Infrastructure of Function Fields

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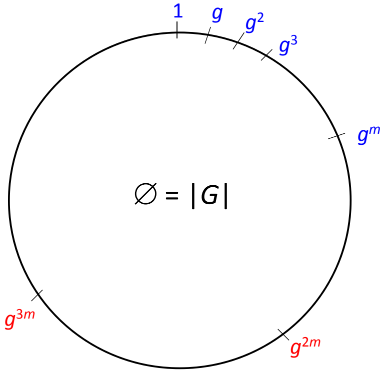

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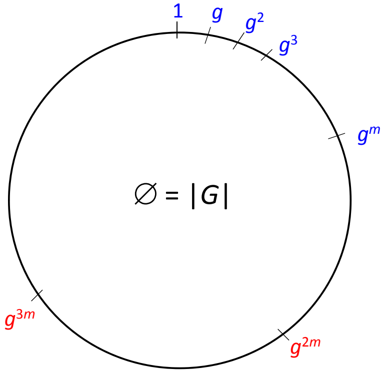

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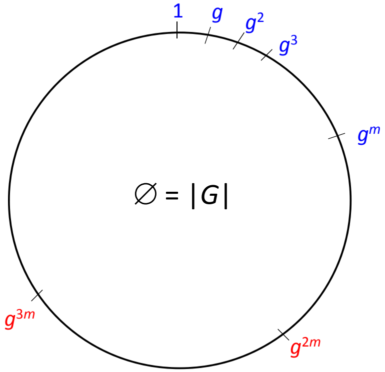

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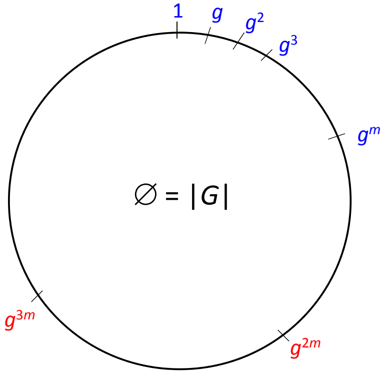

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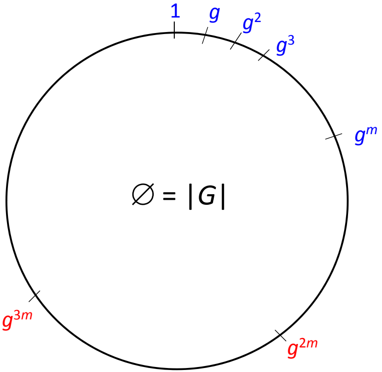

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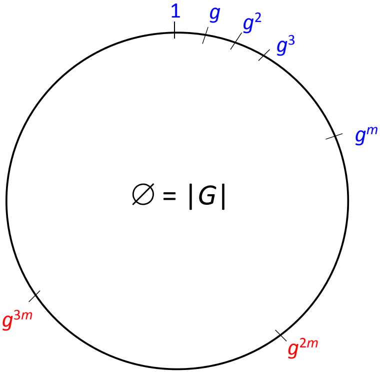

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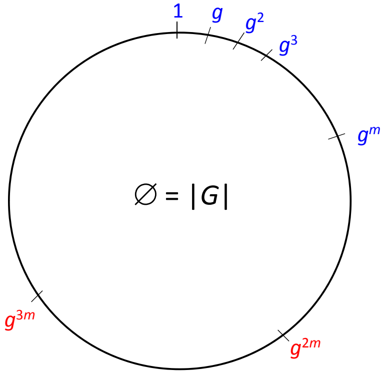

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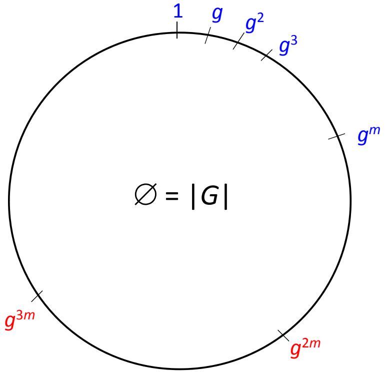

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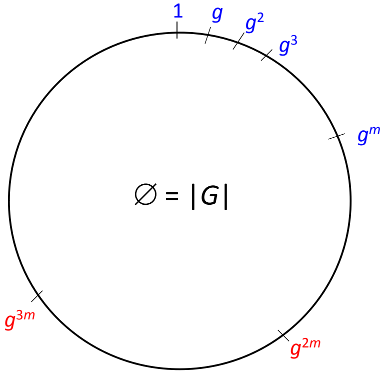

