Abelian varieties with singular theta divisor at an odd 2-torsion point

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0. Introduction and motivation

For

$$X \subset \mathbb{P}^4 \quad \text{smooth cubic threefold}$$

Clemens and Griffiths show non-rationality by looking at the intermediate Jacobian

$$IJ(X) = H^{2,1}(X)^* \big/ H_3(X,\mathbb{Z}) \in \mathcal{A}_5 \qquad (H^{3,0}=0).$$

Theorem (Clemens-Griffiths). The theta divisor Θ of $IJ(X)$ has a unique triple point (at an odd 2-torsion point x_0).

Theorem (Mumford): $X \cong$ proj. of the tangent cone of Θ at x_0.

There is the following solution to the Schottky for intermediate Jacobian

Theorem (Casalaina-Martin, Friedman): The following are equivalent

$$\left\{ \begin{array}{l} (A,\Theta) \text{ is indecomposable ppav of genus } 5 \\ \Theta \text{ has a triple point} \end{array} \right\} \iff \left\{ (A,\Theta) = IJ(X) \right\}.$$

One has a global Torelli theorem for intermediate Jacobians of cubic threefolds, i.e.

$$\begin{array}{ccc} IJ = \{\text{smooth cubic threefolds}\}/\cong & \hookrightarrow & \mathcal{A}_5 . \\ | & & | \\ \dim = \binom{4+3}{3} - 25 = 35-25 = 10 & & \dim = 15. \end{array}$$

Question (van der Geer): What is the class of the cycle $[\overline{IJ}] \in Ch^*(\mathcal{A}_5)$?

One can ask more generally

<u>Question'</u>: What can one say about the locus of ppav's whose theta divisor has a singularity at an odd 2-torsion point?

For this we consider

$$\mathbb{E} = \text{Hodge bundle on } A_g .$$

I.e. let $\pi : \mathcal{X}_g \to A_g$ be the universal family, then

$$\mathbb{E} = \pi_* (\Omega^1_{\mathcal{X}_g / A_g}) .$$

Let

$$\lambda_i = c_i (\mathbb{E}) .$$

These generate the <u>tautological subring</u> of $CH^*(A_g)$. One has

(1) $(1 - \lambda_1 + \lambda_2 - \dots + (-1)^g \lambda_g)(1 + \lambda_1 + \lambda_2 + \dots + \lambda_g) = 1$

(2) $\lambda_g = 0$.

In terms of these classes

<u>Theorem 1.</u> $[\overline{I_g} \cup A_1 \times \theta^{(4)}_{null}] = 2^{g-1}(2^g - 1) \sum_{i=0}^{g} \lambda_{g-i} (\frac{\lambda_1}{2})^i$. $\quad g = 5$

In view of (1), (2) this becomes

$$[\overline{I_g} \cup A_1 \times \theta^{(4)}_{null}] = 93 (4 \lambda_1^2 \lambda_3 + \frac{\lambda_1^5}{2}) .$$

In particular, this class is tautological. We do not know whether $[\overline{I_g}]$ itself is tautological, but we can compute its projection to the tautological subring

<u>Corollary</u> $\quad [\overline{I_g}]^{taut} = 140 \lambda_1^5 - 376 \lambda_1^2 \lambda_3 .$

<u>Remark:</u> $\overline{I_g} \cup A_1 \times \theta^{(4)}_{null}$ are just the ppav's whose theta divisor has a singularity at an odd 2-torsion point.

The next question refers to <u>compactifications</u> of A_g.

Here we work with

A_g^{perf} = perfect cone compactification.

Here the boundary is irreducible. The classes λ_i extend to A_g^{perf}.
One still has a tautological ring with (1), but not (2). Let

$$p: A_g^{perf}(2) \longrightarrow A_g^{perf}$$

be the level-2 cover. Then

- boundary components D_n of $A_g^{perf}(2) \overset{1:1}{\longleftrightarrow} (\mathbb{Z}/2\mathbb{Z})^{2g} \setminus \{0\}$, $\delta_n = [D_n]$
- p is branched of order 2 along the D_n
- $m = (m_1, m_2) \in (\mathbb{Z}/2\mathbb{Z})^{2g}$ is even / odd $:\Leftrightarrow m_1 \cdot m_2 = 0$ resp. 1.
- $b = |Sp(2g, \mathbb{Z}/2\mathbb{Z})|$.

