

# Adaptive Filon methods for the computation of highly oscillatory integrals

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### Oscillatory integrals

$$
I[f] = \int_0^h f(x) e^{i\omega g(x)} dx
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We focus on the particular case

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<span id="page-1-0"></span>If the integrand oscillates rapidly, and unless we use a huge number of function evaluations, the classical  $\nu$ -point Gauss rule



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<span id="page-3-0"></span>If the integrand oscillates rapidly, and unless we use a huge number of function evaluations, the classical  $\nu$ -point Gauss rule is useless.



Gauss rule applied to oscillatory integrands Example :  $f(x) = exp(x)$  and  $h = 1/10$ 

$$
\int_0^h e^x e^{i\omega x} dx = \frac{-1 + e^{h(1+i\omega)}}{1+i\omega}
$$



The absolute error in Gauss-Legendre quadrature for different values of the characteristic frequency  $\psi = \omega h$ .

 $\mathbf{E} = \mathbf{A} \oplus \mathbf{B} + \mathbf{A} \oplus \mathbf{B} + \mathbf{A} \oplus \mathbf{B} + \mathbf{A} \oplus \mathbf{B} + \mathbf{A} \oplus \mathbf{A} + \mathbf{B} + \mathbf{A} \oplus \mathbf{A} + \mathbf{B} + \mathbf{A} \oplus \mathbf{A} + \mathbf{B} + \mathbf{A} + \mathbf$  $2990$ 

#### Asymptotic expansion

$$
\begin{array}{rcl}\nI[f] & = & \int_{a}^{b} f(x)e^{i\omega x} \, dx \\
& = & \frac{1}{i\omega} \left( f(b) e^{i\omega b} - f(a) e^{i\omega a} \right) - \frac{1}{i\omega} I[f'] \\
& = & \frac{1}{i\omega} \left( f(b) e^{i\omega b} - f(a) e^{i\omega a} \right) \\
& - \frac{1}{(i\omega)^2} \left( f'(b) e^{i\omega b} - f'(a) e^{i\omega a} \right) + \frac{1}{(i\omega)^2} I[f'']\n\end{array}
$$

<span id="page-5-0"></span>
$$
I[f] = -\sum_{m=0}^{\infty} \frac{1}{(-i\omega)^{m+1}} \left[ e^{i\omega b} f^{(m)}(b) - e^{i\omega a} f^{(m)}(a) \right]
$$

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\mathsf Q_s^A[f] = -\sum_{m=0}^{s-1} \frac{1}{(-\mathrm i \omega)^{m+1}} \left[ \mathrm e^{\mathrm i \omega b} f^{(m)}(b) - \mathrm e^{\mathrm i \omega a} f^{(m)}(a) \right]
$$

$$
Q_s^A[f] - I[f] \sim O(\omega^{-s-1}) \quad \omega \to +\infty
$$

This asymptotic method is of asymptotic order  $s + 1$ . The asymptotic order gives us the rate at which the error decreases with increasing  $\omega$ .



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## Exponential fitting

M. VAN DAELE, G. VANDEN BERGHE AND H. VANDE VYVER, Exponentially fitted quadrature rules of Gauss type for oscillatory integrands, Appl. Numer. Math., 53 (2005), pp. 509–526.

How to compute

 $\int_0^1$ −1  $F(t)$ dt

whereby  $F(x)$  has an oscillatory behaviour with frequency  $\mu$ ?

$$
I[f] = \int_0^h f(x)e^{i\omega x} dx = \frac{h}{2}e^{i\mu}\int_{-1}^1 f(h(t+1)/2)e^{i\mu t} dt \quad \mu = \frac{\omega h}{2}
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### Exponential fitting

$$
\mathcal{L}[F; x; h; \mathbf{a}] = \int_{x-h}^{x+h} F(z)dz - h \sum_{k=1}^{b} w_k F(x + \hat{c}_k h), \quad \hat{c}_k \in [-1, 1]
$$
  
(put  $x = \mathbf{0}$  and  $h = \mathbf{1}$  to obtain  $\int_{-\mathbf{1}}^{\mathbf{1}} F(t)dt$ )

 $\mathcal{L}[F; x; h; a] = 0$  for a reference set of  $K + 2(P + 1) + 1 = 2\nu$ functions

$$
1, t, t^2, \ldots t^K,
$$

 $\mathsf{exp}(\pm\mathrm{i}\mu t), t\,\mathsf{exp}(\pm\mathrm{i}\mu t), t^2\,\mathsf{exp}(\pm\mathrm{i}\mu t), \ldots, t^P\,\mathsf{exp}(\pm\mathrm{i}\mu t)$ 

In this talk we only consider the case  $K = -1$ ,  $P = \nu - 1$ .