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Similar technique solves **discrete logarithm/distance problem**):

given g^i, find $\delta(g^i) = i$

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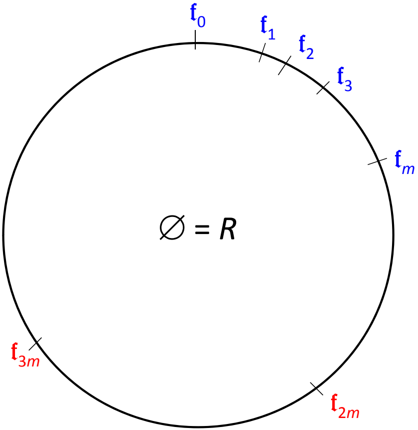

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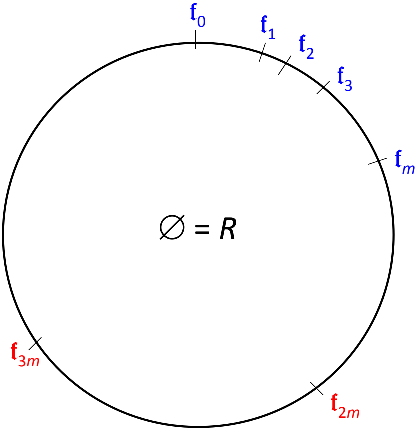

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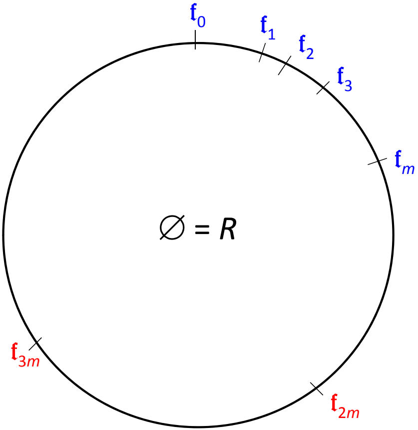

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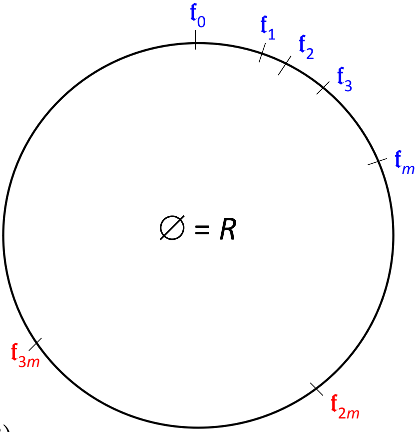

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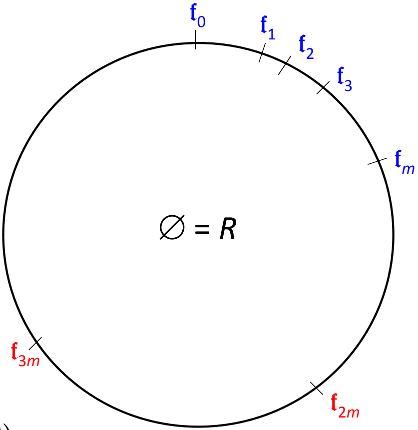

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Can use a similar baby step giant step technique to

- find circumference R of $\mathcal{R}$
- solve distance problem

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Giant Step:

- **Composition** (Gauß): $(A', B', C') \circ (A'', B'', C'') = (A, B, C)$
  where (assuming $\gcd(A', A'', (B' + B'')/2) = 1$):

$$A = A'A'', \qquad B \equiv \begin{cases} 2A' \pmod{B'}, \\ 2A'' \pmod{B''}, \end{cases} \qquad C = \frac{B^2 - D}{4A}$$

- followed by approximately $\log(D)/2$ baby steps

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- R is the regulator of O_D

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A degree 0 divisor $D = D_x - \deg(D_x)\infty + \delta(D)(\overline{\infty} - \infty)$ is **reduced** if

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$\mathcal{R}$ is embeddable into the cyclic group $\langle [\overline{\infty} - \infty] \rangle$ of order R (Fontein 2008)

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The distinguished divisors of a cubic extension of $\mathbb{F}_q(x)$ with two poles at x form an infrastructure:

- Baby steps and giant steps analogous to cubic number fields

(S. & Stein 1998/2000, S. 2001, Landquist 2009, research ongoing)

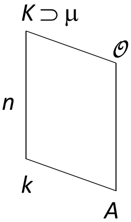
So for what global fields to (circle) infrastructures arise?

