Elliptic Fibrations on K3 surfaces

V. V. NIKULIN

Based on my results 1981, 1996, 1999. Review and applications.

Recently, some new applications emerged, I want to discuss.

K3 surfaces are closely related to Fano, CY var.

Details in arXiv: 1010.3904

Basic results about K3 $X/k=\bar{k}$.

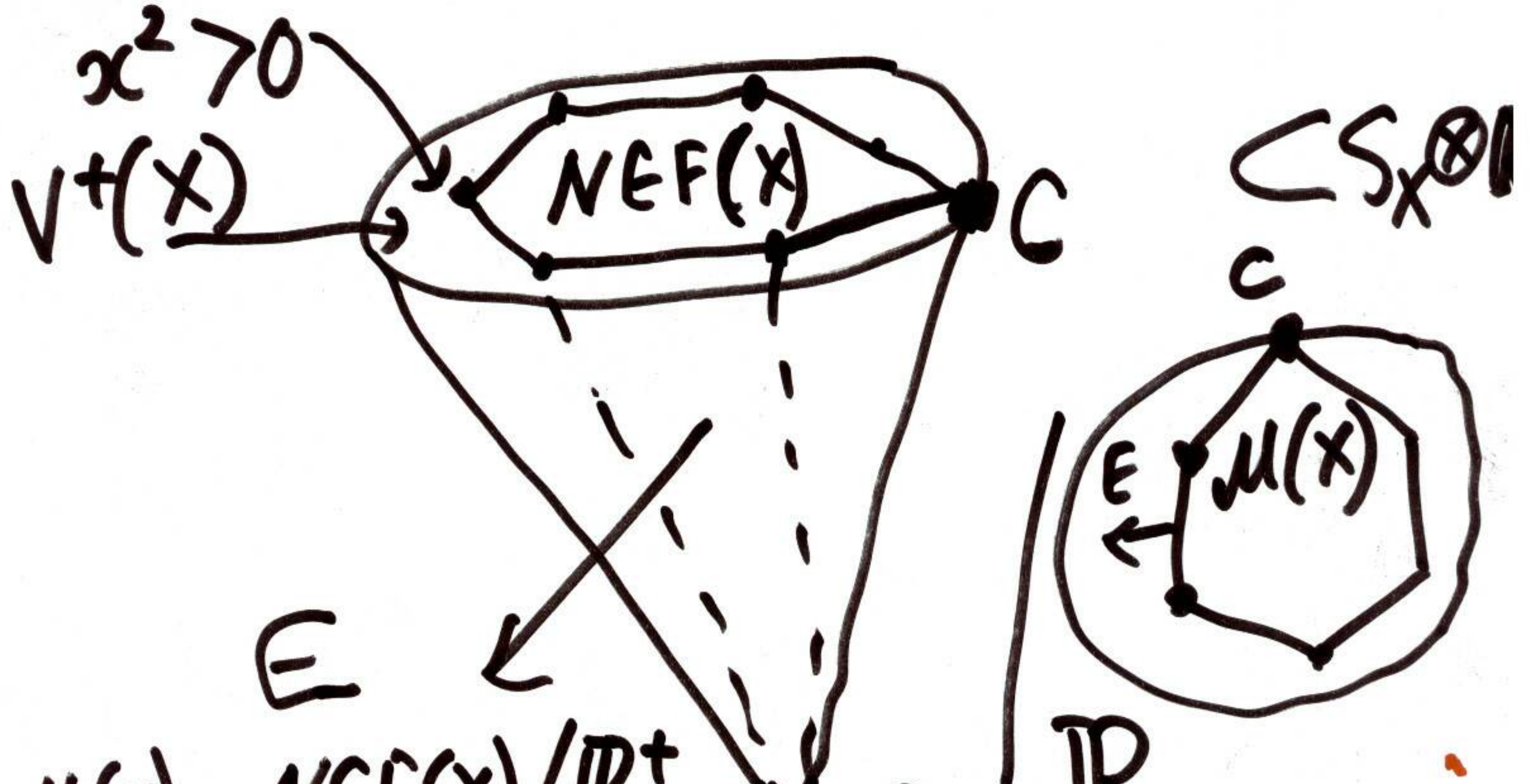
(Piatetsky-Shapiro – Shafarevich, 1971).

