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# Adaptive Filon methods for the computation of highly oscillatory integrals

Marnix Van Daele, Veerle Ledoux

Department of Applied Mathematics and Computer Science Ghent University

SciCADE 2011

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# Oscillatory integrals

$$I[f] = \int_0^h f(x) e^{i\omega g(x)} dx$$

We focus on the particular case

$$I[f] = \int_0^h f(x) e^{\mathrm{i}\omega x} dx$$

If the integrand oscillates rapidly, and unless we use a huge number of function evaluations, the classical  $\nu$ -point Gauss rule is useless.

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Conclusions

# Oscillatory integrals

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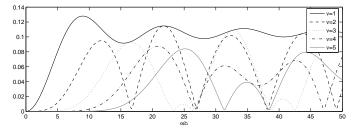
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Gauss rule applied to oscillatory integrands Example : f(x) = exp(x) and h = 1/10

$$\int_0^h e^x e^{i\omega x} dx = \frac{-1 + e^{h(1+i\omega)}}{1 + i\omega}$$



The absolute error in Gauss-Legendre quadrature for different values of the characteristic frequency  $\psi = \omega h$ .

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## Asymptotic expansion

$$I[f] = \int_{a}^{b} f(x)e^{i\omega x} dx$$
  
=  $\frac{1}{i\omega} \left( f(b) e^{i\omega b} - f(a) e^{i\omega a} \right) - \frac{1}{i\omega} I[f']$   
=  $\frac{1}{i\omega} \left( f(b) e^{i\omega b} - f(a) e^{i\omega a} \right)$   
 $- \frac{1}{(i\omega)^{2}} \left( f'(b) e^{i\omega b} - f'(a) e^{i\omega a} \right) + \frac{1}{(i\omega)^{2}} I[f'']$ 

$$I[f] = -\sum_{m=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{m+1}} \left[ e^{\mathrm{i}\omega b} f^{(m)}(b) - e^{\mathrm{i}\omega a} f^{(m)}(a) \right]$$

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$$I[f] = -\sum_{m=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{m+1}} \left[ e^{\mathrm{i}\omega b} f^{(m)}(b) - e^{\mathrm{i}\omega a} f^{(m)}(a) \right]$$

$$= \frac{1}{i\omega} \left( I(b) e^{i\omega b} - I(a) e^{i\omega a} \right) - \frac{1}{i\omega} I[I]$$
  
$$= \frac{1}{i\omega} \left( f(b) e^{i\omega b} - f(a) e^{i\omega a} \right)$$
  
$$- \frac{1}{(i\omega)^2} \left( f'(b) e^{i\omega b} - f'(a) e^{i\omega a} \right) + \frac{1}{(i\omega)^2} I[f'']$$

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## Asymptotic rules

$$I[f] = \int_a^b f(x) e^{i\omega x} dx$$

$$I[f] = -\sum_{m=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{m+1}} \left[ e^{\mathrm{i}\omega b} f^{(m)}(b) - e^{\mathrm{i}\omega a} f^{(m)}(a) \right]$$

$$\mathsf{Q}^{A}_{\mathsf{s}}[f] = -\sum_{m=0}^{\mathsf{s}-1} rac{1}{(-\mathrm{i}\omega)^{m+1}} \left[ e^{\mathrm{i}\omega b} f^{(m)}(b) - e^{\mathrm{i}\omega a} f^{(m)}(a) 
ight]$$

$$\mathsf{Q}^{\mathcal{A}}_{s}[f] - \mathit{I}[f] \sim \mathit{O}(\omega^{-s-1}) \hspace{0.1in} \omega 
ightarrow +\infty$$

This asymptotic method is of asymptotic order s + 1. The asymptotic order gives us the rate at which the error decreases with increasing  $\omega$ .

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# **Exponential fitting**

M. VAN DAELE, G. VANDEN BERGHE AND H. VANDE VYVER, *Exponentially fitted quadrature rules of Gauss type for oscillatory integrands*, Appl. Numer. Math., 53 (2005), pp. 509–526.

How to compute

 $\int_{-1}^{1} F(t) dt$ 

whereby F(x) has an oscillatory behaviour with frequency  $\mu$ ?

$$I[f] = \int_0^h f(x) e^{i\omega x} dx = \frac{h}{2} e^{i\mu} \int_{-1}^1 f(h(t+1)/2) e^{i\mu t} dt \quad \mu = \frac{\omega h}{2}$$

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# **Exponential fitting**

$$\mathcal{L}[F; x; h; \mathbf{a}] = \int_{x-h}^{x+h} F(z) dz - h \sum_{k=1}^{\nu} w_k F(x + \hat{c}_k h), \quad \hat{c}_k \in [-1, 1]$$
(put  $x = \mathbf{0}$  and  $h = \mathbf{1}$  to obtain  $\int_{-1}^{1} F(t) dt$ )

 $\mathcal{L}[F; x; h; \mathbf{a}] = \mathbf{0}$  for a reference set of  $K + \mathbf{2}(P + \mathbf{1}) + \mathbf{1} = \mathbf{2}\nu$  functions

$$1, t, t^2, ...t^K,$$

 $\exp(\pm i\mu t), t \exp(\pm i\mu t), t^2 \exp(\pm i\mu t), \dots, t^P \exp(\pm i\mu t)$ 

In this talk we only consider the case K = -1,  $P = \nu - 1$ .