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### 1-node EF rule

$$
\int_{-1}^1 F(x) dx \approx w_1 F(\hat{c}_1)
$$

 $\int_0^1$ **l\_\_exp**( $\pm$ i $\mu$ x)dx − w<sub>1</sub> exp( $\pm$ i ĉ<sub>1</sub>  $\mu$ ) = 0

$$
w_1 = 2\sin(\mu)/\mu \qquad \qquad \hat{c}_1 = 0
$$

$$
I[f] = \int_0^h f(x) \exp(i\omega x) dx = \int_0^h F(x) dx
$$

<span id="page-17-0"></span>
$$
Q_1^{EF}[F] = \frac{h\sin(\mu)}{\mu}F(h/2) = \frac{e^{ih\omega}-1}{i\omega}f(h/2) \quad \mu = \omega h/2
$$

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#### 2-node EF rule

$$
\int_{-1}^1 F(x) dx \approx w_1 F(\hat{c}_1) + w_2 F(\hat{c}_2)
$$

 $\sqrt{ }$  $\int$  $\overline{\mathcal{L}}$  $\int_0^1$ −**1 exp**(±iµx)dx − w**<sup>1</sup> exp**(±i cˆ**<sup>1</sup>** µ) − w**<sup>2</sup> exp**(±i cˆ**<sup>2</sup>** µ) = **0**  $\int_0^1$ −**1** x **exp**(±iµx)dx − w**<sup>1</sup>** cˆ**<sup>1</sup> exp**(±i cˆ**<sup>1</sup>** µ) − w**2**cˆ**<sup>2</sup> exp**(±i cˆ**<sup>2</sup>** µ) = **0**

Assuming  $w_1 = w_2$  and  $\hat{c}_1 = -\hat{c}_2$ :

 $\Leftrightarrow$   $\begin{cases} w_2\mu\cos(\mu\hat{c}_2) - \sin(\mu) = 0 \\ w_2\hat{c}_2\mu^2\sin(\mu\hat{c}_2) - \sin(\mu) \end{cases}$  $w_2 \hat{c}_2 \mu^2 \sin(\mu \hat{c}_2) - \sin(\mu) + \mu \cos(\mu) = 0$  $\mathrm{Q}_{2}^{EF}[F]=\frac{h}{2}w_{2}\left[F\left(\frac{h(1+\hat{c}_{2})}{2}\right)\right]$  $+ F\left(\frac{h(1-\hat{c}_2)}{2}\right)$  $\left[\frac{(-\hat{c}_2)}{2}\right)\left[\begin{array}{cc} & \mu=\frac{\omega h}{2} \end{array}\right]$ **2 2**K ロ X x 4 D X X 원 X X 원 X 원 X 2 D X Q Q



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### 2-node EF rule

$$
\int_{-1}^1 F(x) dx \approx w_1 F(\hat{c}_1) + w_2 F(\hat{c}_2)
$$

 $\sqrt{ }$  $\int$  $\overline{\mathcal{L}}$  $\int_0^1$ −**1 exp**(±iµx)dx − w**<sup>1</sup> exp**(±i cˆ**<sup>1</sup>** µ) − w**<sup>2</sup> exp**(±i cˆ**<sup>2</sup>** µ) = **0**  $\int_0^1$ −**1** x **exp**(±iµx)dx − w**<sup>1</sup>** cˆ**<sup>1</sup> exp**(±i cˆ**<sup>1</sup>** µ) − w**2**cˆ**<sup>2</sup> exp**(±i cˆ**<sup>2</sup>** µ) = **0**