S_X – Picard lattice

• Ell. fibr. $\longleftrightarrow$ primitive isotropic nef $c\in S_X$:
$0\neq c$, $c^2=0$, $c\cdot D\geqslant 0$,
c – primitive.

$|c|: X \longrightarrow \mathbb{P}^1$, gen. fibre

- Irr. curves $E \subset X$ with $E^2 < 0$ have $E^2 = -2$ and $E \simeq \mathbb{P}^1$. $\Longrightarrow$ $NEF(X) \subset V^+(X)$ is fund. chamber for $W^{(-2)}(S_X)$,



$\mathcal{M}(X) = NEF(X)/\mathbb{R}^+$.

gen. by $s_\delta : x \mapsto x + (x\cdot\delta)\delta$,

$\delta \in S$, $\delta^2 = -2$

• $k=\mathbb{C}$. $\mathrm{Aut}\,X \approx A(S_X) = \{\varphi \in O(S_X) \mid \varphi(\mathcal{M}(X)) = \mathcal{M}(X)\} \approx O(S_X)/W^{(-2)}(S_X)$.

• For any $0 \neq c \in S_X$, primitive, $c^2=0$, ~~$\exists w \in$~~ $\exists w \in W^{(-2)}(S_X): w(c)=c'$ is nef $\Longrightarrow$ $|c'|: X \longrightarrow \mathbb{P}^1$ is ell. fibr.

• X has elliptic fibr. $\Longleftrightarrow$ S_X represents 0.

Number of elliptic fibrations is finite up to $A(S_X)$

When X has ell. fibrations with infinite autom. group?

$$|c| : X \longrightarrow \mathbb{P}^1$$

$$Aut(c) \approx \mathbb{Z}^{r(c)}$$

$r(c) = rk(c^\perp) - rk((c^\perp)^{(2)})$.

$(c^\perp)^{(2)}$ is generated by c and elements with square (-2) $\Longleftrightarrow$ $(c^\perp)^\perp$ is generated by irred. components of fibres, up to finite index.

X has ell. f-s with infinite $Aut(c)$ $\Longleftrightarrow$ $\exists\, 0 \neq c \in S_X$ $c^2 = 0$ and $rk(c^\perp) > rk((c^\perp)^{(2)})$.

<u>Thm</u>(N, 81):

If $\rho = \operatorname{rk} S_X \geq 6$, then
X has (no) ell. f. c with
infinite Aut(c) $\Longleftrightarrow$
Aut X is finite $\Longleftrightarrow$ $[G(S_X) : W^{(-2)}(S_X)] < \infty$ $\Longleftrightarrow$
S_X is one of finite
number of found latt.

$U \oplus E_8 \oplus E_8 \oplus A_1$ ($\rho = 19$)

$U \oplus E_8 \oplus E_8$ ($\rho = 18$)

$\vdots$

($\rho = 6$)

$U \oplus D_4$, $U(2) \oplus D_4$, $U \oplus 4A_1$,
$U(2) \oplus 4A_1$, $U \oplus 2A_1 \oplus A_2$,

6a

Theorem 1. *Let S be an even hyperbolic lattice of the rank $\rho = \operatorname{rk} S \geq 6$ (respectively, X is a K3 surface over an algebraically closed field, and $\rho(X) \geq 6$). Then the following conditions (a), (b), (c) below are equivalent:*

(a) S satisfies the condition (2) (respectively, automorphism groups of all elliptic fibrations on X are finite).

(b) The group $A(\mathcal{M}) \cong O^+(S)/W^{(2)}(S)$ is finite, (respectively, Aut X *is finite).*

(c) The lattice S belongs to the finite list of even hyperbolic lattices below found in [3] (respectively, $S = S_X$ is one of the lattices from the list)

The list of lattices found in [3] is the following (we use notations from [2] and [3], which are now standard):

The list of all even hyperbolic lattices S with $[O(S) : W^{(2)}(S)] < \infty$ and $\operatorname{rk} S \geq 6$ *(see [3]):*