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1-node EF rule

$$\int_{-1}^{1} F(x) dx \approx w_1 F(\hat{c}_1)$$

 $\int_{-1}^{1} \exp(\pm i\mu x) dx - w_1 \exp(\pm i\hat{c}_1 \mu) = \mathbf{0}$ 

$$w_1 = 2\sin(\mu)/\mu$$
  $\hat{c}_1 = 0$ 

$$I[f] = \int_0^h f(x) \exp(i\omega x) dx = \int_0^h F(x) dx$$

$$Q_1^{EF}[F] = \frac{h\sin(\mu)}{\mu}F(h/2) = \frac{e^{ih\omega} - 1}{i\omega}f(h/2) \quad \mu = \omega h/2$$

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## 1-node EF rule

$$\int_{-1}^{1} F(x) dx \approx w_1 F(\hat{c}_1)$$

 $\int_{-1}^{1} \exp(\pm i\mu x) dx - w_1 \exp(\pm i \hat{c}_1 \mu) = \mathbf{0}$ 

$$w_1 = 2\sin(\mu)/\mu$$
  $\hat{c}_1 = 0$ 

$$I[f] = \int_0^h f(x) \exp(i\omega x) dx = \int_0^h F(x) dx$$

$$Q_1^{EF}[F] = \frac{h\sin(\mu)}{\mu}F(h/2) = \frac{e^{ih\omega} - 1}{i\omega}f(h/2) \quad \mu = \omega h/2$$

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# 1-node EF rule

$$\int_{-1}^{1} F(x) dx \approx w_1 F(\hat{c}_1)$$

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$$I[f] = \int_{0}^{h} f(x) \exp(i\omega x) dx = \int_{0}^{h} F(x) dx$$

$$\mathsf{Q}_{\mathbf{1}}^{\textit{EF}}[\textit{F}] = \frac{h\sin(\mu)}{\mu}\textit{F}(h/\mathbf{2}) = \frac{\mathrm{e}^{\mathrm{i}h\omega} - \mathbf{1}}{\mathrm{i}\omega}f(h/\mathbf{2}) \quad \mu = \omega h/\mathbf{2}$$

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## 2-node EF rule

$$\int_{-1}^{1} F(x) dx \approx w_1 F(\hat{c}_1) + w_2 F(\hat{c}_2)$$

 $\begin{cases} \int_{-1}^{1} \exp(\pm i\mu x) dx - w_1 \exp(\pm i \ \hat{c}_1 \ \mu) - w_2 \exp(\pm i \ \hat{c}_2 \ \mu) = \mathbf{0} \\ \int_{-1}^{1} x \exp(\pm i\mu x) dx - w_1 \ \hat{c}_1 \exp(\pm i \ \hat{c}_1 \ \mu) - w_2 \ \hat{c}_2 \exp(\pm i \ \hat{c}_2 \ \mu) = \mathbf{0} \end{cases}$ 

Assuming  $w_1 = w_2$  and  $\hat{c}_1 = -\hat{c}_2$ :

 $\iff \begin{cases} w_2 \mu \cos(\mu \hat{c}_2) - \sin(\mu) = \mathbf{0} \\ w_2 \hat{c}_2 \mu^2 \sin(\mu \hat{c}_2) - \sin(\mu) + \mu \cos(\mu) = \mathbf{0} \end{cases}$  $Q_2^{EF}[F] = \frac{h}{2} w_2 \left[ F\left(\frac{h(1 + \hat{c}_2)}{2}\right) + F\left(\frac{h(1 - \hat{c}_2)}{2}\right) \right] \qquad \mu = \frac{\omega h}{2} \end{cases}$ 

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## 2-node EF rule

$$\int_{-1}^{1} F(x) dx \approx w_1 F(\hat{c}_1) + w_2 F(\hat{c}_2)$$

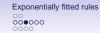
 $\begin{cases} \int_{-1}^{1} \exp(\pm i\mu x) dx - w_1 \exp(\pm i \ \hat{c}_1 \ \mu) - w_2 \exp(\pm i \ \hat{c}_2 \ \mu) = \mathbf{0} \\ \int_{-1}^{1} x \exp(\pm i\mu x) dx - w_1 \ \hat{c}_1 \exp(\pm i \ \hat{c}_1 \ \mu) - w_2 \ \hat{c}_2 \exp(\pm i \ \hat{c}_2 \ \mu) = \mathbf{0} \end{cases}$ 

Assuming  $w_1 = w_2$  and  $\hat{c}_1 = -\hat{c}_2$ :

$$\iff \begin{cases} w_2 \mu \cos(\mu \hat{c}_2) - \sin(\mu) = \mathbf{0} \\ w_2 \hat{c}_2 \mu^2 \sin(\mu \hat{c}_2) - \sin(\mu) + \mu \cos(\mu) = \mathbf{0} \end{cases}$$
$$w_2 \left[ F\left(\frac{h(\mathbf{1} + \hat{c}_2)}{2}\right) + F\left(\frac{h(\mathbf{1} - \hat{c}_2)}{2}\right) \right] \qquad \mu = \frac{\omega h}{2} \end{cases}$$

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#### 2-node EF rule

 $\begin{cases} w_2 \mu \cos(\mu \hat{c}_2) - \sin(\mu) = \mathbf{0} \\ w_2 \hat{c}_2 \mu^2 \sin(\mu \hat{c}_2) - \sin(\mu) + \mu \cos(\mu) = \mathbf{0} \end{cases}$ If  $\cos(\mu \hat{c}_2) \neq \mathbf{0}$  then  $w_2 = \sin \mu / (\mu \cos(\mu \hat{c}_2))$ 

 $m{G}(\hat{m{c}}_{2}):=(m{sin}\,\mu-\mu\,m{cos}\,\mu)\,m{cos}(\mu\hat{m{c}}_{2})-\mu\hat{m{c}}_{2}\,m{sin}\,\mu\,m{sin}(\mu\hat{m{c}}_{2})=m{0}$ 

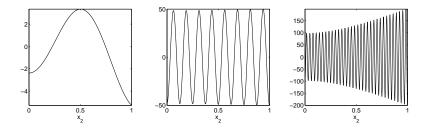


Figure:  $G(x_2)$  for  $\mu = 5$ ,  $\mu = 50$  and  $\mu = 200$ .