Assuming  $w_1 = w_2$  and  $\hat{c}_1 = -\hat{c}_2$ :

$$
\Longleftrightarrow \left\{\begin{array}{l}w_2\mu\cos(\mu\hat{c}_2)-\sin(\mu)=\mathbf{0}\\ w_2\hat{c}_2\mu^2\sin(\mu\hat{c}_2)-\sin(\mu)+\mu\cos(\mu)=\mathbf{0}\end{array}\right.
$$
\n
$$
D_2^{EF}[F]=\frac{h}{2}w_2\left[F\left(\frac{h(1+\hat{c}_2)}{2}\right)+F\left(\frac{h(1-\hat{c}_2)}{2}\right)\right]\qquad\mu=\frac{\omega h}{2}
$$





[Rules of Filon-type](#page-17-0) [Adaptive Filon rules](#page-26-0) [Conclusions](#page-37-0)<br>  $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$  $\frac{00}{000}$ 

### 2-node EF rule

 $\int$   $w_2 \mu \cos(\mu \hat{c}_2) - \sin(\mu) = 0$  $w_2 \hat{c}_2 \mu^2 \sin(\mu \hat{c}_2) - \sin(\mu) + \mu \cos(\mu) = 0$ If  $cos(\mu\hat{c}_2) \neq 0$  then  $w_2 = sin \mu/(\mu cos(\mu\hat{c}_2))$  $G(\hat{c}_2) := (\sin \mu - \mu \cos \mu) \cos(\mu \hat{c}_2) - \mu \hat{c}_2 \sin \mu \sin(\mu \hat{c}_2) = 0$ 



Figure:  $G(x_2)$  for  $\mu = 5$ ,  $\mu = 50$  and  $\mu = 200$ .

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#### 2-node EF rule

 $\int w_2\mu\cos(\mu\hat{c}_2) - \sin(\mu) = 0$  $w_2\hat{c}_2\mu^2\sin(\mu\hat{c}_2)-\sin(\mu)+\mu\cos(\mu)=0$ If  $cos(\mu \hat{c}_2) \neq 0$  then  $w_2 = sin \mu/(\mu cos(\mu \hat{c}_2))$  $G(\hat{c}_2) := (\sin \mu - \mu \cos \mu) \cos(\mu \hat{c}_2) - \mu \hat{c}_2 \sin \mu \sin(\mu \hat{c}_2) = 0$ 



Figure:  $G(x_2)$  for  $\mu = 5$ ,  $\mu = 50$  and  $\mu = 200$ .

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2-node EF rule



Figure: The  $\hat{c}_2(\mu)$  and  $w_2(\mu)$  curve for the EF method with  $\nu = 2$ .

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#### 3-node EF rule

$$
\hat{c}_1 = -\hat{c}_3
$$
  $\hat{c}_2 = 0$   $w_1 = w_3$ 



<span id="page-26-0"></span>Figure: The  $\hat{c}_3(\mu)$ ,  $w_1(\mu) = w_3(\mu)$  and  $w_2(\mu)$  curves for the  $\nu = 3$  EF rule

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#### 4-node EF rule

$$
\hat{c}_1 = -\hat{c}_4
$$
  $\hat{c}_2 = -\hat{c}_3$   $w_1 = w_4$   $w_2 = w_3$ 



Figure: Nodes and weights of the EF rule with  $\nu = 4$  quadrature nodes.

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## Accuracy of EF rules

All EF rules reduce to the classical  $\nu$ -point Gauss(-Legendre) method in the limiting case  $\mu = 0$ .

Thus for small  $\mu: \mathsf{O}(h^{2\nu+1})$ What about the accuracy for larger values of  $\mu = \omega h/2$ ?

for large  $\mu$  :  $O(\mu^{\bar{\nu}-\nu})$  with  $\bar{\nu}=\lfloor (\nu-1)/2 \rfloor$ 

 $\nu=1:\,{\rm O}(\omega^{-1}) \qquad \quad \nu=2,3:\,{\rm O}(\omega^{-2}) \qquad \quad \nu=4,5:\,{\rm O}(\omega^{-3})$ 