$S = U \oplus 2E_8 \oplus A_1$; $U \oplus 2E_8$; $U \oplus E_8 \oplus E_7$; $U \oplus E_8 \oplus D_6$; $U \oplus E_8 \oplus D_4 \oplus A_1$; $U \oplus E_8 \oplus D_4$, $U \oplus D_8 \oplus D_4$, $U \oplus E_8 \oplus 4A_1$; $U \oplus E_8 \oplus 3A_1$, $U \oplus D_8 \oplus 3A_1$, $U \oplus A_3 \oplus E_8$; $U \oplus E_8 \oplus 2A_1$, $U \oplus D_8 \oplus 2A_1$, $U \oplus D_4 \oplus D_4 \oplus 2A_1$, $U \oplus A_2 \oplus E_8$; $U \oplus E_8 \oplus A_1$, $U \oplus D_8 \oplus A_1$, $U \oplus D_4 \oplus D_4 \oplus A_1$, $U \oplus D_4 \oplus 5A_1$; $U \oplus E_8$, $U \oplus D_8$, $U \oplus E_7 \oplus A_1$, $U \oplus D_4 \oplus D_4$, $U \oplus D_6 \oplus 2A_1$, $U(2) \oplus D_4 \oplus D_4$, $U \oplus D_4 \oplus 4A_1$, $U \oplus 8A_1$, $U \oplus A_2 \oplus E_6$; $U \oplus E_7$, $U \oplus D_6 \oplus A_1$, $U \oplus D_4 \oplus 3A_1$, $U \oplus 7A_1$, $U(2) \oplus 7A_1$, $U \oplus A_7$, $U \oplus A_3 \oplus D_4$, $U \oplus A_2 \oplus D_5$, $U \oplus D_7$, $U \oplus A_1 \oplus E_6$; $U \oplus D_6$, $U \oplus D_4 \oplus 2A_1$, $U \oplus 6A_1$, $U(2) \oplus 6A_1$, $U \oplus 3A_2$, $U \oplus 2A_3$, $U \oplus A_2 \oplus A_4$, $U \oplus A_1 \oplus A_5$, $U \oplus A_6$, $U \oplus A_2 \oplus D_4$, $U \oplus A_1 \oplus D_5$, $U \oplus E_6$; $U \oplus D_4 \oplus A_1$, $U \oplus 5A_1$, $U(2) \oplus 5A_1$, $U \oplus A_1 \oplus 2A_2$, $U \oplus 2A_1 \oplus A_3$, $U \oplus A_2 \oplus A_3$, $U \oplus A_1 \oplus A_4$, $U \oplus A_5$, $U \oplus D_5$; $U \oplus D_4$, $U(2) \oplus D_4$, $U \oplus 4A_1$, $U(2) \oplus 4A_1$, $U \oplus 2A_1 \oplus A_2$, $U \oplus 2A_2$, $U \oplus A_1 \oplus A_3$, $U \oplus A_4$, $U(4) \oplus D_4$, $U(3) \oplus 2A_2$.

Thus, a K3 surface X over an algebraically closed field and with $\rho(X) \geq 6$ has an elliptic fibration with infinite automorphism group if and only if its Picard lattice S_X is different from each lattice of this finite list. If the Picard lattice S_X of X is one of lattices from the list, then not only automorphism groups of all elliptic fibrations on X are finite, but the full automorphism

<u>Thm (N, 81)</u> If $\rho = \mathrm{rk}\, S_X = 5$, ~~th~~

X has no ell. fibr c with

infinite $\mathrm{Aut}(c) \iff$ ~~Aut X~~

Aut X is finite $\iff$

$[G(S_X) : W^{(-2)}(S_X)] < \infty \iff$

S_X is one of finite number of found lattices:

$U \oplus 3A_1$, $U(2) \oplus 3A_1$, $U \oplus A_1 \oplus A_2$,

$U \oplus A_3$, $U(4) \oplus 3A_1$, $\langle 2^k \rangle \oplus D_4$,

$k = 2, 3, 4$, $\langle 6 \rangle \oplus 2A_2$

X has f. number of el. f.

OR

S_X is one of

$\langle 2^m \rangle \oplus D_4$, $m \geqslant 5$,

$\langle 2 \cdot 3^{2m-1} \rangle \oplus 2A_2$, $m \geqslant 2$.

$\rho = 3$ ... has ∞ number of

Later, ~~If~~ $\exists\alpha$ $char\, k = 0$,

For $\rho = 4$ ~~and $char\, k = 0$~~

there are 12 latt. S_X

with finite $Aut\, X$

(Vinberg) (1982).

For $\rho = 3$ there are

26 latt. S_X with finite $Aut\, X$

(N.) (1983)

For some of these S_X,

S_X has no isotropic elements.

Thus, ~~it~~ finding of such S_X

... out methods

• what about number of e.f.?
~~wt~~ number of el. fibr. with
infinite $Aut(c)$?