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#### 2-node EF rule

 $\begin{cases} w_2 \mu \cos(\mu \hat{c}_2) - \sin(\mu) = 0\\ w_2 \hat{c}_2 \mu^2 \sin(\mu \hat{c}_2) - \sin(\mu) + \mu \cos(\mu) = 0 \end{cases}$ If  $\cos(\mu \hat{c}_2) \neq 0$  then  $w_2 = \sin \mu / (\mu \cos(\mu \hat{c}_2))$  $G(\hat{c}_2) := (\sin \mu - \mu \cos \mu) \cos(\mu \hat{c}_2) - \mu \hat{c}_2 \sin \mu \sin(\mu \hat{c}_2) = 0$ 

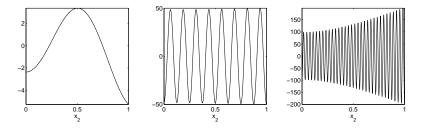


Figure:  $G(x_2)$  for  $\mu = 5$ ,  $\mu = 50$  and  $\mu = 200$ .

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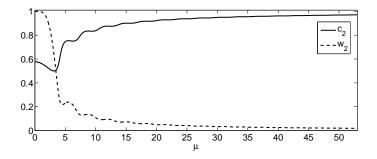


Figure: The  $\hat{c}_2(\mu)$  and  $w_2(\mu)$  curve for the EF method with  $\nu = 2$ .

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#### 3-node EF rule

$$\hat{c}_1 = -\hat{c}_3$$
  $\hat{c}_2 = 0$   $w_1 = w_3$ 

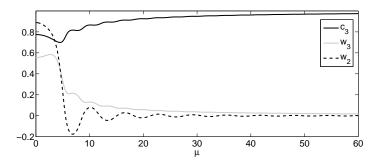


Figure: The  $\hat{c}_3(\mu)$ ,  $w_1(\mu) = w_3(\mu)$  and  $w_2(\mu)$  curves for the  $\nu = 3$  EF rule

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# 4-node EF rule

$$\hat{c}_1 = -\hat{c}_4$$
  $\hat{c}_2 = -\hat{c}_3$   $w_1 = w_4$   $w_2 = w_3$ 

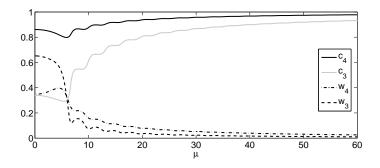
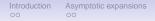


Figure: Nodes and weights of the EF rule with  $\nu = 4$  quadrature nodes.

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# Accuracy of EF rules

All EF rules reduce to the classical  $\nu$ -point Gauss(-Legendre) method in the limiting case  $\mu = 0$ .

Thus for small  $\mu$  :  $O(h^{2\nu+1})$ What about the accuracy for larger values of  $\mu = \omega h/2$ ?

J. P. COLEMAN AND L. GR. IXARU, *Truncation errors in exponential fitting for oscillatory problems*, SIAM. J. Numer. Anal., 44 (2006), pp. 1441–1465.

for large  $\mu$  :  $O(\mu^{ar{
u}u})$  with  $ar{
u} = \lfloor (
u - 1)/2 
floor$ 

 $\nu = 1: O(\omega^{-1})$   $\nu = 2, 3: O(\omega^{-2})$   $\nu = 4, 5: O(\omega^{-3})$ 

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for large  $\mu$  :  $O(\mu^{\bar{\nu}-\nu})$  with  $\bar{\nu} = \lfloor (\nu-1)/2 \rfloor$ 

 $\nu = 1: O(\omega^{-1})$   $\nu = 2, 3: O(\omega^{-2})$   $\nu = 4, 5: O(\omega^{-3})$ 

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for large  $\mu$  :  $O(\mu^{\bar{\nu}-\nu})$  with  $\bar{\nu} = \lfloor (\nu-1)/2 \rfloor$ 

$$u = 1: O(\omega^{-1})$$
  $u = 2, 3: O(\omega^{-2})$   $u = 4, 5: O(\omega^{-3})$ 

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$$\int_{-1}^{1} F(t) dt \approx \int_{-1}^{1} \bar{F}(t) dt$$

 $\bar{F}(t) \in \text{span}\{\exp(\pm i\mu t), t \exp(\pm i\mu t), t^2 \exp(\pm i\mu t), \dots, t^P \exp(\pm i\mu t)\}$ 

$$I[f] = \int_{0}^{h} f(x)e^{j\omega x} dx = \frac{h}{2}e^{j\frac{\omega h}{2}} \int_{-1}^{1} f(\frac{h}{2}(t+1))e^{j\frac{\omega h}{2}t} dt$$
  
If  $\frac{\omega h}{2} = \mu$  then  $I[f] \approx I[\bar{f}]$  with  $\bar{f}(x) \in \text{span}\{1, x, x^{2}, \dots, x^{\nu-1}\}$   
 $Q_{\nu}^{EF}[f] - I[f] = I[\bar{f}] - I[f] = I[\nu]$   $\nu(x) := \bar{f}(x) - f(x)$ 

