s_b, s_t be continuous functions of $\mathbf{x}$. Then*

$$\Gamma_{\mathbf{x}_0} \;=\; \{\mathbf{x} \mid \mathbf{x}_0 \in L_\pi(\mathbf{x})\}.$$

The artifact occurs at all points $\mathbf{x}$ whose π-lines intersect the support of f.

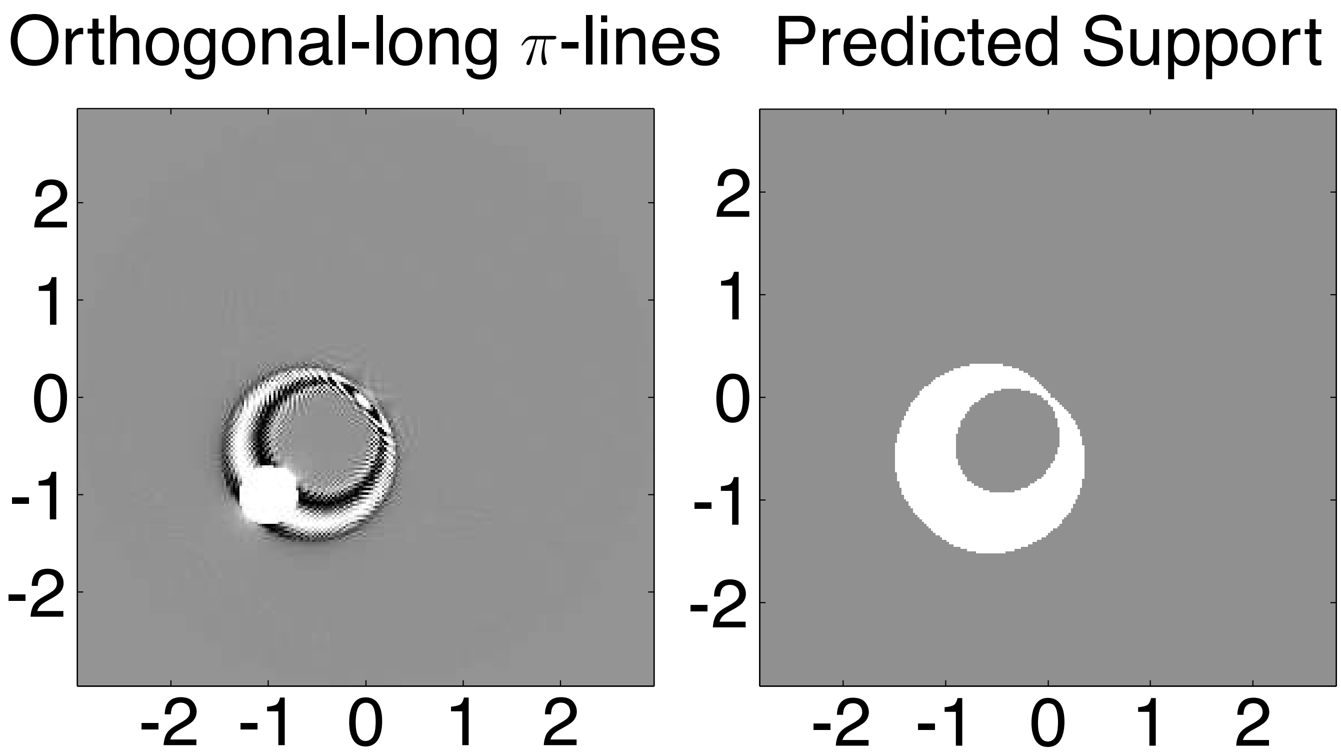
Artifact for orthog.-long π-lines

For $\mathbf{x}_0, \mathbf{x} \in S$,

$$\begin{aligned} \mathbf{x}_0 \in L_\pi(\mathbf{x}) &\Leftrightarrow \mathbf{x} \perp (\mathbf{x} - \mathbf{x}_0) \\ &\Leftrightarrow |\mathbf{x} - \mathbf{x}_0/2| = |\mathbf{x}_0|/2 \end{aligned}$$

Hence $\Gamma_{\mathbf{x}_0}$ is the circle with center $\mathbf{x}_0/2$ and radius $|\mathbf{x}_0|/2$.

Orthogonal-long π-lines



Reconstruction with artifact (left) and predicted support of artifact (right).

