Domain Decomposition approaches for grid generation via the Equidistribution Principle

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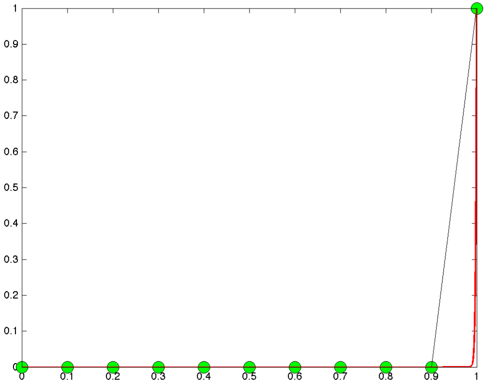
Plotting a function with a shock

Consider

$$u(x) = \frac{e^{\lambda x} - 1}{e^{\lambda} - 1}, \qquad \text{for large } \lambda.$$

Boundary layer is difficult to resolve on a uniform grid.



A variable transformation

Transform the independent variable

$$x(\xi) = \frac{1}{\nu} \ln(1 + (e^{\nu} - 1)\xi),$$

and we have,

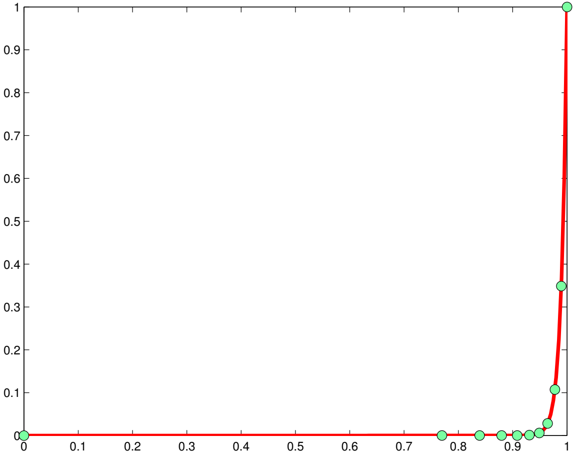
$$u(\xi) = u(x(\xi)) = \frac{(1 + (e^{\nu} - 1)\xi)^{\lambda/\nu} - 1}{e^{\lambda} - 1} \approx \xi, \qquad \text{for } \nu \approx \lambda.$$

Easy to represent on a uniform grid $\xi_i = \frac{i}{N}$

The transformation gives us an appropriate mesh for the original function:

$$x_i = x(\xi_i) = \frac{1}{\nu} \ln(1 + (e^{\nu} - 1)\xi_i).$$

Transformed Mesh



r–Refinement

- r–refinement methods typically start with a uniform mesh and then moves or relocates the mesh, keeping the number of mesh points $x_i(t)$, $i = 0, \ldots, N$ and mesh topology fixed.
- Solution and mesh locations are coupled and determined simultaneously.

How is this done?

Solve

$$u_t = \mathcal{L}(u) \qquad 0 < x < 1, \quad t > 0$$

by choosing a mesh transformation $x = x(\xi, t)$ so that a uniform mesh

$$\xi_i = \frac{i}{N}, \quad i = 0, 1, \ldots, N \quad \text{is sufficient.}$$

Given some measure of the error M in the solution we require

Equidistribution Principle (EP) – DeBoor (1973)

$$\int_{x_{i-1}(t)}^{x_i(t)} M(t, \tilde{x}, u)\, d\tilde{x} = \frac{1}{N} \int_0^1 M(t, \tilde{x}, u)\, d\tilde{x}.$$

1D Differential Form

$$\frac{\partial}{\partial \xi}\left\{M(t, x(\xi, t), u)\frac{\partial}{\partial \xi}x(\xi, t)\right\} = 0.$$

subject to boundary conditions

$$x(0, t) = 0 \quad \text{and} \quad x(1, t) = 1.$$

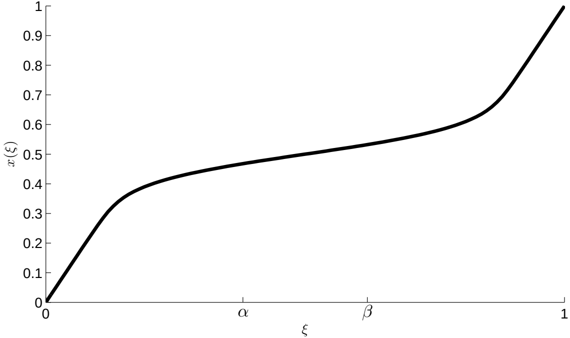
Relaxed EP at $t + \tau$

$$\frac{\partial x}{\partial t} = \frac{1}{\tau}\frac{\partial}{\partial \xi}\left(M\frac{\partial x}{\partial \xi}\right) \qquad \text{(MMPDE5)}$$

Grid Generation via DD

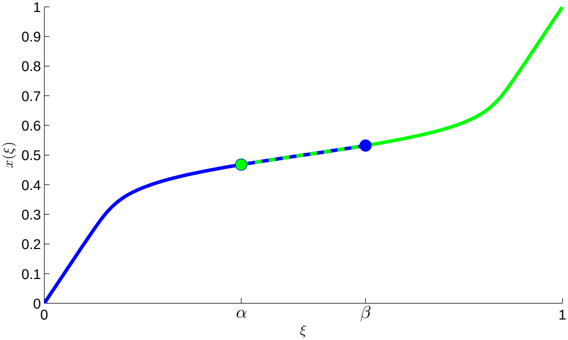
$$\frac{d}{d\xi}\left(M(x)\frac{dx}{d\xi}\right) = 0$$

$$x(0) = 0, \quad x(1) = 1$$



$$\frac{d}{d\xi}\left(M(x_1)\frac{dx_1}{d\xi}\right) = 0 \qquad \frac{d}{d\xi}\left(M(x_2)\frac{dx_2}{d\xi}\right) = 0$$