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 $\bar{F}(t) \in \operatorname{span}\{\exp(\pm i\mu t), t \exp(\pm i\mu t), t^2 \exp(\pm i\mu t), \dots, t^P \exp(\pm i\mu t)\}$ 

$$I[f] = \int_0^h f(x)e^{i\omega x} dx = \frac{h}{2}e^{i\frac{\omega h}{2}}\int_{-1}^1 f(\frac{h}{2}(t+1))e^{i\frac{\omega h}{2}t} dt$$

$$h^2 = \mu \text{ then } I[f] \approx I[\overline{f}] \text{ with } \overline{f}(x) \in \text{span}\{1, x, x^2, \dots, x^{\nu-1}\}$$

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$$\int_{-1}^{1} F(t) dt \approx \int_{-1}^{1} \bar{F}(t) dt$$

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 $Q_{\nu}^{EF}[f] - I[f] = I[\bar{f}] - I[f] = I[\nu]$   $\nu(x) := \bar{f}(x) - f(x)$ 

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$$\int_{-1}^{1} F(t) dt \approx \int_{-1}^{1} \bar{F}(t) dt$$

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$$I[f] = \int_{0}^{h} f(x)e^{i\omega x} dx = \frac{h}{2}e^{i\frac{\omega h}{2}} \int_{-1}^{1} f(\frac{h}{2}(t+1))e^{i\frac{\omega h}{2}t} dt$$
  
If  $\frac{\omega h}{2} = \mu$  then  $I[f] \approx I[\bar{f}]$  with  $\bar{f}(x) \in \text{span}\{1, x, x^{2}, \dots, x^{\nu-1}\}$   
 $Q_{\nu}^{EF}[f] - I[f] = I[\bar{f}] - I[f] = I[\nu]$   $\nu(x) := \bar{f}(x) - f(x)$ 

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$$c_j = a + \lambda_j / \omega$$
  $c_{\nu-j+1} = b - \lambda_j / \omega$   $j = 1, \dots, \nu/2$ 

$$v(x) = \frac{f^{(\nu)}(\xi(x))}{\nu!} \prod_{i=1}^{\nu} (x - c_i)$$

$$v(x) = s(x) \prod_{i=1}^{\nu/2} (x - b + \lambda_i/\omega)$$
  $s(x) = \frac{f^{(\nu)}(\xi(x))}{\nu!} \prod_{j=1}^{\nu/2} (x - a - \lambda_j/\omega)$ 

$$\mathbf{v}(\mathbf{b}) = \mathbf{s}(\mathbf{b}) \prod_{i=1}^{\nu/2} (\lambda_i/\omega) = \mathbf{O}(\omega^{-\nu/2})$$

$$v'(b) = s(b)\omega^{-\nu/2+1} \sum_{k=1}^{\nu/2} \prod_{i \neq k} \lambda_i + O(\omega^{-\nu/2}) = O(\omega^{-\nu/2+1})$$

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$$v(x) = \frac{f^{(\nu)}(\xi(x))}{\nu!} \prod_{i=1}^{\nu} (x - c_i)$$

$$V(x) = s(x) \prod_{i=1}^{\nu/2} (x - b + \lambda_i/\omega)$$
  $s(x) = \frac{f^{(\nu)}(\xi(x))}{\nu!} \prod_{j=1}^{\nu/2} (x - a - \lambda_j/\omega)$ 

$$\mathbf{v}(\mathbf{b}) = \mathbf{s}(\mathbf{b}) \prod_{i=1}^{\nu/2} (\lambda_i/\omega) = \mathbf{O}(\omega^{-\nu/2})$$

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$$c_j = a + \lambda_j / \omega$$
  $c_{\nu-j+1} = b - \lambda_j / \omega$   $j = 1, \dots, \nu/2$ 

$$v(x) = \frac{f^{(\nu)}(\xi(x))}{\nu!} \prod_{i=1}^{\nu} (x - c_i)$$

$$v(x) = s(x) \prod_{i=1}^{\nu/2} (x - b + \lambda_i/\omega)$$
  $s(x) = \frac{f^{(\nu)}(\xi(x))}{\nu!} \prod_{j=1}^{\nu/2} (x - a - \lambda_j/\omega)$ 

$$\mathbf{v}(\mathbf{b}) = \mathbf{s}(\mathbf{b}) \prod_{i=1}^{\nu/2} (\lambda_i / \omega) = \mathbf{O}(\omega^{-\nu/2})$$

$$v'(b) = s(b)\omega^{-\nu/2+1} \sum_{k=1}^{\nu/2} \prod_{i \neq k} \lambda_i + O(\omega^{-\nu/2}) = O(\omega^{-\nu/2+1})$$

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Suppose  $\nu$  is even and  $a < c_1 < c_2 < \ldots < c_{\nu} < b$ 

$$c_j = a + \lambda_j / \omega$$
  $c_{\nu-j+1} = b - \lambda_j / \omega$   $j = 1, \dots, \nu/2$ 

$$v(x) = rac{f^{(
u)}(\xi(x))}{
u!} \prod_{i=1}^{
u} (x - c_i)$$

$$v(x) = s(x) \prod_{i=1}^{\nu/2} (x - b + \lambda_i/\omega)$$
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$$v(b) = s(b) \prod_{i=1}^{\nu/2} (\lambda_i/\omega) = O(\omega^{-\nu/2})$$
$$V(b) = s(b) \omega^{-\nu/2+1} \sum_{i=1}^{\nu/2} \prod \lambda_i + O(\omega^{-\nu/2}) = O(\omega^{-\nu/2+1})$$

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 $k=1 i \neq k$ 

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 $v(b) = O(\omega^{-\nu/2})$   $v'(b) = O(\omega^{-\nu/2+1})$  $v^{(n)}(b) = O(\omega^{-\nu/2+n}), n = 0, 1, ..., \nu/2 - 1$ 