<u>Def:</u> $x \in S_X$ is exceptio (for $Aut\,X$) if $Aut\,X\,(x)$
is finite $\Leftrightarrow [Aut\,X : (Aut\,X)_x] < \infty$.
~~is finite~~

$E(S_X) \subset S_X$ is sublattice
of exceptional elements

<u>Cases:</u>

• $E(S_X)$ is hyperbolic $\Leftrightarrow |Aut\,X| < \infty$

• $E(S_X)$ is parabolic $\Leftrightarrow E(S_X)$
has 1-dim. kernel

If X has ell. fibr. with C infinite $Aut(C)$, then

$$E(S_X) = \bigcap_{\substack{C \text{ el.f.} \\ Aut(C) \text{ inf.}}} (C^{\perp})^{(2)}$$

$\Longrightarrow$

$E(S_X)$ parabolic $\Longleftrightarrow$ X has only one el. fibr. with $Aut(C)$ infinite.

In all other cases $E(S_X) < 0$ (elliptic)

<u>Thm (N, ~~1996~~, 99):</u> If ($k = \mathbb{C}$) $\rho \geq 3$, then $E(S_X) = \{0\}$

finite number of S_X

The main idea of the proof: If $E(S_X)\neq\{0\}$, then $\mathcal{M}(X)$, the fund. chamber for $W^{(-2)}(S_X)$, has **Narrow part**:

~~The set~~

$\exists$ classes $\delta_1, \delta_2, \dots, \delta_\rho$ $(\delta_i^2=-2)$ of $\mathbb{P}^1$ ($\perp$ to $\mathcal{M}(X)$) such that

- $\delta_1, \dots, \delta_\rho$ generate $S_X\otimes\mathbb{Q}$.
- $\delta_i \cdot \delta_j \leq A$ absolute

<u>Thm (N, 96, 99)</u>: If $k = \mathbb{C}$ or X has ell. fibration c with $|Aut(c)| = \infty$, ~~then~~ and $\rho \geqslant 3$ ~~then~~ $E(S_X) = E(S_X) = \{0\}$ except finite number of $S_X \in SEK3$

$SEK3 = \{ S_X \mid E(S_X) \neq \{0\},$
$\rho = rk\, S_X \geqslant 3.$

<u>Thm 1</u>: If $k = \mathbb{C}$, X has e.f. c, and $\rho \geqslant 3$, then X has <u>infinite</u> number of ell. fibrations except finite number of $S_X \in SEK3$ ~~$S_X \in SEK3$~~ e

1) $|Aut\, X| < \infty$, el. fibrat.

2) X has unique ... Aut(c) infinite

By recent preprint (arXiv) by
R. van Luijk, surfaces
having 2 el. fibrations
have especially ~~nic~~
many rational points.

Other applications
of $E(S_X)$ and finiteness
Theorem:

• K3 with finite,
not empty set of $\mathbb{P}^1$
have $S_X \in$ SEK3 (finite)

Aut X is finite if

• K3 with finite, ≥ 1, number of Enriques involutions σ ~~$(\neq \emptyset)$~~ have S_X from SEK3 (finite number)

$$(S_X)^{\sigma} = U(2) \oplus E_8(2).$$

If $\rho = 10$, X has only one σ.

If $\rho > 10$,

$$\left((S_X)^{\sigma}\right)^{\perp}_{S_X} < 0$$

is exceptional.

$\Rightarrow S_X \in$ SEK3,

finite

Def: Aut X is naturally arithmetic if $\exists K \subset S_X$, sublattice, such that $\mathrm{Aut}\, X \approx O(K)$

(related to preprint by Totaro)

Thm: $k = \mathbb{C}$. Aut X is nat. arithmetic $\Longleftrightarrow$

(1) X has no $\mathbb{P}^1$

(2) X has $\mathbb{P}^1$ and $\rho \neq 2$

(3) X has $\mathbb{P}^1$, $\rho \geq 3$, $S_X \in \overline{\mathrm{SEK3}} \underset{\neq}{\subset} \mathrm{SEK3}$, finite set. $K \neq S_X$) $K^\perp < 0$

Proof: K is hyp. $\Rightarrow$

Problem: Enumerate the finite set SEK3. of Sx. (It describes K3 with exotic structures.

If CY is fibered by such K3, then (expected) its quantum cohomology are related to classical automorphic