Infrastructure from the Unit Lattice
(Fontein 2011)

$k = \mathbb{Q}$ or $\mathbb{F}_q(x)$, $A = \mathbb{Z}$ or $\mathbb{F}_q[x]$, $\mu \subset K^*$ roots of unity

K a finite algebraic extension of k of degree n

$\mathcal{O}$ the integral closure of A in K (Dedekind domain)



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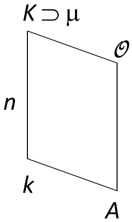

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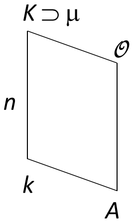

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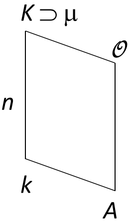

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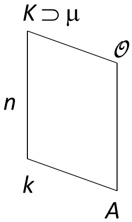

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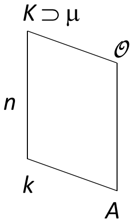

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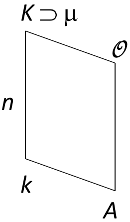

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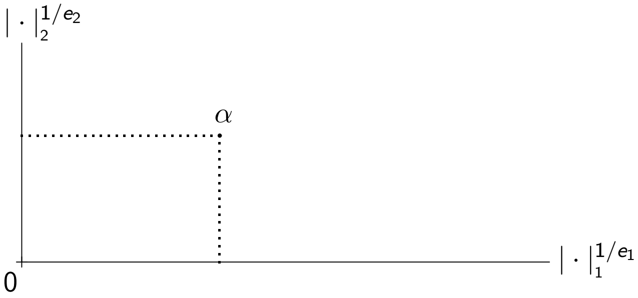

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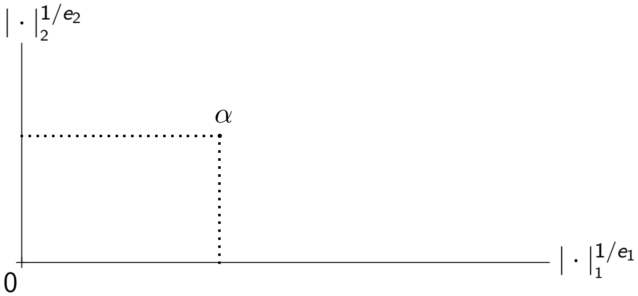

Length function on K: $B(\alpha) = \max_{1 \leq i \leq r+1} |\alpha|_i^{1/e_i}$

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(Minkowski 1910, Mahler 1986, Tang 2011)

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Successive minima depend only on $\mathfrak{f}$, not on $\omega_1, \ldots, \omega_n, \alpha$

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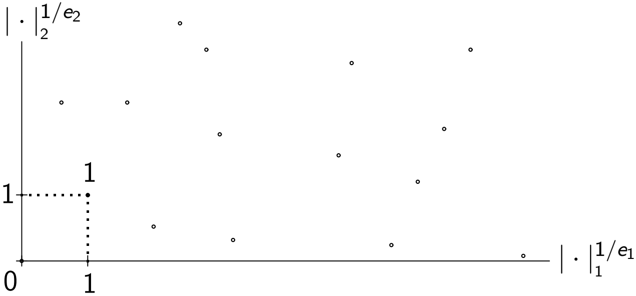

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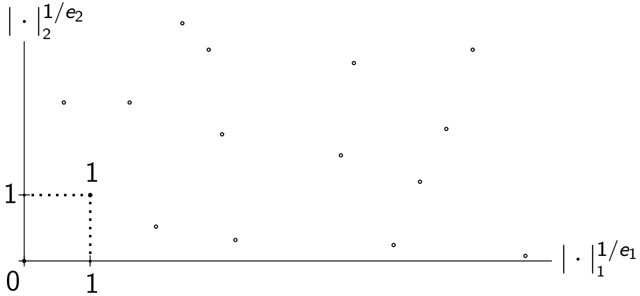

Properties: Suppose $M_1(\mathfrak{f}) = B(\alpha)$ with $\alpha \in \mathfrak{f}$

- $M_1(\alpha^{-1}\mathfrak{f}) = 1$
- $\mathfrak{f}$ distinguished $\iff$ $\alpha \in \mathbb{F}_q^*$ (so $M_1(\mathfrak{f}) = 1$) and $M_2(\mathfrak{f}) > 1$

Neighbours

i-**neighbour** of 1 in $\mathfrak{f}$ — next lattice point from 1 in $|\cdot|_i$-direction
without increasing all the other dimensions of the box