 $v^{(n)}(a) = O(\omega^{-\nu/2+n}), \ n = 0, 1, \dots, \nu/2 - 1$ 

 $\begin{aligned} Q_{\nu}^{EF}[f] - I[f] &= I[\nu] \\ &= -\sum_{m=0}^{\infty} \frac{1}{(-i\omega)^{m+1}} \left[ e^{i\omega b} \nu^{(m)}(b) - e^{i\omega a} \nu^{(m)}(a) \right] \\ &= -\sum_{m=0}^{\nu/2-1} \frac{1}{(-i\omega)^{m+1}} O(\omega^{-\nu/2+m}) + O(\omega^{-\nu/2-1}) \\ &= O(\omega^{-\nu/2-1}) = O(\omega^{\lfloor (\nu-1)/2 \rfloor - \nu}) & \text{ for all } \nu \in \mathbb{R}$ 

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 $v(b) = O(\omega^{-\nu/2}) \qquad v'(b) = O(\omega^{-\nu/2+1})$  $v^{(n)}(b) = O(\omega^{-\nu/2+n}), \ n = 0, 1, \dots, \nu/2 - 1$ 

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### Filon-type

L. N. G FILON, On a quadrature formula for trigonometric integrals, Proc. Royal Soc. Edinburgh, 49 (1928), pp. 38–47.

Interpolate only the function f(x) at  $c_1 h, \ldots, c_{\nu} h$  by a polynomial  $\overline{f}(x)$ 

$$I[f] \approx Q_{\nu}^{F}[f] = \int_{0}^{h} \overline{f}(x) e^{i\omega x} dx = h \sum_{l=1}^{\nu} b_{l}(ih\omega) f(c_{l}h)$$

$$b_l(\mathrm{i}\hbar\omega) = \int_0^1 \ell_l(x) e^{\mathrm{i}\hbar\omega x} dx$$

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 $\ell_I$  is the *I*th cardinal polynomial of Lagrangian interpolation.



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## 1-node Filon-type rule

$$I[f] = \int_0^h F(x) dx = \int_0^h f(x) \exp(i\omega x) dx$$
$$Q_1^F[f] = \frac{\exp(ih\omega) - 1}{i\omega} f(c_1 h)$$

$$Q_1^{EF}[F] = \frac{e^{ih\omega} - 1}{i\omega} f(h/2)$$
$$Q_1^F[f] = Q_1^{EF}[F] \text{ iff } c_1 = \frac{1}{2}$$

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#### 2-node Filon-type rule

$$I[f] = \int_0^h F(x) dx = \int_0^h f(x) \exp(i\omega x) dx$$

If f is interpolated at  $c_1 h$  and  $c_2 h$ , then

$$Q_{2}^{F}[f] = h\left[\left(\frac{i\left((e^{i\psi}-1)c_{2}-e^{i\psi}\right)}{(c_{1}-c_{2})\psi}+\frac{e^{i\psi}-1}{(c_{1}-c_{2})\psi^{2}}\right)f(c_{1}h)\right.\\\left.+\left(\frac{i\left((e^{i\psi}-1)c_{1}-e^{i\psi}\right)}{(c_{2}-c_{1})\psi}+\frac{e^{i\psi}-1}{(c_{2}-c_{1})\psi^{2}}\right)f(c_{2}h)\right]$$

 $Q_2^F[f] = Q_2^{EF}[F]$  iff the same nodes are used

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# Accuracy of Filon-type rules

A. ISERLES, On the numerical quadrature of highly-oscillating integrals. I. Fourier transforms, IMA J. Numer. Anal., 24 (2004), pp. 365–391.

For small  $\omega$ , a Filon-type quadrature method has an order as if  $\omega = 0$ .

Legendre nodes : order 2  $\nu$  Lobatto nodes : order 2  $\nu$  – 2 For large  $\omega$  :

$$egin{aligned} {\sf Q}^F_{
u}[f] - {\it I}[f] &\sim egin{cases} {\sf O}(\omega^{-1}) & {\it c}_1 > 0 ext{ or } {\it c}_
u < 1 \ {\sf O}(\omega^{-2}) & {\it c}_1 = 0, {\it c}_
u = 1 \end{aligned}$$

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$$Q_{\nu}^{F}[f] - I[f] = I[\bar{f}] - I[f] = I[v]$$
  
=  $-\sum_{m=0}^{\infty} \frac{1}{(-i\omega)^{m+1}} \left[ e^{i\omega h} v^{(m)}(h) - v^{(m)}(0) \right]$ 

If  $(c_1, c_\nu) = (0, 1)$  then v(h) = v(0) = 0 $\implies Q_\nu^F[f] - I[f] = O(\omega^{-2}).$ 

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## How to improve the accuracy of Filon-rules ?

 by using Hermite interpolation : asymptotic order p + 1 can be reached where p is the number of derivatives at the endpoints:

 $\overline{f}^{(l)}(h) = f^{(l)}(h), \overline{f}^{(l)}(0) = f^{(l)}(0), l = 0, \dots, p-1$ 

- by using adaptive Filon-type methods : allowing the interpolation points to depend on  $\omega$  (is discussed later)
- by using nodes in the complex plane (=method of steepest descent)

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 $\mathsf{Q}_{\nu}^{\mathsf{F}}[f] - I[f] = O(\omega^{-p-1})$ 

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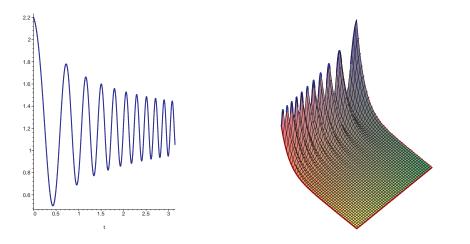
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### Method of steepest descent

D. HUYBRECHS AND S. VANDEWALLE, On the evaluation of highly oscillatory integrals by analytic continuation, SIAM J. Numer. Anal., 44 (2007) pp 1026–1048.