$$x_1(0) = 0, \quad x_1(\beta) = ? \qquad x_2(\alpha) = ?, \quad x_2(1) = 1$$



An iterative approach

We partition $\xi \in [0,1]$ into two overlapping subdomains $\Omega_1 = (0,\beta)$ and $\Omega_2 = (\alpha, 1)$ with $\alpha < \beta$. Let x_1^n and x_2^n solve

$$\begin{aligned} \frac{d}{d\xi}\left(M(x_1^n)\frac{dx_1^n}{d\xi}\right) &= 0 \quad \text{on } \Omega_1 \\ x_1^n(0) &= 0 \\ x_1^n(\beta) &= x_2^{n-1}(\beta) \end{aligned} \qquad \begin{aligned} \frac{d}{d\xi}\left(M(x_2^n)\frac{dx_2^n}{d\xi}\right) &= 0 \quad \text{on } \Omega_2 \\ x_2^n(\alpha) &= x_1^{n-1}(\alpha) \\ x_2^n(1) &= 1. \end{aligned}$$

Define $f : (a,b) \to \mathbb{R}$ by $||f||_\infty := \sup_{x\in(a,b)} |f(x)|$.

Theorem

Suppose $0 < a \le M(x) \le \tilde{A}$. The overlapping ($\beta > \alpha$) parallel Schwarz iteration converges for any initial guess $x_1^0(\alpha)$, $x_2^0(\beta)$, and we have the linear convergence estimates

$$\begin{aligned} ||x - x_1^{2n+1}||_\infty &\le \rho^n \frac{\tilde{A}}{a^2}|x(\beta) - x_2^0(\beta)|, \\ ||x - x_2^{2n+1}||_\infty &\le \rho^n \frac{\tilde{A}}{a^2}|x(\alpha) - x_2^0(\alpha)|, \end{aligned}$$

with contraction factor $\rho := \frac{\alpha}{\beta}\frac{1-\beta}{1-\alpha} < 1$.

Proof: Two approaches

- Write the subdomain solutions $x^n_{1,2}$ implicitly and show $x^n_{1,2} \to x$
- Define the subdomain errors

$$e^n_{1,2} = \int_{x^n_{1,2}}^{x} M(\tilde{x})\, d\tilde{x}.$$

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$$e^n_{1,2} = \int_{x^n_{1,2}}^{x} M(\tilde{x})\, d\tilde{x}.$$

Technique one

The solutions $x^n_{1,2}(\xi)$ are given implicitly as

$$\int_0^{x_1^n(\xi)} M(\tilde{x})\, d\tilde{x} = \frac{\xi}{\beta} \int_0^{x_2^{n-1}(\beta)} M(\tilde{x})\, d\tilde{x} \tag{1}$$

and

$$\int_{x_2^n(\xi)}^1 M(\tilde{x})\, d\tilde{x} = \frac{1-\xi}{1-\alpha} \int_{x_1^{n-1}(\alpha)}^1 M(\tilde{x})\, d\tilde{x}. \tag{2}$$

This gives the iteration

$$\int_0^{x_1^n(\alpha)} M(\tilde{x})\, d\tilde{x} = \frac{\alpha}{\beta} \int_0^{x_2^{n-1}(\beta)} M(\tilde{x})\, d\tilde{x}$$

$$= \frac{\alpha}{\beta} \frac{\beta - 1}{\alpha - 1} \left(\int_0^{x_1^{n-2}(\alpha)} M(\tilde{x})\, d\tilde{x} - \int_0^1 M(\tilde{x})\, d\tilde{x} \right) + \frac{\alpha}{\beta} \int_0^1 M(\tilde{x})\, d\tilde{x}.$$

If we let

$$K_1^n = \int_0^{x_1^n(\alpha)} M(\tilde{x})\,d\tilde{x} \text{ and } C = \int_0^1 M(\tilde{x})\,d\tilde{x}$$

then we have a fixed point iteration of the form

$$K_1^{n} = F_1(K_1^{n-2}) \equiv \frac{\alpha}{\beta}\frac{\beta-1}{\alpha-1}\left(K_1^{n-2} - C\right) + \frac{\alpha}{\beta}C.$$

- If $\alpha < \beta$ then $K_1^n \to K_1^*$
- If $M \geq \alpha > 0$ then MVT $\Longrightarrow x_1^n(\alpha) \to x(\alpha)$, BVP well–posed implies $x_1^n(\xi) \to x(\xi)$.

Technique two

Define

$$e^n_{1,2}(\xi) = \int_{x^n_{1,2}(\xi)}^{x(\xi)} M(\tilde{x})\, d\tilde{x}$$

and use

$$M(x)x_\xi - M(x^n_1)x^n_{1,\xi} = \mathcal{C}$$

to obtain the BVP

$$\frac{de^n_1}{d\xi} = \mathcal{C},\, e^n_1(0) = 0,\, e^n_1(\beta) = e^{n-1}_2(\beta).$$

Solving directly we have

$$e^n_1(\xi) = \frac{\xi}{\beta} e^{n-1}_2(\beta).$$

Likewise

$$e^n_2(\xi) = \frac{\xi - 1}{\alpha - 1} e^{n-1}_1(\alpha).$$

Combining we have

$$e_1^n(\alpha) = \frac{\alpha}{\beta}\frac{1-\beta}{1-\alpha}e_1^{n-2}(\alpha)$$

and the contraction results if $\alpha < \beta$.

