Conductivity imaging from one interior measurement

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A convergent algorithm to solve

$$u = \text{argmin}\{\int_\Omega |J||\nabla v| : v \in H^1(\Omega), \quad v|_{\partial\Omega} = f\}.$$

Joint work with A. Nachman and A. Timonov

Let $u_f \in H^1(\Omega)$ with $u_f |_{\Omega} = f$. Then our weighted minimization problem can be written as

$$(P) \qquad \inf_{v \in H_0^1(\Omega)} \int_{\Omega} |J| |\nabla v + \nabla u_f|.$$

The dual problem is

$$(D) \quad \sup\{ < \nabla u_f, b >: \quad b \in (L^2(\Omega))^n, \quad |b(x)| \leq |J(x)| \;\; a.e. \;\; \text{and} \;\; \nabla \cdot b \equiv 0 \}.$$

Theorem (M, A. Nachman, A. Timonov (2011))

Assume that the data $(|J|, f)$ *is admissible. Then*

$$\inf_{v \in H_0^1(\Omega)} \int_\Omega |J||\nabla v + \nabla u_f|$$

$$=$$

$$\sup\{< \nabla u_f, b >: \quad b \in (L^2(\Omega))^n, \quad |b(x)| \leq |J(x)| \;\; a.e. \;\; and \;\; \nabla \cdot b \equiv 0\}$$

and the current density J *corresponding to the voltage potential* f *on* $\partial\Omega$ *is the unique solution of the dual problem.*

Let $E : (L^2(\Omega))^n \to \mathbb{R}$ and $G : H_0^1(\Omega) \to \mathbb{R}$ be defined by

$$E(d) = \int_\Omega |J||d + \nabla u_f| \quad \text{and} \quad G(v) \equiv 0.$$

Then the dual problem can be written in the form

$$(D) \qquad - \min_{b \in (L^2(\Omega))^n} \{E^*(b) + G^*(-\nabla \cdot b)\}.$$

Since J is the solution of the dual problem

$$0 \in \partial E^*(J) + \partial[G^* o(-\nabla \cdot)](J).$$

Let $A := \partial E^*(J)$ and $B := \partial[G^* o(-\nabla \cdot)]$. Then above can be written as

$$0 \in A(J) + B(J),$$

where A and B are maximal monotone set-valued operators.

To solve

$$0 \in A(J) + B(J)$$

we apply a Douglas-Rachford algorithm. This algorithm produces two sequences p_k and x_k such that

$$p_k \rightharpoonup J \quad \text{and} \quad x_k \rightharpoonup \nabla u.$$

Theorem (Lions and Mercier (1979), Svaiter (2010))

Let H be a Hilbert space and A, B be maximal monotone operators and assume that a solution of (1) exists. Then, for any initial elements x_0 and p_0 the sequences p_k and x_k generated by the following algorithm

$$\begin{aligned} x_{k+1} &= R_A(2p_k - x_k) + x_k - p_k \\ p_{k+1} &= R_B(x_{k+1}), \end{aligned}$$

converges weakly to some $\hat{x}$ and $\hat{p}$ respectively. Furthermore, $\hat{p} = R_B(\hat{x})$ and $\hat{p}$ satisfies

$$0 \in A(\hat{p}) + B(\hat{p}). \tag{1}$$

$$R_A = (Id + A)^{-1}$$

Let $u_f \in H^1(\Omega)$ with $u_f|_{\partial\Omega} = f$, and initialize $b^0, d^0 \in (L^2(\Omega))^n$. For $k \geq 1$:

1. Solve
$$\Delta u^{k+1} = \nabla \cdot (d^k(x) - b^k(x)), \quad u^{k+1}|_{\partial\Omega} = f.$$
2. Compute
$$d^{k+1} := \begin{cases} \max\{|\nabla u^{k+1} + b^k| - |J|, 0\} \frac{\nabla u^{k+1} + b^k}{|\nabla u^{k+1} + b^k|} & \text{if } |\nabla u^{k+1}(x) + b^k(x)| \neq 0, \\ 0 & \text{if } |\nabla u^{k+1}(x) + b^k(x)| = 0. \end{cases}$$
3. Let
$$b^{k+1}(x) = b^k(x) + \nabla u^{k+1}(x) - d^{k+1}(x).$$

This is an alternating split Bregman algorithm of Goldstein and Osher applied to the primal problem (P).

Theorem (M, A. Nachman, A. Timonov (2011))

The sequences b^k, d^k, and u^k produced by the above algorithm converge weakly to J, ∇u, and u, respectively.

So we are simultaneously solving the primal and the dual problem.