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#### Method of steepest descent

$$\int_{a}^{b} f(x)e^{i\omega x} dx$$

$$= e^{i\omega a} \int_{0}^{\infty} f(a+ip)e^{-\omega p} dp - e^{i\omega b} \int_{0}^{\infty} f(b+ip)e^{-\omega p} dp$$

$$= \frac{e^{i\omega a}}{\omega} \int_{0}^{\infty} f(a+i\frac{q}{\omega})e^{-q} dq - \frac{e^{i\omega b}}{\omega} \int_{0}^{\infty} f(b+i\frac{q}{\omega})e^{-q} dq$$

This leads to the numerical evaluation of the two resulting integrals with classical Gauss-Laguerre quadrature.

High asymptotic order is obtained : using  $\nu$  points for each integral, the error behaves as  $O(\omega^{-2\nu-1})$ .

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$$= \frac{e^{i\omega a}}{\omega} \int_{0}^{\infty} f(a+i\frac{q}{\omega})e^{-q} dq - \frac{e^{i\omega b}}{\omega} \int_{0}^{\infty} f(b+i\frac{q}{\omega})e^{-q} dq$$

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### Method of steepest descent

$$\int_{a}^{b} f(x)e^{i\omega x} dx$$

$$= e^{i\omega a} \int_{0}^{\infty} f(a+ip)e^{-\omega p} dp - e^{i\omega b} \int_{0}^{\infty} f(b+ip)e^{-\omega p} dp$$

$$= \frac{e^{i\omega a}}{\omega} \int_{0}^{\infty} f(a+i\frac{q}{\omega})e^{-q} dq - \frac{e^{i\omega b}}{\omega} \int_{0}^{\infty} f(b+i\frac{q}{\omega})e^{-q} dq$$

This leads to the numerical evaluation of the two resulting integrals with classical Gauss-Laguerre quadrature.

High asymptotic order is obtained : using  $\nu$  points for each integral, the error behaves as  $O(\omega^{-2\nu-1})$ .

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### Method of steepest descent

$$\int_{a}^{b} f(x)e^{i\omega x}dx$$
  
=  $\frac{e^{i\omega a}}{\omega}\int_{0}^{\infty} f(a+i\frac{q}{\omega})e^{-q}dq - \frac{e^{i\omega b}}{\omega}\int_{0}^{\infty} f(b+i\frac{q}{\omega})e^{-q}dq$ 

One ends up evaluating f at the points

$$a+\mathrm{i}rac{\mathbf{x}_{nj}}{\omega}, \ \mathrm{and} \ b+\mathrm{i}rac{\mathbf{x}_{nj}}{\omega}, \ j=1,...,n,$$

where  $x_{nj}$  are the *n* roots of the Laguerre polynomial of degree *n*.

This approach is equivalent to using a Filon rule with the same interpolation points.

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### Method of steepest descent

$$\int_{a}^{b} f(x)e^{i\omega x}dx$$
  
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, and  $b + i \frac{x_{nj}}{\omega}$ ,  $j = 1, ..., n$ ,

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Conclusions

# Adaptive Filon-type rules

Idea : combine best properties of EF and Filon quadrature

- EF
  - + accurate for small  $\omega$  *h* since the method reduces to Gauss-Legendre quadrature
  - + good results for large  $\omega$  *h* since the nodes tend to the endpoints (at a rate proportional to  $\omega^{-1}$ )
  - but : difficult to determine the nodes and weights for a given  $\omega h$  (iteration needed and ill-conditioned)
- Filon
  - + any set of nodes can be used
  - there is no optimal set of nodes for all  $\omega$  h
    - most accurate for small  $\omega$  *h* if the method is built on Legendre nodes
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# Adaptive Filon-type rules

Idea : create quadrature rules with  $\omega$ -dependent nodes that

- reduce to Legendre-nodes for small  $\omega$
- reduce to Lobatto-nodes for large  $\boldsymbol{\omega}$
- for given value of  $\omega$  are easy to compute

To do so, we introduce S-shaped functions.

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### Adaptive Filon-type methods

$$S(\psi; r; n) = rac{1 - rac{\psi^n - r^n}{1 + |\psi^n - r^n|}}{1 + rac{r^n}{1 + r^n}}$$

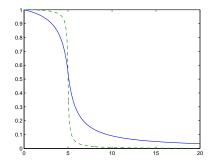


Figure: S(x, r, 1) and S(x, r, 2) (dashed) for r = 5 in [0, 20]

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# Adaptive Filon-type methods • $\nu = 2$ : $c_1(\psi) = \frac{3 - \sqrt{3}}{6} S(\psi; 2\pi; 1); c_2(\psi) = 1 - c_1(\psi)$

• 
$$\nu = 3$$
:  $c_1(\psi) = \frac{10 - \sqrt{15}}{5}S(\psi; 3\pi; 1); c_3(\psi) = 1 - c_1(\psi)$ 

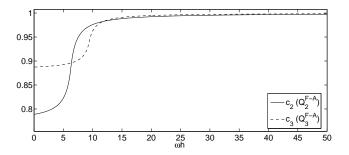


Figure:  $c_2(\psi)$  of the adaptive Filon method  $Q_2^{F-A}$  and  $c_3(\psi)$  of the adaptive Filon method  $Q_2^{F-A}$ . (日) (日) (日) (日) (日) (日) (日)

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Asymptotic analysis for  $Q_2^{F-A}$ 