Special cases

Corollary *If the family of π-lines satisfies Property B, then*

$$\Gamma_{\mathbf{x}_0} = L_\pi(\mathbf{x}_0) \cap S \quad \textit{for all } \mathbf{x}_0 \in S.$$

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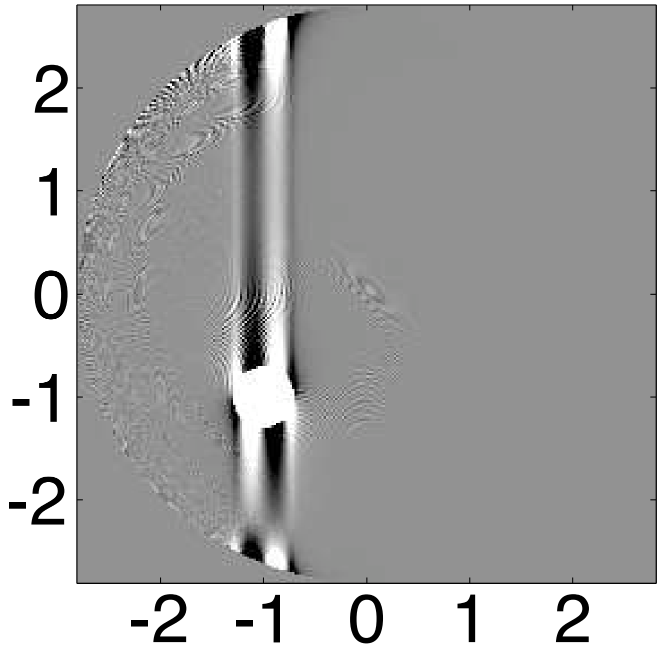
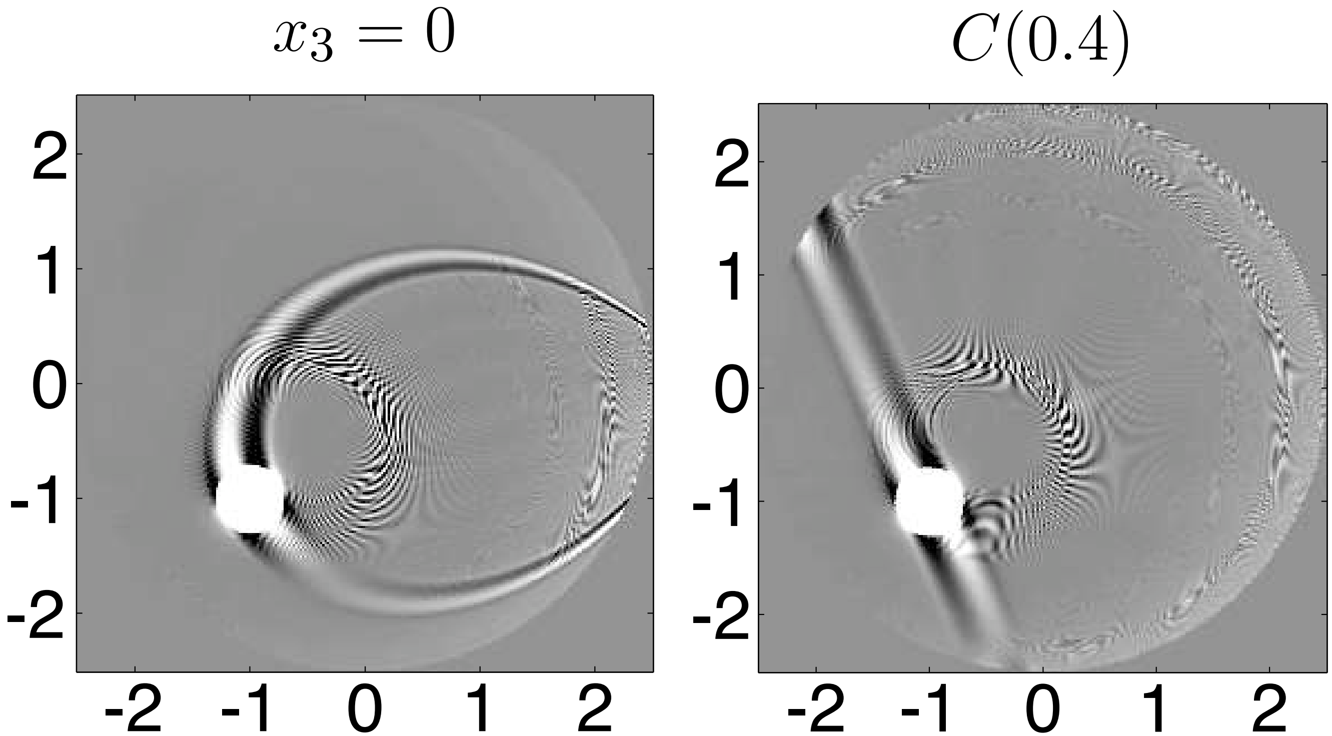
In such a case, for example the parallel π-lines and the helix, the artifact will appear on the union of all π-lines of points in S that intersect the support of f.