If we compute sequentially (Alternating Schwarz) we obtain the expected (improved) result

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Optimal Schwarz

We decompose $\xi \in [0, 1]$ into two non–overlapping subdomains $\Omega_1 = [0, \beta]$ and $\Omega_2 = [\beta, 1]$ and consider the iteration

$$\begin{aligned} (M(x_1^n)x_{1,\xi}^n)_\xi &= 0, \quad \xi \in \Omega_1 \\ x_1^n(0) &= 0 \\ B_1(x_1^n(\beta)) &= B_1(x_2^{n-1}(\beta)) \end{aligned} \qquad \begin{aligned} (M(x_2^n)x_{2,\xi}^n)_\xi &= 0, \quad \xi \in \Omega_2 \\ B_2(x_2^n(\beta)) &= B_2(x_1^{n-1}(\beta)) \\ x_2^n(1) &= 1, \end{aligned}$$

where the transmission operators B_1 and B_2 are given by

$$B_1(\cdot) \equiv M(\cdot)\partial_\xi(\cdot) - B(\cdot), \qquad B_2(\cdot) \equiv M(\cdot)\partial_\xi(\cdot) - \tilde{B}(\cdot)$$

and

$$B(\cdot) = \frac{1}{1-\beta}\int_{(\cdot)}^{1} M(\tilde{x})\,d\tilde{x}, \qquad \text{and} \qquad \tilde{B}(\cdot) = \frac{1}{\beta}\int_{0}^{(\cdot)} M(\tilde{x})\,d\tilde{x}.$$

Convergence is obtained in two iterations!!

Optimal Schwarz

We decompose $\xi \in [0,1]$ into two non–overlapping subdomains $\Omega_1 = [0,\beta]$ and $\Omega_2 = [\beta,1]$ and consider the iteration

$$
\begin{aligned}
&(M(x_1^n)x_{1,\xi}^n)_\xi = 0, \quad \xi \in \Omega_1 && (M(x_2^n)x_{2,\xi}^n)_\xi = 0, \quad \xi \in \Omega_2 \\
&x_1^n(0) = 0 && B_2(x_2^n(\beta)) = B_2(x_1^{n-1}(\beta)) \\
&B_1(x_1^n(\beta)) = B_1(x_2^{n-1}(\beta)) && x_2^n(1) = 1,
\end{aligned}
$$

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Convergence is obtained in two iterations!!

Optimized Schwarz

We decompose $\xi \in [0,1]$ into two non–overlapping subdomains $\Omega_1 = [0,\beta]$ and $\Omega_2 = [\beta, 1]$ and consider the iteration

$$(M(x_1^n)x_{1,\xi}^n)_\xi = 0,$$
$$x_1^n(0) = 0$$
$$M(x_1^n)\partial_\xi x_1^n + px_1^n\Big|_{\xi=\beta} = M(x_2^{n-1})\partial_\xi x_2^{n-1} + px_2^{n-1}\Big|_{\xi=\beta}$$

and

$$(M(x_2^n)x_{2,\xi}^n)_\xi = 0,$$
$$M(x_2^n)\partial_\xi x_2^n - px_2^n\Big|_{\xi=\beta} = M(x_1^n)\partial_\xi x_1^{n-1} - px_1^{n-1}\Big|_{\xi=\beta}$$
$$x_2^n(1) = 1,$$

where p is a constant to be chosen to improve the convergence rate.

Theorem

The subdomain solutions are given implicitly by the formulas

$$\int_0^{x_1^n(\xi)} M(\tilde{x})\, d\tilde{x} = R_1(x_1^n(\beta))\xi \;\&\; \int_0^{x_2^n(\xi)} M(\tilde{x})\, d\tilde{x} = R_2(x_2^n(\beta))(\xi-1)+C,$$

where $C = \int_0^1 M(\tilde{x})\, d\tilde{x}$. The operators R_1 and R_2 are given by

$$R_1(x) = \frac{1}{\beta}\int_0^x M\, d\tilde{x} \quad \text{and} \quad R_2(x) = \frac{1}{\beta - 1}\left(\int_0^x M\, d\tilde{x} - C\right).$$

And $x_1^{n+1}(\beta)$ and $x_2^n(\beta)$ satisfy

$$\begin{aligned} (pI - R_2)x_2^n(\beta) &= (pI - R_1)x_1^{n-1}(\beta), \\ (pI + R_1)x_1^{n+1}(\beta) &= (pI + R_2)x_2^n(\beta). \end{aligned}$$

Theorem

The optimized Schwarz iteration converges globally to the exact solution $x(\beta)$ for all $p > 0$.

Outline of Proof

- R_1 and $-R_2$ are continuous and uniformly monotonic (increasing) since

$$R_1'(x) = \frac{1}{\beta}M(x) \geq \frac{1}{\beta}a > 0 \quad \text{and} \quad -R_2'(x) = \frac{1}{1-\beta}M(x) \geq \frac{1}{1-\beta}a > 0$$

- $p > 0 \Longrightarrow pI - R_2$ and $pI + R_1$ are continuous and uniformly monotonic $\Longrightarrow$ invertible.
- $x_2^n(\beta)$ and $x_1^{n+1}(\beta)$ are well–defined.
-

$$x_1^{n+1}(\beta) = Gx_1^{n-1}(\beta)$$

where

$$G \equiv (pI + R_1)^{-1}(pI + R_2)(pI - R_2)^{-1}(pI - R_1).$$

G is a contraction for all $p > 0$.

Parallel Linearized Schwarz

$$\begin{aligned}
\frac{d}{d\xi}\left(M(x_1^{n-1})\frac{dx_1^n}{d\xi}\right) &= 0 \\
x_1^n(0) &= 0 \\
x_1^n(\beta) &= x_2^{n-1}(\beta)
\end{aligned}
\qquad
\begin{aligned}
\frac{d}{d\xi}\left(M(x_2^{n-1})\frac{dx_2^n}{d\xi}\right) &= 0 \\
x_2^n(\alpha) &= x_1^{n-1}(\alpha) \\
x_2^n(1) &= 1.
\end{aligned}$$

Lemma

The subdomain solutions are given by

$$x_1^n(\xi) = x_2^{n-1}(\beta)\frac{\int_0^\xi \frac{d\xi}{M(x_1^{n-1}(\xi))}}{\int_0^\beta \frac{d\xi}{M(x_1^{n-1}(\xi))}},$$

and

$$x_2^n(\xi) = x_1^{n-1}(\alpha) + (1 - x_1^{n-1}(\alpha))\frac{\int_\alpha^\xi \frac{d\xi}{M(x_2^{n-1}(\xi))}}{\int_\alpha^1 \frac{d\xi}{M(x_2^{n-1}(\xi))}}.$$

Theorem

The parallel linearized Schwarz iteration converges for any smooth initial guesses $x_1^0(\xi)$ and $x_2^0(\xi)$.

