Degenerations of K3 Surfaces of Degree Two

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This talk is based upon my recent work on the explicit study of degenerations of K3 surfaces of degree two. Its contents may be found in more detail in the preprint [Tho10] and in my doctoral thesis [Tho11], a copy of which is currently available on my website:

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Recall. Let $\pi\colon X \to \Delta$ be a semistable degeneration of K3 surfaces (i.e. a proper, flat, surjective morphism $\pi\colon X \to \Delta$ whose general fibre $X_t = \pi^{-1}(t)$ for $t \in \Delta^* = \Delta - \{0\}$ is a smooth K3 surface, such that X is smooth and $X_0 := \pi^{-1}(0)$ is reduced with normal crossings). Then Kulikov [Kul77] [Kul81] and Persson-Pinkham [PP81] show that we can perform birational modifications that affect only the central fibre X_0

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr, "\pi"'] & & X' \arrow[dl, "\pi'"] \\ & \Delta & \end{tikzcd}$$

so that $\pi'\colon X' \to \Delta$ is semistable and has $\omega_{X'} \sim \mathcal{O}_{X'}$. Such $\pi'\colon X' \to \Delta$ is called a *Kulikov model* of our degeneration.

Kulikov models are classified by the following theorem:

Theorem 1. [Per77], [Kul77] [FM83] *Let $\pi\colon X \to \Delta$ be a semistable degeneration of K3 surfaces with $\omega_X \cong \mathcal{O}_X$, such that all components of X_0 are Kähler. Then either*

(I) *X_0 is a smooth K3 surface;*

(II) *X_0 is a chain of elliptic ruled components with rational surfaces at each end, and all double curves are smooth elliptic curves;*

(III) *X_0 consists of rational surfaces meeting along rational curves which form cycles in each component. If Γ is the dual graph of X_0, then $|\Gamma|$, the topological support of Γ, is homeomorphic to the sphere S^2.*

Aim. Use this classification to study the geometric behaviour at the boundary of the moduli space $\mathcal{F}_2$ of K3 surfaces of degree two.

We begin by studying the moduli space $\mathcal{P}_2$ of pairs (S, D), where S is a K3 surface and D is an ample divisor on S with $D^2 = 2$. Let $\pi\colon X \to \Delta$ be a semistable degeneration of K3 surfaces with $\omega_X \sim \mathcal{O}_X$ and let D be an effective divisor on X that is flat over Δ^* and that induces an ample divisor $D_t = D \cap X_t$ with $D_t^2 = 2$ on a general fibre.

Theorem 2. [SB83] *There exists an effective or zero divisor Z supported on X_0 such that $D - Z = H$ is effective and flat over Δ. Furthermore, after a sequence of elementary modifications have been performed on X we may assume that H is nef.*

Using this, we have a naive description of the fibres at the boundary of $\mathcal{P}_2$:

- X_0 is a degenerate fibre of Type I, II or III;
- $H_0 = H \cap X_0$ is a nef divisor on X_0 with $H_0^2 = 2$.

We henceforth call these conditions $(*)$.

However, there is a problem with this description of the fibres on the boundary: Kulikov models of a given degeneration are not unique (i.e. the same $\pi^*\colon X^* \to \Delta^*$ can be completed to several different Kulikov models $\pi\colon X \to \Delta$). Elementary modifications can be used to move between these birationally equivalent models. This means that if we use the above description of the boundary to compactify our moduli space the resulting space will not be separated.

Solution. We proceed to the relative log canonical model of the pair (X, H):

$$\phi\colon X - \to X^c := \mathbf{Proj}_\Delta \bigoplus_{n\geq 0} \pi_*\mathcal{O}_X(nH).$$

Results of the minimal model program show that ϕ is an isomorphism over Δ^* and that all of the birationally equivalent Kulikov models map to the same relative log canonical model. So a better description of the fibres on the boundary of $\mathcal{F}_2$ would be "those $(X^c)_0$ that are the central fibres in the relative log canonical models of pairs (X, H) satisfying the conclusion of Theorem 2".

It "just" remains to calculate these images.

