CONFERENCE PROBLEMS

1. Proposed by Damien Roy (University of Ottawa): Are there solutions $(x_0, x_1, x_2) \in \mathbb{Z}^3$ of the equation $x_0^2 x_2 - x_1^3 = 1$ with x_0, x_1, x_2 approximately the same size and $x_0 \to \infty$? More precisely, are there such integer solutions which converge in the projective sense?

Solution by John Dixon (Carleton University): We address the problem in the form: Does there exist a sequence $\{(x_2^{(k)}, x_1^{(k)}, x_0^{(k)})\}$ of integer solutions to $X_2^3 + 1 = X_1^2 X_0$ such that $x_2^{(k)} : x_1^{(k)} : x_0^{(k)}$ converges to a limit?

To simplify the notation put $X_2 = Z$. We have $Z^3+1 = (Z+1)(Z^2-Z+1)$ and try to write $Z + 1\tilde{}\lambda X_0$ and $Z^2 - Z + 1\tilde{}(1/\lambda)X_1^2$. Since $4(Z^2 - Z + 1) = (2Z-1)^2+3$ we put $X := 2Z-1$ and consider the Pell equation $X^2+3 = mY^2$ (eventually we shall take $X_1 = \frac{1}{2}Y$ where the $\frac{1}{2}$ comes up because of the multiplier 4 which we introduced). The smallest value of m which seems to work well is $m = 3$.

Solving $X^2 - 3Y^2 = -3$. An obvious solution is $(3, 2)$. The equation $U^2 - 3V^2 = 1$ has a solution $(2, 1)$. Since $(a+b\sqrt{3})(2+\sqrt{3}) = (2a+3b)+(a+2b)\sqrt{3}$ we put

$$A := \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

and then obtain the infinite sequence $\{(a_n, b_n)\}$ of solutions to $X^2-3Y^2 = -3$ given by $(a_n, b_n) := (3, 2)A^n$ $(n = 0, 1, ...)$. By the previous paragraph we need to have X odd so we restrict to $n = 0, 2, 4, ...$; for these values a_n is odd and b_n is even. Now putting $X_2 = \frac{1}{2}(a_{2n} + 1)$, $X_1 = \frac{1}{2}b_{2n}$ we get

$$\left(\frac{a_{2n}+1}{2}\right)^3 + 1 = \left(\frac{a_{2n}+3}{2}\right) 3 \left(\frac{b_{2n}}{2}\right)^2$$

and so can take $X_0 = \left(\frac{a_{2n}+3}{2}\right) 3$. This gives infinitely many integers solutions with a triple ratio which converges to $1 : \frac{1}{\sqrt{3}} : 3$.

Note. The first few solutions are $(2, 1, 3)$, $(23, 13, 72)$, $(314, 181, 945)$, $(4367, 2521, 13104)$, ...

2. Proposed by Abdellah Sebbar (University of Ottawa): Let n be a positive integer and x a complex variable. What is known about the polynomials $P_n(x) := \sum_{d|n} x^{d+(n/d)}$?

3. Proposed by Todd Cochrane (Kansas State University): Does there exist an absolute positive integer n such that if $S \subseteq F_p$ is such that $S - S = F_p$ then $nS = F_p$?

4. Proposed by Todd Cochrane (Kansas State University): Is the congruence $x_1^p + x_2^p + x_3^p \equiv a \pmod{p^2}$ solvable for every integer a and every prime $p > 59$?

5. Proposed by Kumar Murty (University of Toronto): Let $E/\mathbb{Q}$ be an elliptic curve. By the Taniyama conjecture (now a theorem due to the fundamental work of Wiles, Taylor, Breull, Conrad and Diamond) E is modular. Therefore there exist Hecke eigenforms f of weight 2 all of whose eigenvalues are integers. Is there a way to produce such f without using the Taniyama conjecture?

6. Proposed by Hester Graves (Queen's University): There are beautiful bases for $E_2(\Gamma(N))$ and $E_2(\Gamma_1(N))$, where the coefficients are variations on divisor functions. What about $E_2(\Gamma_0(N))$?