Proof

For any $\xi \in (0, \beta]$

$$x_1^n(\xi) = \mathcal{C}_\xi^n x_1^{n-2}(\alpha) + \mathcal{D}_\xi^n.$$

where

$$\mathcal{C}_\xi^n = \frac{\int_\beta^1 \frac{d\tilde{\xi}}{M(x_2^{n-2}(\tilde{\xi}))}}{\int_\alpha^1 \frac{d\tilde{\xi}}{M(x_2^{n-2}(\tilde{\xi}))}} \frac{\int_0^\xi \frac{d\tilde{\xi}}{M(x_1^{n-1}(\tilde{\xi}))}}{\int_0^\beta \frac{d\tilde{\xi}}{M(x_1^{n-1}(\tilde{\xi}))}}, \quad \mathcal{D}_\xi^n = \frac{\int_\alpha^\beta \frac{d\tilde{\xi}}{M(x_2^{n-2}(\tilde{\xi}))}}{\int_\alpha^1 \frac{d\tilde{\xi}}{M(x_2^{n-2}(\tilde{\xi}))}} \frac{\int_0^\xi \frac{d\tilde{\xi}}{M(x_1^{n-1}(\tilde{\xi}))}}{\int_0^\beta \frac{d\tilde{\xi}}{M(x_1^{n-1}(\tilde{\xi}))}}.$$

The quantities $\mathcal{C}_\xi^n$ and $\mathcal{D}_\xi^n$ satisfy

$$0 < \mathcal{C}_\xi^n \leq \rho_\xi < 1, \text{ and } 0 < \mathcal{D}_\xi^n \leq \gamma_\xi < 1,$$

where

$$\rho_\xi := \frac{1}{1 + \frac{a}{A}\frac{\beta-\alpha}{1-\beta}} \frac{1}{1 + \frac{a}{A}\frac{\beta-\xi}{\xi}} \quad \text{and} \quad \gamma_\xi := \frac{1}{1 + \frac{a}{A}\frac{\beta-\xi}{\xi}} \frac{1}{1 + \frac{a}{A}\frac{1-\beta}{\beta-\alpha}}.$$

Furthermore, these quantities are uniformly bounded

$$\rho_\xi \leq \frac{1}{1 + \frac{a}{A}\frac{\beta-\alpha}{1-\beta}} := \rho < 1 \quad \text{and} \quad \gamma_\xi \leq \frac{1}{1 + \frac{a}{A}\frac{1-\beta}{\beta-\alpha}} := \gamma < 1.$$

If n is even

$$x_1^n(\xi) = \mathcal{C}_\xi^n \prod_{k=1}^{\frac{n-2}{2}} \mathcal{C}_\alpha^{2k} x_1^0(\alpha) + \mathcal{D}_\xi^n + \mathcal{C}_\xi^n \sum_{k=1}^{\frac{n-2}{2}} \mathcal{D}_\alpha^{2k} \left(\prod_{l=k+1}^{\frac{n-2}{2}} \mathcal{C}_\alpha^{2l} \right).$$

- $\{x_1^n(\xi)\}$ and $\{x_2^n(\xi)\}$ converge uniformly to $\tilde{x}_1, \tilde{x}_2$.
- $\tilde{x}_1(\alpha) = \tilde{x}_2(\alpha)$ and $\tilde{x}_1(\beta) = \tilde{x}_2(\beta)$,
- $\Longrightarrow \tilde{x}_1 = \tilde{x}_2$ on $[\alpha, \beta]$.
- Define

$$\bar{x} = \begin{cases} \tilde{x}_1(\xi), & \xi \in [0, \beta], \\ \tilde{x}_1(\xi) = \tilde{x}_2(\xi), & [\alpha, \beta] \\ \tilde{x}_2(\xi), & [\beta, 1] \end{cases}$$

- By uniqueness $\bar{x}(\xi) = x(\xi)$.

Time Dependent Case

For a given $u(x,t)$ a time dependent mesh transformation may be found by solving

$$x_t = \frac{1}{\tau}(M(x)x_\xi)_\xi,$$

subject to boundary and initial conditions

$$x(0,t) = 0, \quad x(1,t) = 1, x(\xi,0) = x_0(\xi).$$

Discretize this problem in time implicitly and then solve the sequence of elliptic problems using a DD approach.
The exact solution at time step k satisfies

$$x^k - \frac{\Delta t}{\tau}(M(x^k)x^k_\xi)_\xi = x^{k-1}(\xi)$$
$$x^k(0) = 0 \qquad x^k(1) = 1.$$

Assuming two spatial domains for each time step ($k = 1, 2, \ldots$) we consider the iteration

$$x_1^{n,k} - \frac{\Delta t}{\tau}(M(x_1^{n,k})x_{1,\xi}^{n,k})_\xi = x^{k-1}(\xi) \quad x_2^{n,k} - \frac{\Delta t}{\tau}(M(x_2^{n,k})x_{2,\xi}^{n,k})_\xi = x^{k-1}(\xi)$$

$$x_1^{n,k}(0) = 0 \qquad x_2^{n,k}(\alpha) = x_1^{n-1,k}(\alpha)$$

$$x_1^{n,k}(\beta) = x_2^{n-1,k}(\beta) \qquad x_2^{n,k}(1) = 1$$

Subtracting the equation for $x_1^{n,k}$ from the equation for x^k we have

$$x^k - x_1^{n,k} - \frac{\Delta t}{\tau}\left(M(x^k)x_\xi^k - M(x_1^{n,k})x_{1,\xi}^{n,k}\right)_\xi = 0.$$