Lemma 3. [Tho10] *The map ϕ is a birational morphism and furthermore, writing*

$$(X_0)^c := \mathrm{Proj} \bigoplus_{n \geq 0} H^0(X_0, \mathcal{O}_{X_0}(nH_0))$$

for the log canonical model of X_0, we have that $(X_0)^c$ and $(X^c)_0$ agree.

Sketch proof. This is a consequence of the base point free theorem [Anc87] and the theorem on cohomology and base change. $\square$

In light of this, we set $X_0^c := (X_0)^c = (X^c)_0$. This allows us to restrict our attention to finding the log canonical models of pairs satisfying $(*)$.

Example 4. We begin by calculating the log canonical model when X_0 is a fibre of Type I (i.e. a smooth K3). Suppose first that H_0 is base point free. Then a simple Riemann-Roch calculation shows that $\phi_0 := \phi|_{X_0}$ is a birational morphism

$$\phi_0 \colon X_0 \longrightarrow X_0^c \cong X_6 \subset \mathbb{P}(1,1,1,3)$$

that contracts finitely many curves to Du Val singularities. This surface is the traditional "double cover of $\mathbb{P}^2$" that one normally associates with K3 surfaces of degree two.

Example 5. Suppose next that H_0 has base points. Then Mayer [May72] shows that $|2H_0|$ is base point free and a Riemann-Roch calculation shows that ϕ_0 is a birational morphism

$$\phi_0 \colon X_0 \longrightarrow X_0^c \cong X_{2,6} \subset \mathbb{P}(1,1,1,2,3),$$

where the degree two relation does not involve the degree two variable, that contracts finitely many curves to Du Val singularities. Note that X_0^c cannot be expressed as a double cover of $\mathbb{P}^2$. Instead, it can be seen as a double cover of the singular rational surface $X_2 \subset \mathbb{P}(1,1,1,2)$.

In fact, we find that these two cases are essentially all that can occur:

Theorem 6. [Tho10] *Let $\pi \colon X \to \Delta$ be a semistable degeneration of K3 surfaces, with $\omega_X \cong \mathcal{O}_X$. Let H be a divisor on X that is effective, nef and flat over Δ, and suppose that H induces an ample divisor H_t on X_t satisfying $H_t^2 = 2$ for $t \in \Delta^*$.*

Then the morphism $\phi \colon X \to X^c$ taking X to the relative log canonical model of the pair (X, H) maps X_0 to one of:

- (*Hyperelliptic Case*) *A sextic hypersurface*

$$\{z^2 - f_6(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,3)}[x_1, x, x_3, z].$$

- (*Unigonal Case*) *A complete intersection*

$$\{z^2 - f_6(x_i, y) = f_2(x_i) = 0\} \subset \mathbb{P}_{(1,1,1,2,3)}[x_1, x_2, x_3, y, z],$$

where $f_6(0, 0, 0, 1) \neq 0$.

Furthermore, we have the following tables that explicitly classify the possible central fibres:

Table 1: $\phi(X_0) = \{z^2 - f_6(x_i) = 0\} \subset \mathbb{P}(1, 1, 1, 3)$ hyperelliptic.

Type	Name	$f_6(x_i)$	Comments
I	h	Reduced	f_6 has at worst A-D-E's.
II	0h	Reduced	f_6 has exactly one $\tilde{E}_7$ or $\tilde{E}_8$.
	1	$l^2(x_i)f_4(x_i)$	l linear, $\lvert l \cap f_4\rvert = 4$, f_4 may have an $\tilde{E}_7$.
	2	$q^2(x_i)f_2(x_i)$	q smooth quadric, $\lvert q \cap f_2\rvert = 4$.
	3	$f_3^2(x_i)$	f_3 smooth cubic.
III	0h	Reduced	f_6 has exactly one $T_{2,3,r}$ with $r \geq 7$ or $T_{2,q,r}$ with $q \geq 4$ and $r \geq 5$.
	1	$l^2(x_i)f_4(x_i)$	l linear, $\lvert l \cap f_4\rvert \leq 3$ with multiplicities ≤ 2.
	2	$q^2(x_i)f_2(x_i)$	q (possibly nodal) quadric, $\lvert q \cap f_2\rvert \leq 4$ (< 4 if q smooth) with multiplicities ≤ 2.
	3	$f_3^2(x_i)$	f_3 cubic with nodal singularities.