Numerical simulations

To simulate the internal data $|J|$ we use a CT (Computed Tomography) image of human abdomen rescaled to a realistic range of tissue conductivities.

Figure: Original image (left) and reconstructed image with 60 iterations (right).

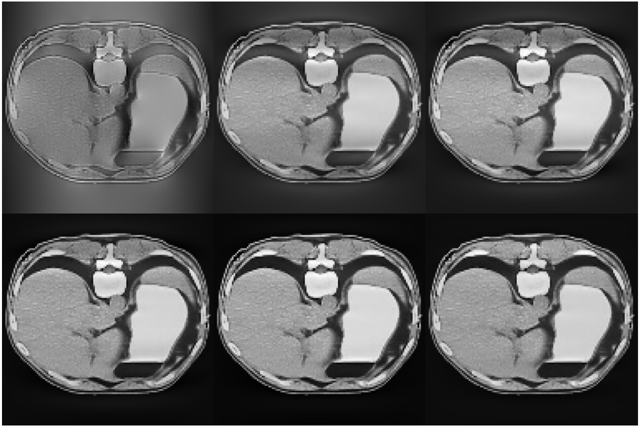


Figure: Conductivity reconstruction with the boundary condition $f(x, y) = y$ for $N = 1, 5, 10, 30, 50, 100$ iterations.

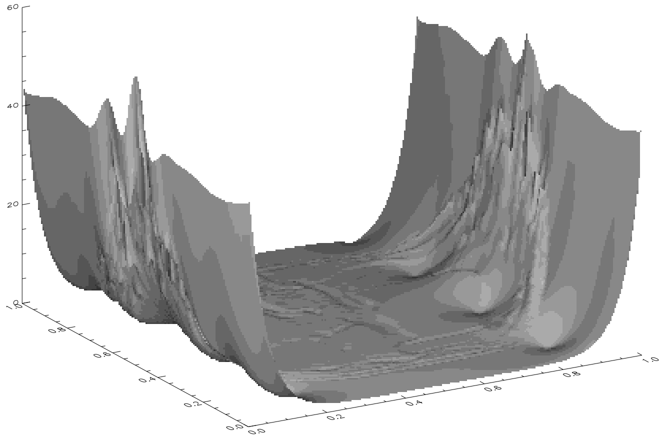


Figure: Magnitude of the current density $|J|$ for the non two-to-one boundary data $f(x, y) = y + 2\sin(7\pi y)$.

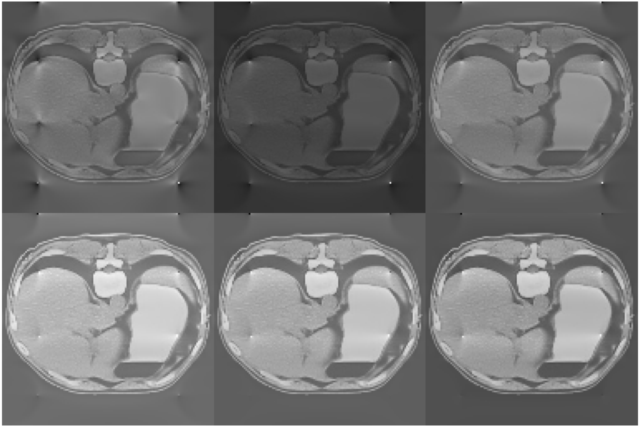
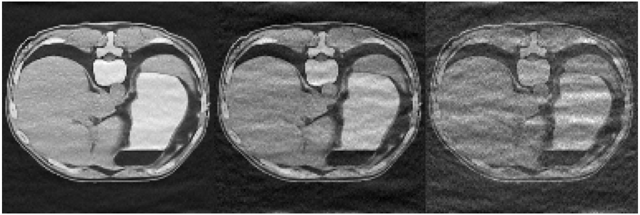


Figure: Conductivities constructed using the alternating split Bregman algorithm with $N = 1, 5, 10, 30, 50, 100$ iterates for the non two-to-one boundary data $f(x, y) = y + 2\sin(7\pi y)$.

Numerical errors for 100 iterations.

Low Noise (Level=0.01)	Moderate Noise (Level=0.035)	Higher Noise (Level=0.06)
0.026	0.080	0.152



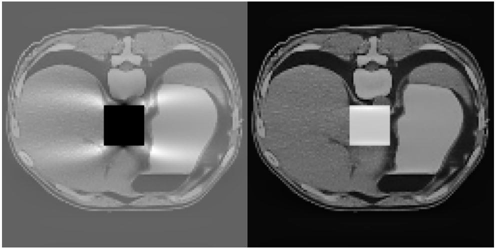


Figure: Reconstruction in the presence of the perfectly conducting (right) and insulating (left) inclusions.