<u>Theorem 2</u>: The following formulae hold

(a) $$\overline{[I] \cup A_1 \times \Theta_{null}^{(4)}} \overset{g=5}{=} \frac{1}{b} \Big(\sum_{\substack{m \in (\mathbb{Z}/2\mathbb{Z})^{2g} \\ m \text{ odd}}} \sum_{i=0}^{g} p_* \big(\lambda_{g-i} \big(\tfrac{\lambda_1}{2} - \tfrac{1}{4} \sum_{n,\, m+n \text{ even}} \delta_n \big)^i \big) \Big).$$

(b) $$\overline{[I] \cup A_1 \times \Theta_{null}^{(4)}} \overset{g=5}{=} \frac{(-1)^{g-1}(g-1)!}{8g(1-2g)} \lambda_g + 2^{g-1}(2^g - 1) \sum_{i=0}^{g} \lambda_{g-i} \left(\tfrac{\lambda_1}{2}\right)^i.$$

II. <u>Theta functions and gradients</u>

$$\mathbb{H}_g = \{ \tau \in Mat(g \times g, \mathbb{C});\ \tau = {}^t\tau,\ \operatorname{Im} \tau > 0 \} \quad \text{(Siegel space)}.$$

The group $\Gamma_g = Sp(2g, \mathbb{Z})$ acts on $\mathbb{H}_g$ by

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \tau \mapsto (A\tau + B)(C\tau + D)^{-1}$$

and analytically

$$A_g = \Gamma_g \backslash \mathbb{H}_g.$$

We also consider the groups

$$\Gamma_g(n) = \{\gamma \in \Gamma_g;\ \gamma \equiv \mathbb{1} \bmod n\}$$

as well as

$$\Gamma_g(n,2n) = \{\gamma \in \Gamma_g(n),\ \operatorname{diag} {}^tA\,B \equiv \operatorname{diag} {}^tC\,D \equiv 0 \bmod 2n\}$$

with corresponding quotients

$$A_g(n) = \Gamma_g(n) \backslash \mathbb{H}_g,\quad A_g(n,2n) = \Gamma_g(n,2n) \backslash \mathbb{H}_g .$$

The Riemann theta function is defined by

$$\Theta(\tau, z) = \sum_{n \in \mathbb{Z}^g} e^{(\pi i\, {}^tn\, \tau n + 2\pi i\, {}^tn\, z)} \qquad \text{(even function)}.$$

To each $\tau \in \mathbb{H}_g$ one associates the torus

$$A_\tau = \mathbb{C}^g / (\mathbb{Z}^g \tau + \mathbb{Z}^g)$$

Then

$$\Theta = \{\Theta(\tau, z) = 0\} \subset A_\tau$$

is a symmetric theta divisor and (A_τ, Θ) define a ppav. The 2-torsion points of A_τ are

$$m = \frac{\tau \varepsilon + \delta}{2}\,; \qquad \varepsilon, \delta \in \{0,1\}^g$$

and m is even/odd if $\varepsilon . \delta$ is even/odd. Let

$$\boxed{f_m(\tau) = \operatorname{grad}_z \Theta(\tau, z)\Big|_{z=m}} \qquad (m \text{ odd, otherwise } f_m \equiv 0).$$

Lemma: $f_m \in H^0(A_g(4,8), \mathbb{E} \otimes \det \mathbb{E}^{1/2})$.

The group $\Gamma_g(2)/\Gamma_g(4,8)$ acts on these gradients by certain signs and thus

$$G_m = G_{\varepsilon,\delta} = \{[\tau];\ f_m(\tau) = 0\} \subset A_g(2)$$

is well defined. Using the level-2 cover $p: A_g(2) \to A_g$

we define

$$I^{(g)} = p(G_m) \subset A_g$$

This is independent of m and by definition

$$I^{(g)} = \{(\Pi, \Theta);\ \Theta \text{ is pp with a singularity at some odd 2-torsion point}\}$$

Clearly

$$(*) \qquad \operatorname{codim}_{A_g} I^{(g)} \leq g.$$

<u>Conjecture</u> (Grushevsky, Salvati-Manni). The locus $I^{(g)}$ is empty for $g \leq 2$ and has pure codimension g in A_g for $g \geq 3$.

<u>Theorem</u> (Casalaina-Martin, Friedman): This holds for $g \leq 5$.

<u>Remark</u>: Independent proof by G.-H.