We define an error measure

$$e_{1,2}^{n,k} = \int_{x_{1,2}^{n,k}}^{x^k} M \, d\tilde{x} \Longrightarrow \frac{de_{1,2}^{n,k}}{d\xi} = M(x^k)\frac{dx^k}{d\xi} - M(x_{1,2}^{n,k})\frac{dx_{1,2}^{n,k}}{d\xi}.$$

The mean value theorem for integrals implies

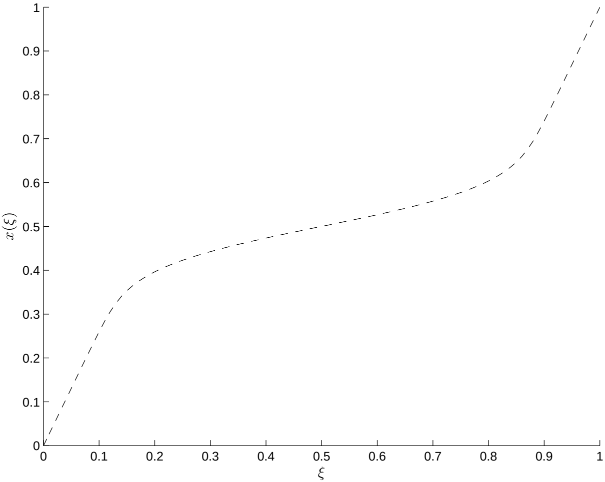
$$e_{1,2}^{n,k} = M(x^*)(x^k - x_{1,2}^{n,k}) \text{ for } x^* \text{ between } x^k \,\&\, x_{1,2}^{n,k}.$$

So on subdomain one we have

$$\frac{d^2 e_1^{n,k}}{d\xi^2} - \frac{\tau}{\Delta t}\frac{1}{M(x^*)} e_{1,2}^{n,k} = 0.$$

Since $M, \tau, \Delta t > 0$ the classical results for second order elliptic problems tells us that $e_1^{n,k}$ satisfies a maximum principle.

Numerical Example: steady mesh pde



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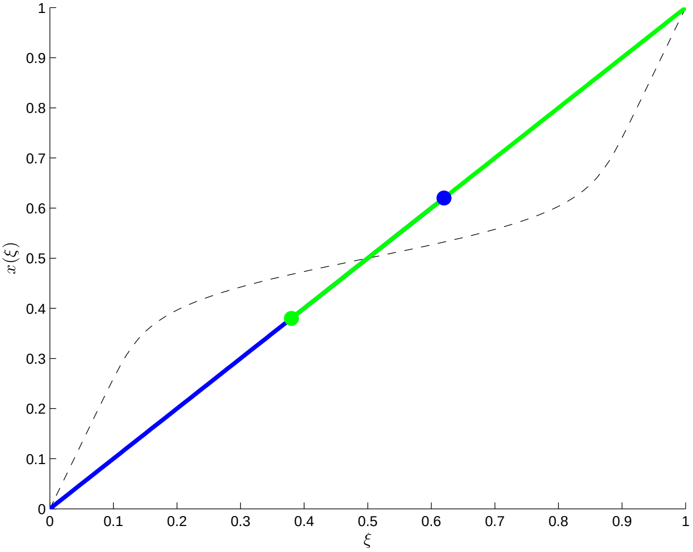

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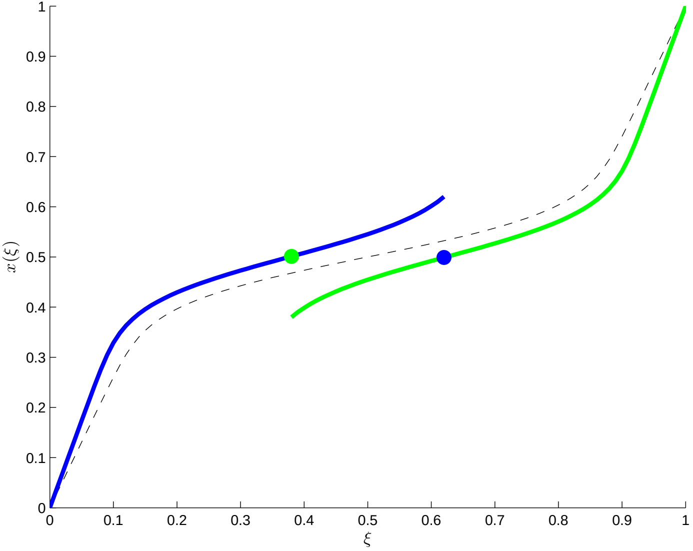

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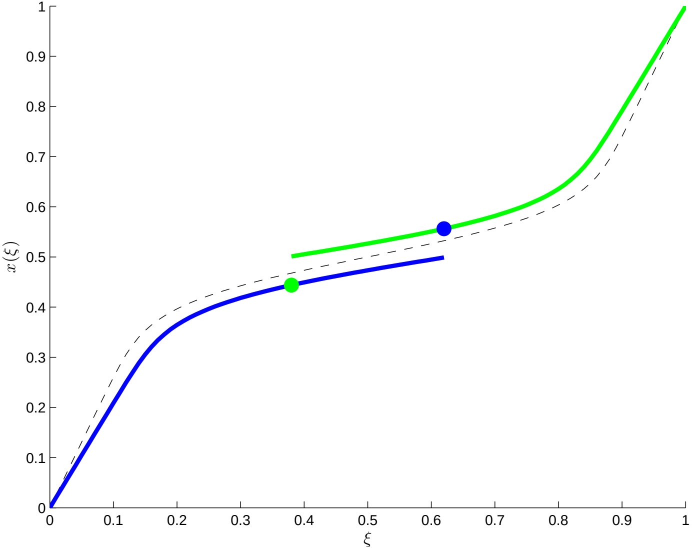