 $\tilde{c}_1 = c_1 h = \sigma_1(\omega)$  and  $\tilde{c}_2 = c_2 h = h + \sigma_2(\omega)$  with  $\sigma_{1,2}(\omega) \sim \omega^{-1}$ 

 $\begin{array}{lll} v(x) &=& s_h(x)(x-h-\sigma_2) & s_h(x) = \frac{f''(\xi_h(x))}{2}(x-\sigma_1) \\ v'(x) &=& s_h(x) + s'_h(x)(x-h-\sigma_2) \\ v''(x) &=& 2s'_h(x) + s''_h(x)(x-h-\sigma_2) \\ &\vdots \end{array}$ 

 $v(h) = -s_h(h)\sigma_2$   $v'(h) = s_h(h) - s'_h(h)\sigma_2$  $v''(h) = 2s'_h(h) - s''_h(h)\sigma_2$ 

Similar results for the other endpoint.

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# Asymptotic analysis for $Q_2^{F-A}$

 $\tilde{c}_1 = c_1 h = \sigma_1(\omega)$  and  $\tilde{c}_2 = c_2 h = h + \sigma_2(\omega)$  with  $\sigma_{1,2}(\omega) \sim \omega^{-1}$ 

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# Asymptotic analysis for $Q_2^{F-A}$

$$\tilde{c}_1 = c_1 h = \sigma_1(\omega)$$
 and  $\tilde{c}_2 = c_2 h = h + \sigma_2(\omega)$  with  $\sigma_{1,2}(\omega) \sim \omega^{-1}$ 

$$\begin{aligned} \mathbf{v}(\mathbf{x}) &= s_h(\mathbf{x})(\mathbf{x} - \mathbf{h} - \sigma_2) & s_h(\mathbf{x}) = \frac{f''(\xi_h(\mathbf{x}))}{2}(\mathbf{x} - \sigma_1) \\ \mathbf{v}'(\mathbf{x}) &= s_h(\mathbf{x}) + s'_h(\mathbf{x})(\mathbf{x} - \mathbf{h} - \sigma_2) \\ \mathbf{v}''(\mathbf{x}) &= 2s'_h(\mathbf{x}) + s''_h(\mathbf{x})(\mathbf{x} - \mathbf{h} - \sigma_2) \\ &\vdots \end{aligned}$$

$$v(h) = -s_h(h)\sigma_2$$
  
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Similar results for the other endpoint.

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Asymptotic analysis for  $Q_2^{F-A}$ 

$$Q_2^{\mathcal{F}-\mathcal{A}}[f] - I[f] = I[v] \sim \sum_{m=0}^{\infty} \frac{1}{(-i\omega)^{m+1}} \left[ e^{i\omega h} v^{(m)}(h) - v^{(m)}(0) \right]$$

Reordering for  $s_h(h)$ ,  $s'_h(h)$ , ...

$$I[v] \sim s_h(h)e^{i\psi}\left[\frac{\sigma_2}{i\omega} - \frac{1}{\omega^2}\right] + s'_h(h)e^{i\psi}\left[\frac{\sigma_2}{\omega^2} + \frac{2}{i\omega^3}\right] + \dots + s_0(0)\left[\frac{\sigma_1}{i\omega} - \frac{1}{\omega^2}\right] + s'_0(0)\left[\frac{\sigma_1}{\omega^2} + \frac{2}{i\omega^3}\right] + \dots$$

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A complex adaptive Filon-rule :  $Q_2^{F-C}$ 

Are there better options than choosing  $\sigma_2 = -\sigma_1$ ?

$$I[v] \sim s_h(h)e^{i\psi}\left[\frac{\sigma_2}{i\omega} - \frac{1}{\omega^2}\right] + s'_h(h)e^{i\psi}\left[\frac{\sigma_2}{\omega^2} + \frac{2}{i\omega^3}\right] + \dots + s_0(0)\left[\frac{\sigma_1}{i\omega} - \frac{1}{\omega^2}\right] + s'_0(0)\left[\frac{\sigma_1}{\omega^2} + \frac{2}{i\omega^3}\right] + \dots$$

Yes : Suppose  $\sigma_1 = \sigma_2 = i/\omega \Longrightarrow Q_2^{F-C}[f] - I[f] \sim O(\psi^{-3}).$ 

$$\mathsf{Q}_2^{ extsf{F-C}} = rac{\mathrm{i}h\left[f(\mathrm{i}h/\psi) - e^{\mathrm{i}\psi}f\left((i+\psi)h/\psi
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A complex adaptive Filon-rule :  $Q_2^{F-C}$ 

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$$I[v] \sim s_h(h)e^{i\psi}\left[\frac{\sigma_2}{i\omega} - \frac{1}{\omega^2}\right] + s'_h(h)e^{i\psi}\left[\frac{\sigma_2}{\omega^2} + \frac{2}{i\omega^3}\right] + \dots + s_0(0)\left[\frac{\sigma_1}{i\omega} - \frac{1}{\omega^2}\right] + s'_0(0)\left[\frac{\sigma_1}{\omega^2} + \frac{2}{i\omega^3}\right] + \dots$$

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A complex adaptive Filon-rule :  $Q_2^{F-C}$ 

Are there better options than choosing  $\sigma_2 = -\sigma_1$ ?