<u>Remark</u>: For small g:

$$I^{(3)} = \operatorname{Sym}^3 A_1, \quad I^{(4)} = A_1 \times \Theta^{(3)}_{null}, \quad I^{(4)} = \overline{IJ} \cup A_1 \times \Theta^{(4)}_{null}.$$

<u>Proof of theorem 1</u>: This is a Chern class calculation

$$[I^{(g)}] = c_g(\mathbb{E} \otimes \det \mathbb{E}^{1/2}).$$

<u>Extension to the boundary</u>

We work on $A_g^{perf}(8)$. The boundary components correspond to

$$\{D_n\} \longleftrightarrow \pm n \in (\mathbb{Z}/8\mathbb{Z})^{2g} \setminus \{0\}, \ n \text{ primitive}$$

Let $n_2 := n \bmod 2$.

<u>Proposition</u>: The sections f_m extend to sections

$$\bar{f}_m \in H^0(A_g^{perf}(8), \mathbb{E} \otimes \det \mathbb{E}^{1/2} \otimes \mathcal{O}(-\sum_{\substack{\pm n \text{ primitive} \\ m+n_2 \text{ even}}} D_n))$$

and these sections do not vanish on the generic point of any boundary component.

We define

$$\overline{G_m^{(g)}} = \{\tilde{f}_m = 0\} \subset A_g^{perf}(8)$$

and using the Galois cover $p: A_g^{perf}(8) \to A_g^{perf}$ we define

$$\overline{G^{(g)}} = p(\overline{G_m^{(g)}}) \subset A_g^{perf}. \qquad \text{(independent of } m\text{)}.$$

Clearly one has

$$(**) \qquad \overline{I^{(g)}} \subset \overline{G^{(g)}}.$$

<u>Conjecture</u>: $\overline{I^{(g)}} = \overline{G^{(g)}}$ for all g (and this is of pure codimension g)

<u>Theorem</u>: This conjecture is true for $g \leq 5$.

<u>Idea of proof</u>. One has to prove that $\overline{G^{(g)}}$ does not "pick up" extra components in the boundary. For this one has to study degenerations of abelian varieties (semi-abelic varieties) of low torus rank, and analyse the theta divisor and the gradient of its defining equation at limits of odd 2-torsion points.

<u>Proof of theorem 2</u>: (a) This is again a Chern class calculation, if there is equality in $(**)$:

$$[\overline{I_m^{(g)}}] = [\overline{G_m^{(g)}}] = c_g\Big(E \otimes \det E^{1/2} \otimes \mathcal{O}\big(-\sum_{\substack{\pm n \text{ primitive} \\ m+n_1 \text{ even}}}\big)\Big) \quad (\text{in } A_g^{perf}(8))$$

(b) Here one has to show that many classes in the above expression project to 0 in the tautological ring, and obtains (if equality holds in $(**)$):

$$[\overline{I^{(g)}}]^{taut} = [\overline{G^{(g)}}]^{taut} = \frac{(-1)^{g-1}(g-1)!}{8\,\zeta(1-2g)}\lambda_g + 2^{g-1}(2^g-1)\cdot \sum_{i=0}^{g} \lambda_{g-i}\left(\frac{\lambda_1}{2}\right)^i$$

III Low genus

$g=3$. Here $A_3^{\text{perf}} = A_3^{\text{Vor}} = A_3^{\text{Centr}}$ and the Chow ring was computed by van der Geer.

$$[\overline{G^{(3)}}] = [\overline{I^{(3)}}] = [\overline{\operatorname{Sym}^3 A_1}] = -35\lambda_3 + \frac{35}{2}\lambda_1^3 - \frac{25}{4}\lambda_1^2\sigma_1$$
$$+ \frac{5}{8}\lambda_1\sigma_2 + \frac{5}{8}\lambda_1\sigma_1^2 - \frac{1}{12}\sigma_1\sigma_2$$

where the σ_i are the symmetric polynomials in the boundary components.

$g=4$ Here

$$\underset{\text{smooth}}{A_4^{\text{Vor}}} \longrightarrow \underset{\text{singular}}{A_4^{\text{perf}}} = A_4^{\text{Centr}}$$

We have

$$[\overline{G^{(4)}}] = [\overline{I^{(4)}}] = [\overline{A_1 \times \Theta_{\text{null}}^{(3)}}] = 180\lambda_1\lambda_3 + \frac{45}{2}\lambda_1^4 +$$

15 further terms supported on the boundary

$g=5$ Here

$$[\overline{G^{(5)}}] = [\overline{I^{(5)}}] = [\overline{IJ \cup A_1 \times \Theta_{\text{null}}^{(4)}}] = 4596\lambda_5 + 372\lambda_3\lambda_1^2 + \frac{53}{2}\lambda_1^5$$

+ many terms supported on the boundary