All EF rules reduce to the classical  $\nu$ -point Gauss(-Legendre) method in the limiting case  $\mu = 0$ . Thus for small  $\mu$  :  $\mathsf{O}(h^{2\nu+1})$ 

What about the accuracy for larger values of  $\mu = \omega h/2$ ?

for large  $\mu$  :  $O(\mu^{\bar{\nu}-\nu})$  with  $\bar{\nu}=\lfloor (\nu-1)/2 \rfloor$ 

 $\nu=1:\,{\rm O}(\omega^{-1}) \qquad \quad \nu=2,3:\,{\rm O}(\omega^{-2}) \qquad \quad \nu=4,5:\,{\rm O}(\omega^{-3})$ 

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All EF rules reduce to the classical  $\nu$ -point Gauss(-Legendre) method in the limiting case  $\mu = 0$ . Thus for small  $\mu$  :  $\mathsf{O}(h^{2\nu+1})$ What about the accuracy for larger values of  $\mu = \omega h/2$ ?

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 $\nu=1:\,{\rm O}(\omega^{-1}) \qquad \quad \nu=2,3:\,{\rm O}(\omega^{-2}) \qquad \quad \nu=4,5:\,{\rm O}(\omega^{-3})$ 

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J. P. COLEMAN AND L. GR. IXARU, Truncation errors in exponential fitting for oscillatory problems, SIAM. J. Numer. Anal., 44 (2006), pp. 1441–1465.

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for large  $\mu$  :  $O(\mu^{\bar{\nu}-\nu})$  with  $\bar{\nu} = \lfloor (\nu-1)/2 \rfloor$ 

 $\nu=$  1 :  ${\sf O}(\omega^{-1}$ )  $\nu = 2, 3: O(\omega^{-2})$  $\nu=4,5:O(\omega^{-3})$ 

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Proof

$$
(\mathcal{O}_{\mathcal{A}}^{\mathcal{A}})^{\mathcal{A}}_{\mathcal{A}}(\mathcal{O}_{\mathcal{A}}^{\mathcal{A}})
$$

$$
\int_{-1}^1 F(t)dt \approx \int_{-1}^1 \bar{F}(t)dt
$$

 $\bar{F}(t) \in \text{span}\{\text{exp}(\pm \mathrm{i} \mu t), t \exp(\pm \mathrm{i} \mu t), t^2 \exp(\pm \mathrm{i} \mu t), \dots, t^P \exp(\pm \mathrm{i} \mu t)\}$ 

$$
I[f] = \int_0^h f(x)e^{i\omega x} dx = \frac{h}{2} e^{i\frac{\omega h}{2}} \int_{-1}^1 f(\frac{h}{2}(t+1))e^{i\frac{\omega h}{2}t} dt
$$
  
If  $\frac{\omega h}{2} = \mu$  then  $I[f] \approx I[\bar{f}]$  with  $\bar{f}(x) \in \text{span}\{1, x, x^2, ..., x^{\nu-1}\}$   
 $Q_{\nu}^{EF}[f] - I[f] = I[\bar{f}] - I[f] = I[\nu]$   $v(x) := \bar{f}(x) - f(x)$ 

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Proof

$$
\int_{-1}^1 F(t)dt \approx \int_{-1}^1 \bar{F}(t)dt
$$

 $\bar{F}(t) \in \text{span}\{\text{exp}(\pm \mathrm{i} \mu t), t \exp(\pm \mathrm{i} \mu t), t^2 \exp(\pm \mathrm{i} \mu t), \dots, t^P \exp(\pm \mathrm{i} \mu t)\}$ 

$$
I[f] = \int_0^h f(x)e^{i\omega x} dx = \frac{h}{2}e^{i\frac{\omega h}{2}} \int_{-1}^1 f(\frac{h}{2}(t+1))e^{i\frac{\omega h}{2}t} dt
$$

If  $\frac{\omega h}{2} = \mu$  then  $I[f] \approx I[\bar{f}]$  with  $\bar{f}(x) \in \text{span}\{1, x, x^2, \dots, x^{\nu-1}\}\$ 

 $Q_{\nu}^{EF}[f] - I[f] = I[\bar{f}] - I[f] = I[\nu]$   $v(x) := \bar{f}(x) - f(x)$ 