Comet tail artifact for parallel π-lines



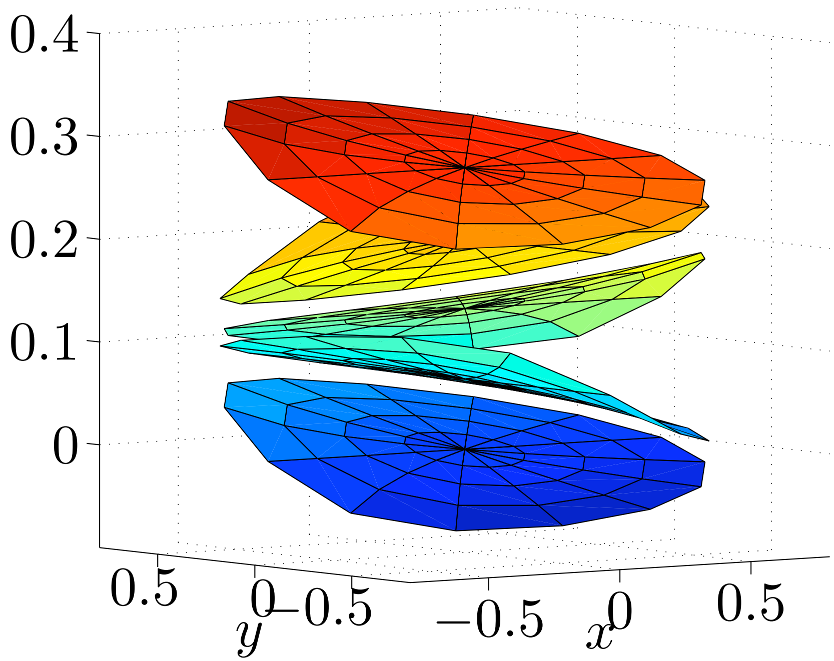
Comet tail artifact for the helix



- $\mathbf{y}(.4)$

The structure of the artifact is not apparent when shown in a plane (left) but is clear when shown on a surface of π-lines called a chip (right).

Helical Chips (Izen 07)



$$C(t) = \{\mathbf{x} \in S : I_\pi(\mathbf{x}) = [t - \alpha, t + \alpha],\ 0 < \alpha < \pi\}$$

Relationship to Hilbert image

Recall

$$\frac{-1}{2\pi}\int_{I_\pi(\mathbf{x})}\frac{1}{|\mathbf{x}-\mathbf{y}(s)|}\frac{\partial}{\partial q}\mathcal{D}f(\mathbf{y}(q),\boldsymbol{\beta}(s,\mathbf{x}))\Big|_{q=s}ds \;=\; H_{\boldsymbol{\beta}(s_b(\mathbf{x}),\mathbf{x})}f(\mathbf{x})$$

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The 'Hilbert image' $H_{\boldsymbol{\beta}(s_b(\mathbf{x}),\mathbf{x})}f(\mathbf{x})$ and the comet tail artifact have the same support.

Summary

- π-line formulas hold in 2D and 3D and generate filtered backprojection (FBP) and backprojection-filtration algorithms based on exact inversion formulas.
- In 2D, the π-line FBP algorithm with orthogonal-long π-lines outperforms standard FBP when the x-ray source is close to the object.
- High sensitivity to data misalignment causes strong comet tail artifacts. Can be used in some cases to determine the correct alignment.
- Provided some theory for the region of backprojection.
- Determined the support of the comet tail artifact, which equals the support of the Hilbert image.

Some references

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