Neighbours

i-**neighbour** of 1 in $\mathfrak{f}$ — next lattice point from 1 in $|\cdot|_i$-direction
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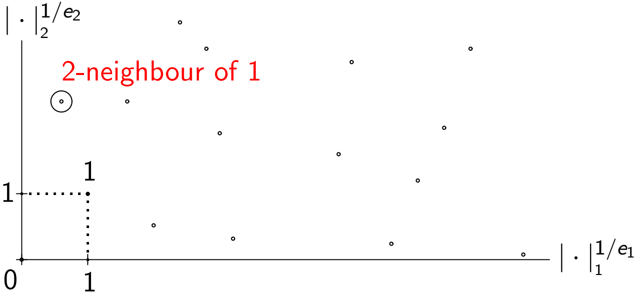

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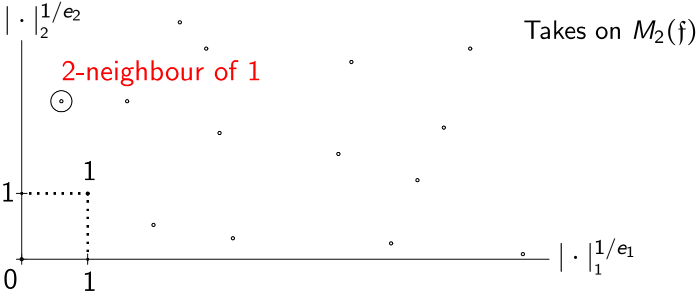

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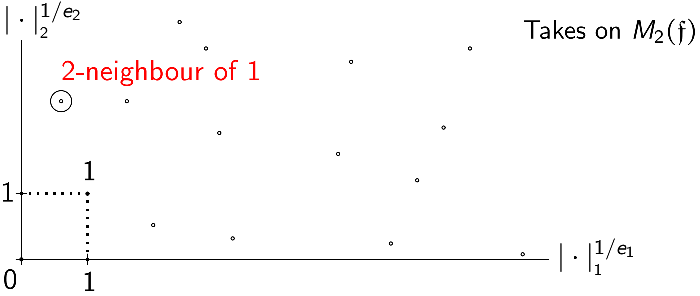

Obtained via a **0-reduced** B-**ordered** $\mathbb{F}_q[x]$-basis of $\mathfrak{f}$

- very technical definition (Schörnig 1996, A. Lenstra 1985)
- computationally highly useful
- takes on the n successive minima of $\mathfrak{f}$
- efficiently computable for $r = 1$, $e_1 = 1$, $e_2 = n - 1$ (Tang 2011)

Infrastructure, Ideal-Theoretic Description (Tang 2011)

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Giant step $\mathfrak{f}' * \mathfrak{f}''$:

1. Compute ideal product $\mathfrak{f}'\mathfrak{f}''$, 0-reduce & B-order resulting basis
2. Divide by ω where $B(\omega) = M_1(\mathfrak{f})$, 0-reduce & B-order resulting basis
3. Apply one baby step

Infrastructure, Divisor-Theoretic Description

Distinguished fractional ideal $\mathfrak{f}$ of distance $\delta(\mathfrak{f})$

$\updownarrow$

Distinguished integral ideal $\mathfrak{a} = \mathrm{denom}(\mathfrak{f})\mathfrak{f}$ of distance $\delta(\mathfrak{a}) = \delta(\mathfrak{f})$

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Distinguished degree 0 divisor $D = D_x - \deg(D_x)\infty_1 + \delta(D)(\infty_2 - \infty_1)$
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- Run time ratio giant steps/baby steps proportional to n^2

Higher-Dimensional Infrastructures

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$r = 2$, purely cubic extensions $K = k(\sqrt[3]{D})$

- Number fields: H. C. Williams et al (1970s and 80s), Buchmann (1980s)
- Function fields: Lee, S. & Yarrish (2003); Fontein, Landquist & S. (in progress)

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Arbitrary r:

- Number Fields: Buchmann (*Habilitationsschrift* 1987)
- Number fields in function field language (Arakelov theory): Schoof (2008)
- Global Fields: Fontein (2011, ongoing)

Wrap-Up

- There are better regulator/class number algorithm than straightforward baby step giant step that use truncated Euler products — $O(|D|^{1/5}) = O(R^{2/5})$
  - Real quadratic number fields: Lenstra 1982, Schoof 1982
  - Real hyperelliptic curves: Stein & Williams 1999, Stein & Teske 2002/2005
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- *Lots* left to do:
  - Improvements to and implementation of Tang's algorithms
  - Other signatures (splitting of infinite place of $\mathbb{F}_q(x)$)
  - Low degree extensions with special arithmetic (cubics? quartics?)
  - ...

$\ast \ \ast \ \ast$ **Thank You!** — **Questions (or Answers)?** $\ast \ \ast \ \ast$