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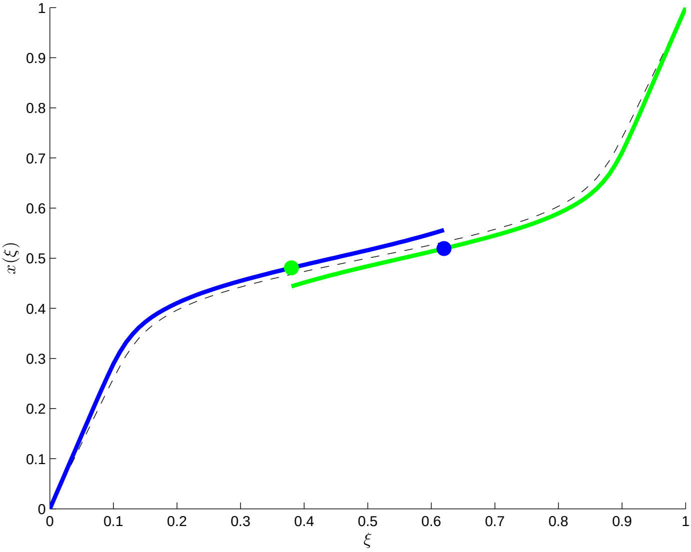

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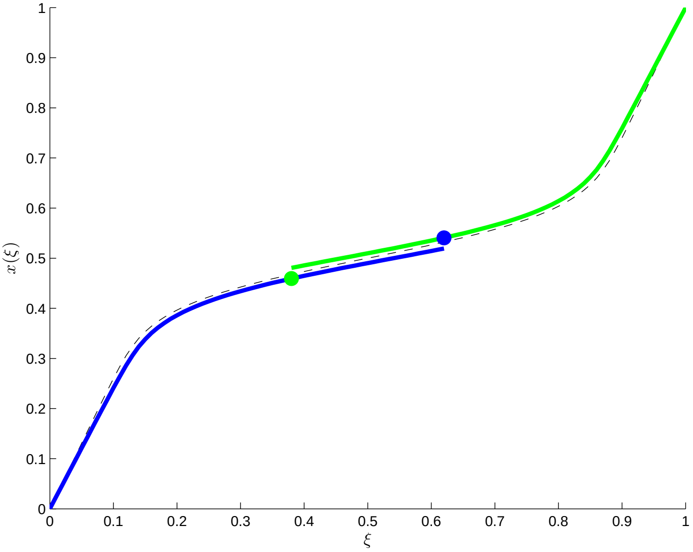

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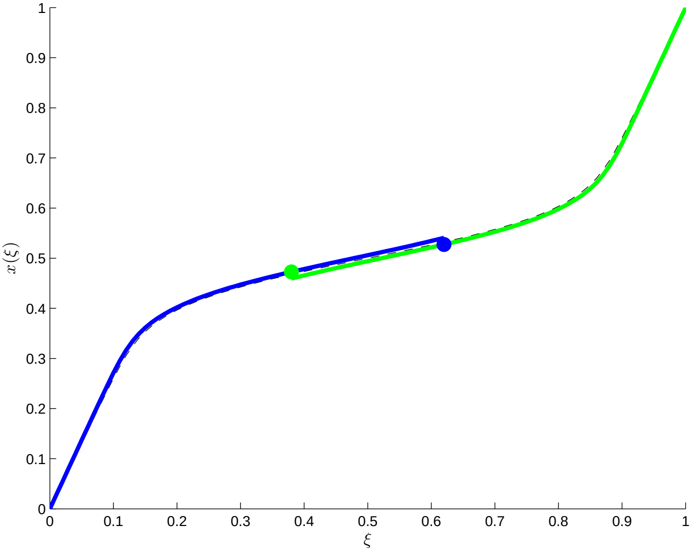


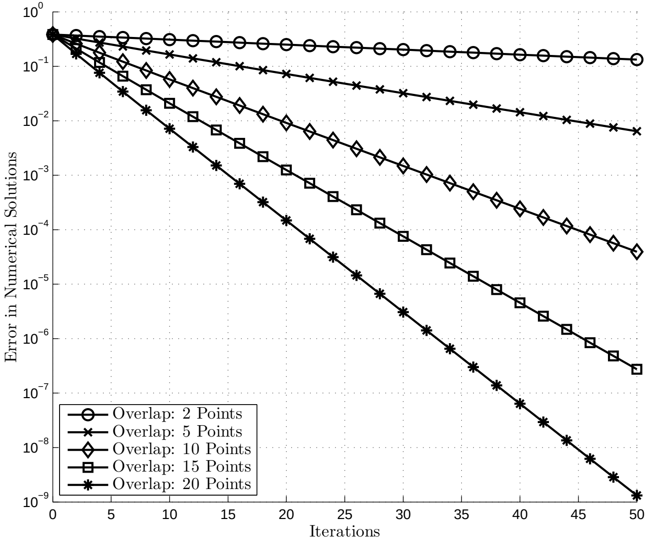


Figure: Convergence of grid generation by classical Schwarz with varying overlap for $u(x) = (1 - exp(20x))/(1 - exp(20))$.