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Proof

$$
f_{\rm{max}}
$$

$$
\int_{-1}^1 F(t)dt \approx \int_{-1}^1 \bar{F}(t)dt
$$

 $\bar{F}(t) \in \text{span}\{\text{exp}(\pm \mathrm{i} \mu t), t \exp(\pm \mathrm{i} \mu t), t^2 \exp(\pm \mathrm{i} \mu t), \dots, t^P \exp(\pm \mathrm{i} \mu t)\}$ 

$$
I[f] = \int_0^h f(x)e^{i\omega x} dx = \frac{h}{2}e^{i\frac{\omega h}{2}} \int_{-1}^1 f(\frac{h}{2}(t+1))e^{i\frac{\omega h}{2}t} dt
$$
  
If  $\frac{\omega h}{2} = \mu$  then  $I[f] \approx I[\bar{f}]$  with  $\bar{f}(x) \in \text{span}\{1, x, x^2, \dots, x^{\nu-1}\}$   
 $Q_{\nu}^{EF}[f] - I[f] = I[\bar{f}] - I[f] = I[\nu]$   $V(x) := \bar{f}(x) - f(x)$ 

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$$

$$
\int_{-1}^1 F(t)dt \approx \int_{-1}^1 \bar{F}(t)dt
$$

 $\bar{F}(t) \in \text{span}\{\text{exp}(\pm \mathrm{i} \mu t), t \exp(\pm \mathrm{i} \mu t), t^2 \exp(\pm \mathrm{i} \mu t), \dots, t^P \exp(\pm \mathrm{i} \mu t)\}$ 

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$$
  
If  $\frac{\omega h}{2} = \mu$  then  $I[f] \approx I[\bar{f}]$  with  $\bar{f}(x) \in \text{span}\{1, x, x^2, ..., x^{\nu-1}\}$   
 $Q_{\nu}^{EF}[f] - I[f] = I[\bar{f}] - I[f] = I[\nu]$   $v(x) := \bar{f}(x) - f(x)$ 

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Suppose  $\nu$  is even and  $a < c_1 < c_2 < \ldots < c_\nu < b$ 

$$
c_j = a + \lambda_j/\omega \qquad c_{\nu-j+1} = b - \lambda_j/\omega \quad j = 1, \ldots, \nu/2
$$

$$
v(x) = \frac{f^{(\nu)}(\xi(x))}{\nu!} \prod_{i=1}^{\nu} (x - c_i)
$$

$$
v(x) = s(x) \prod_{i=1}^{\nu/2} (x - b + \lambda_i/\omega) \qquad s(x) = \frac{f^{(\nu)}(\xi(x))}{\nu!} \prod_{j=1}^{\nu/2} (x - a - \lambda_j/\omega)
$$

$$
v(b) = s(b) \prod_{i=1}^{\nu/2} (\lambda_i/\omega) = O(\omega^{-\nu/2})
$$

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$$
v'(b) = s(b)\omega^{-\nu/2+1} \sum_{k=1}^{\nu/2} \prod_{i \neq k} \lambda_i + O(\omega^{-\nu/2}) = O(\omega^{-\nu/2+1})
$$

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Suppose  $\nu$  is even and  $a < c_1 < c_2 < \ldots < c_\nu < b$ 

$$
c_j = a + \lambda_j/\omega \qquad c_{\nu-j+1} = b - \lambda_j/\omega \quad j = 1, \ldots, \nu/2
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$$

$$
v(x) = s(x) \prod_{i=1}^{\nu/2} (x - b + \lambda_i/\omega) \qquad s(x) = \frac{f^{(\nu)}(\xi(x))}{\nu!} \prod_{j=1}^{\nu/2} (x - a - \lambda_j/\omega)
$$

$$
v(b) = s(b) \prod_{i=1}^{\nu/2} (\lambda_i/\omega) = O(\omega^{-\nu/2})
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$$
v'(b) = s(b)\omega^{-\nu/2+1} \sum_{k=1}^{\nu/2} \prod_{i \neq k} \lambda_i + O(\omega^{-\nu/2}) = O(\omega^{-\nu/2+1})
$$