$$\begin{split} I[v] &\sim s_h(h)e^{\mathrm{i}\psi}\left[\frac{\sigma_2}{\mathrm{i}\omega}-\frac{1}{\omega^2}\right]+s'_h(h)e^{\mathrm{i}\psi}\left[\frac{\sigma_2}{\omega^2}+\frac{2}{\mathrm{i}\omega^3}\right]+\dots\\ &+ s_0(0)\left[\frac{\sigma_1}{\mathrm{i}\omega}-\frac{1}{\omega^2}\right]+s'_0(0)\left[\frac{\sigma_1}{\omega^2}+\frac{2}{\mathrm{i}\omega^3}\right]+\dots\\ \end{split}$$

$$\begin{aligned} \mathsf{Yes}: \text{Suppose } \sigma_1=\sigma_2=i/\omega \Longrightarrow Q_2^{\mathsf{F}-\mathsf{C}}[f]-I[f]\sim O(\psi^{-3}). \end{split}$$

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A complex adaptive Filon-rule :  $Q_2^{F-C}$ 

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$$\end{split}$$

$$\begin{split} \mathsf{Yes: Suppose } \sigma_1 &= \sigma_2 = i/\omega \Longrightarrow \mathsf{Q}_2^{F-C}[f]-I[f]\sim \mathsf{O}(\psi^{-3}),\\ \mathsf{Q}_2^{F-C} &= \frac{\mathrm{i}h\left[f(\mathrm{i}h/\psi)-e^{\mathrm{i}\psi}f\left(((i+\psi)h/\psi)\right]\right]}{\omega^4}, \quad \psi = \omega h \end{split}$$

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### Illustration

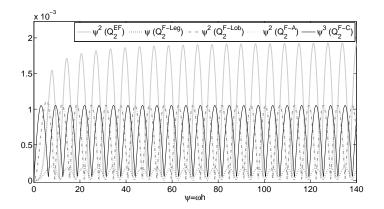


Figure: The normalised errors in some  $\nu = 2$  Filon-type schemes for  $f(x) = e^x$ , h = 1/10 and different values of  $\omega$ .

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Error control for  $Q_2^{F-C}$ 

$$\mathsf{Q}_2^{\mathsf{F}\text{-}\mathsf{C}} = \frac{\mathrm{i}h\left[f(\mathrm{i}h/\psi) - e^{\mathrm{i}\psi}f\left((i+\psi)h/\psi\right)\right]}{\psi}, \quad \psi = \omega h.$$

Obtained by replacing *f* by interpolating polynomial  $\overline{f}$  in nodes i  $h/\omega$  and  $h + i h/\omega$  (for large  $\psi : \sim \psi^{-3}$ )

Similarly :  $Q_3^{F-C}$  by replacing f by interpolating polynomial  $\tilde{f}$  in nodes i  $h/\omega$ , h/2 and  $h + i h/\omega$  (for large  $\psi$  : also  $\sim \psi^{-3}$  but about 100 times more accurate)

$$\begin{split} I[f] - I[ar{f}] &\approx I[ar{f}] - I[ar{f}] = rac{(1 - e^{\mathrm{i}\psi})2h}{\psi^2(4 + \psi^2)} imes \ \left( (2 - \mathrm{i}\psi) \, f(rac{\mathrm{i}}{\omega}) - (2 + \mathrm{i}\psi) \, f(h + rac{\mathrm{i}}{\omega}) + (2\mathrm{i}\psi) f(rac{h}{2}) 
ight) \end{split}$$

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Rules of Filon-type

Adaptive Filon rules

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Conclusions

Error control for  $Q_2^{F-C}$ 

$$\mathsf{Q}_2^{\mathsf{F}\text{-}\mathsf{C}} = \frac{\mathrm{i}h\left[f(\mathrm{i}h/\psi) - e^{\mathrm{i}\psi}f\left((i+\psi)h/\psi\right)\right]}{\psi}, \quad \psi = \omega h.$$

Obtained by replacing *f* by interpolating polynomial  $\overline{f}$  in nodes i  $h/\omega$  and  $h + i h/\omega$  (for large  $\psi : \sim \psi^{-3}$ ) Similarly :  $Q_3^{F-C}$  by replacing *f* by interpolating polynomial  $\tilde{f}$  in nodes i  $h/\omega$ , h/2 and  $h + i h/\omega$  (for large  $\psi$  : also  $\sim \psi^{-3}$  but about 100 times more accurate)

$$\begin{split} I[f] - I[\bar{f}] &\approx I[\tilde{f}] - I[\bar{f}] = \frac{(1 - \mathrm{e}^{\mathrm{i}\psi})2h}{\psi^2(4 + \psi^2)} \times \\ &\left( (2 - \mathrm{i}\psi) f(\frac{\mathrm{i}}{\omega}) - (2 + \mathrm{i}\psi) f(h + \frac{\mathrm{i}}{\omega}) + (2\mathrm{i}\psi)f(\frac{h}{2}) \right) \end{split}$$

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### Illustration

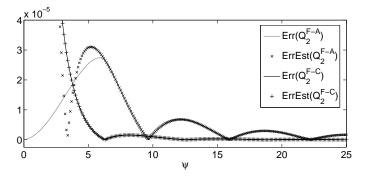


Figure: Error estimations for the  $Q_2^{F-A}$  and  $Q_2^{F-C}$  method applied on the problem with  $f(x) = e^x$ , h = 2.

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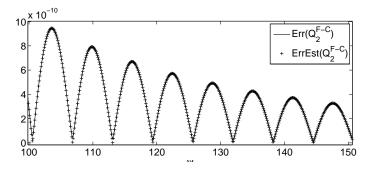


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- Filon rules, EF rules, and steepest descent rules are built up starting from different points of view, the basic underlying idea is the same : f(x) is interpolated by a polynomial.
- Different choices can be made for the interpolation nodes.
- A choice of the (complex) interpolation nodes can improve the asymptotic behaviour of the quadrature rule.
- Even better asymptotic behaviour is obtained if the nodes are frequency dependent.
- Cheap error estimation is possible.



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