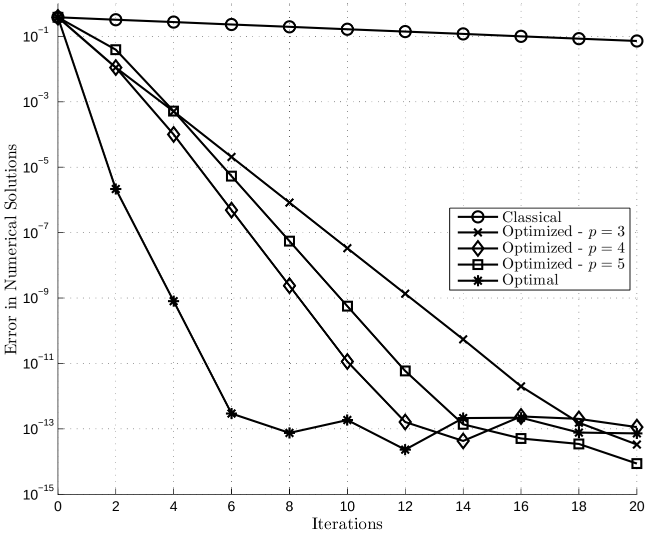


Figure: Grid generation problem for $u(x) = (1 - exp(20x))/(1 - exp(20))$.

Iterations	1	3	5	7	9	11	∞
Nonlinear Parallel	0.3625	0.0520 (5)	0.0498 (10)	0.0478 (15)	0.0462 (21)	0.0448 (27)	0.0366
Linearized Parallel	0.3625	0.1291 (3)	0.1006 (5)	0.0571 (7)	0.0479 (9)	0.0471 (11)	0.0366
Optimized	0.3625	0.1402 (9)	0.0367 (23)	0.0366 (30)	0.0366 (36)	0.0366 (40)	0.0366
Optimal	0.3625	0.0367 (12)	0.0366 (19)	0.0366 (24)	0.0366 (27)	0.0366 (29)	0.0366

Table: Interpolation errors.

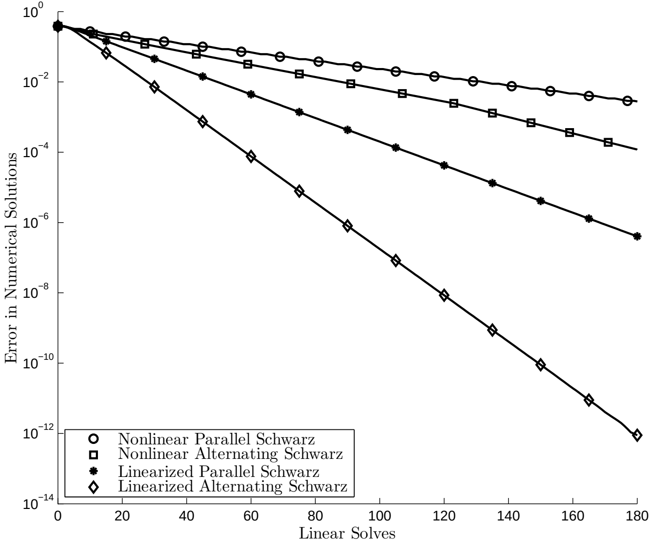
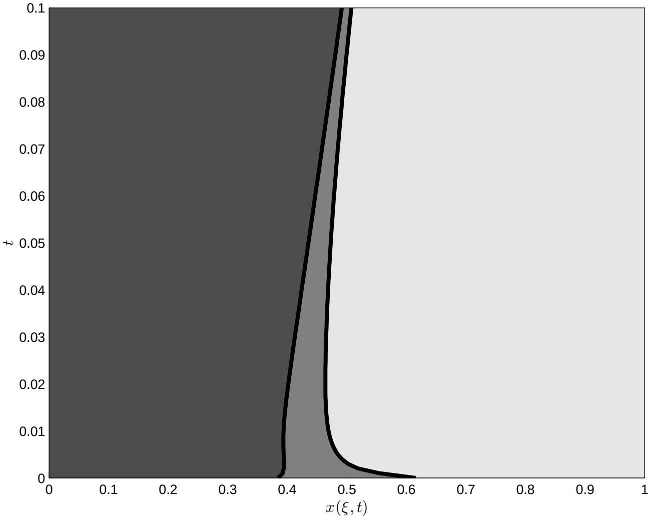
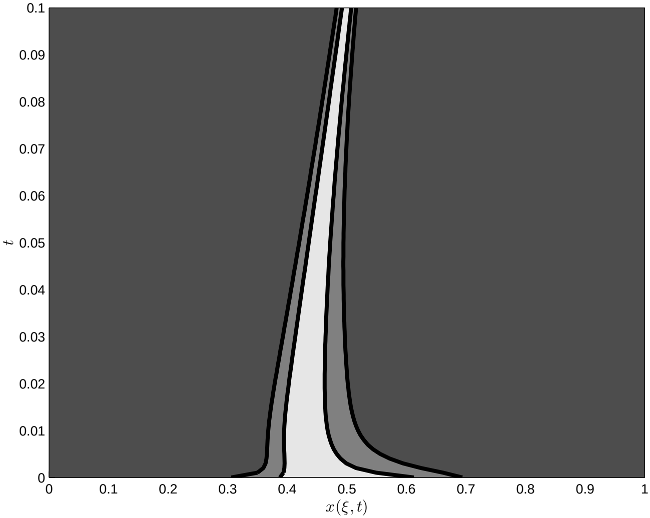


Figure: Grid generation problem for
$u(x) = u(x) = (1 - exp(20x))/(1 - exp(20))$.

2 moving subdomains!