Table 2: $\phi(X_0) = \{z^2 - f_6(x_i, y) = f_2(x_i) = 0\} \subset \mathbb{P}(1, 1, 1, 2, 3)$ unigonal.

Type	Name	$f_2(x_i)$	Comments
I	u	Irreducible	$\phi(X_0)$ has at worst RDP's.
II	0u	Irreducible	$\phi(X_0)$ has exactly one $\tilde{E}_7$ or $\tilde{E}_8$.
	4	$l_1(x_i)l_2(x_i)$	l_i linear, $\lvert l_1 \cap l_2 \cap f_6\rvert = 3$, where $\phi(X_0)$ may have an $\tilde{E}_8$.
III	0u	Irreducible	$\phi(X_0)$ has exactly one $T_{2,3,r}$ with $r \geq 7$ or $T_{2,q,r}$ with $q \geq 4$ and $r \geq 5$.
	4	$l_1(x_i)l_2(x_i)$	l_i linear, $\lvert l_1 \cap l_2 \cap f_6\rvert = 2$, where the curve $\{f_6 = l_i = 0\}$ may be non-reduced for exactly one choice of $i \in \{1, 2\}$.

Note the relationship between the entries in this table and other known compactifications of the moduli space of K3 surfaces of degree two:

- (II.1)-(II.4) correspond to the four Type II boundary components appearing the Baily-Borel-Satake compactification [Fri84].
- All cases except (III.0) appear in Shah's [Sha80] GIT compactification, although several of our cases map to the same GIT points.

Sketch proof of Theorem 6. Recall that, by Lemma 3, we just have to analyse the log canonical model of the pair (X_0, H_0). Write X_0 as a union of irreducible components $X_0 = V_1 \cup \cdots \cup V_r$ and let $H_i = H \cap V_i$. Then we have:

Lemma 7. [Tho10] *If $H_i^2 = 0$, then V_i is contracted by ϕ.*

This allows us to focus our attention on components V_i with $H_i^2 > 0$. We have:

Theorem 8. [Tho10] *After performing a birational modification on X_0 that does not affect the form of its log canonical model, we may assume that for any surface V_i with $H_i^2 > 0$, the linear system $|nH_i|$*

- *has no fixed components or base locus for $n \geq 2$;*
- *defines a morphism to projective space that is birational onto its image for $n \geq 3$.*

Sketch proof. This follows from known facts about anticanonical pairs [Fri83] and elliptic ruled surfaces [Tho11] if one can prove that $\mathrm{Fix}(|H_i|)$ does not contain any component of the double locus on V_i. This can be proved for the central fibre of a degeneration of K3 surfaces of degree two, but the proof does not work for other polarisations (it relies upon the fact that the only partitions of 2 are (2) and $(1, 1)$). □

To finish proving the theorem, one just has to explicitly calculate cases corresponding to different positions of surfaces that have $H_i^2 > 0$ within X_0. For instance, in the Type II case we have 5 possibilities:

1. There is one component V_i with $H_i^2 = 2$, that is rational. This case gives rise to cases (II.0), (II.2) and (II.4), distinguished by the intersection number $H_i.K_{V_i}$.
2. There is one component V_i with $H_i^2 = 2$, that is elliptic ruled. This case gives rise to case (II.4), where $\phi(X_0)$ has a $\tilde{E}_8$ singularity.
3. There are two components V_i and V_j with $H_i^2 = H_j^2 = 1$, that are both rational. This gives rise to cases (II.1) and (II.3).

4. There are two components V_i and V_j with $H_i^2 = H_j^2 = 1$, one of which is rational and the other of which is elliptic ruled. This case gives rise to case (II.1), where $\phi(X_0)$ has a $\tilde{E}_7$ singularity.

5. There are two components V_i and V_j with $H_i^2 = H_j^2 = 1$, that are both elliptic ruled. This case leads to a contradiction and cannot occur.

$\square$

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