3 moving subdomains



Two dimensions

Determine equidistributing mesh for

$$u(x, y) = \left(1 - e^{10(x-1)}\right) \sin(\pi y), \quad (x, y) \in [0, 1] \times [0, 1].$$

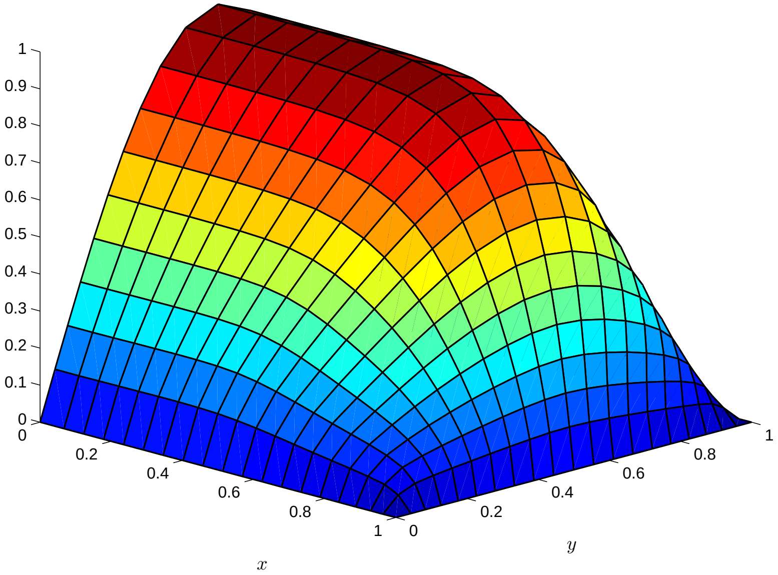


Figure: Surface plot of the function $u(x, y)$ over the equidistributed mesh.

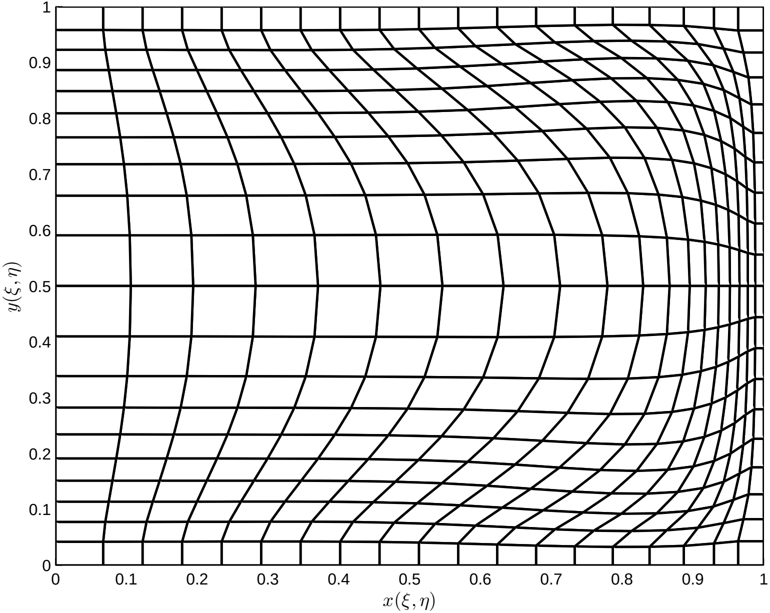


Figure: Single domain equidistributed mesh.

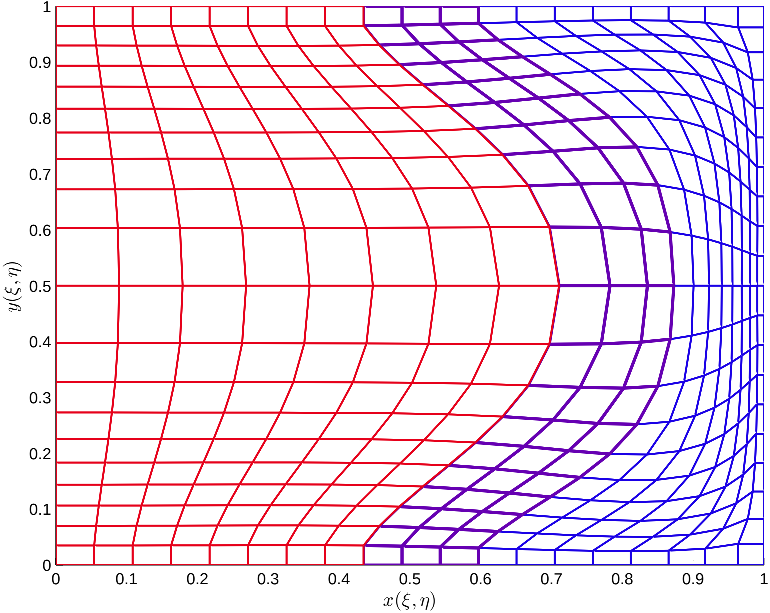


Figure: Equidistributed mesh obtained by DD.

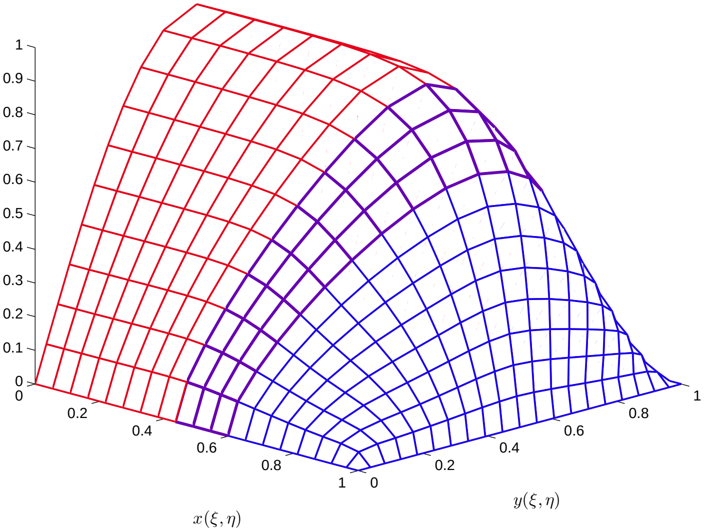


Figure: $u(x, y)$ on DD mesh.

Summary

- Presented and analyzed a parallel DD framework for grid generation via equidistribution in 1D.
- 2D numerics – analysis in progress.
- Theoretical and practical assessment of the many possible flavours of DD for solution of PDEs on equidistributing grids is in progress.

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