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Suppose  $\nu$  is even and  $a < c_1 < c_2 < \ldots < c_\nu < b$ 

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$$
v(x) = \frac{f^{(\nu)}(\xi(x))}{\nu!} \prod_{i=1}^{\nu} (x - c_i)
$$

$$
v(x) = s(x) \prod_{i=1}^{\nu/2} (x - b + \lambda_i/\omega) \qquad s(x) = \frac{f^{(\nu)}(\xi(x))}{\nu!} \prod_{j=1}^{\nu/2} (x - a - \lambda_j/\omega)
$$

$$
v(b) = s(b) \prod_{i=1}^{\nu/2} (\lambda_i/\omega) = O(\omega^{-\nu/2})
$$

$$
v'(b) = s(b)\omega^{-\nu/2+1} \sum_{k=1}^{\nu/2} \prod_{i \neq k} \lambda_i + O(\omega^{-\nu/2}) = O(\omega^{-\nu/2+1})
$$

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Suppose  $\nu$  is even and  $a < c_1 < c_2 < \ldots < c_\nu < b$ 

$$
c_j = a + \lambda_j/\omega \qquad c_{\nu-j+1} = b - \lambda_j/\omega \quad j = 1, \ldots, \nu/2
$$

$$
v(x) = \frac{f^{(\nu)}(\xi(x))}{\nu!} \prod_{i=1}^{\nu} (x - c_i)
$$

$$
v(x) = s(x) \prod_{i=1}^{\nu/2} (x - b + \lambda_i/\omega) \qquad s(x) = \frac{f^{(\nu)}(\xi(x))}{\nu!} \prod_{j=1}^{\nu/2} (x - a - \lambda_j/\omega)
$$

$$
v(b) = s(b) \prod_{i=1}^{\nu/2} (\lambda_i/\omega) = O(\omega^{-\nu/2})
$$

$$
v'(b) = s(b)\omega^{-\nu/2+1} \sum_{k=1}^{\nu/2} \prod_{i \neq k} \lambda_i + O(\omega^{-\nu/2}) = O(\omega^{-\nu/2+1})
$$



Suppose  $\nu$  is even and  $a < c_1 < c_2 < \ldots < c_\nu < b$ 

$$
c_j = a + \lambda_j/\omega \qquad c_{\nu-j+1} = b - \lambda_j/\omega \quad j = 1, \ldots, \nu/2
$$

$$
v(x) = \frac{f^{(\nu)}(\xi(x))}{\nu!} \prod_{i=1}^{\nu} (x - c_i)
$$

$$
v(x) = s(x) \prod_{i=1}^{\nu/2} (x - b + \lambda_i/\omega) \qquad s(x) = \frac{f^{(\nu)}(\xi(x))}{\nu!} \prod_{j=1}^{\nu/2} (x - a - \lambda_j/\omega)
$$

$$
v(b) = s(b) \prod_{i=1}^{\nu/2} (\lambda_i/\omega) = O(\omega^{-\nu/2})
$$

$$
v'(b) = s(b)\omega^{-\nu/2+1} \sum_{k=1}^{\infty} \prod_{i \neq k} \lambda_i + O(\omega^{-\nu/2}) = O(\omega^{-\nu/2+1})
$$



 $v(b) = O(\omega^{-\nu/2})$ )  $v'(b) = O(\omega^{-\nu/2+1})$ 

 $v^{(n)}(b) = O(\omega^{-\nu/2+n}), n = 0, 1, ..., \nu/2 - 1$ 

 $v^{(n)}(a) = O(\omega^{-\nu/2+n}), n = 0, 1, ..., \nu/2 - 1$ 

 $Q_{\nu}^{EF}[f] - I[f] = I[\nu]$  $=-\sum^{\infty}$ 1  $(-i\omega)^{m+1}$  $\left[e^{i\omega b}v^{(m)}(b)-e^{i\omega a}v^{(m)}(a)\right]$  $\sum$ 2−1 1  $\frac{1}{(-i\omega)^{m+1}}O(\omega^{-\nu/2+m})+O(\omega^{-\nu/2-1})$  $= {\mathsf{O}}(\omega^{-\nu/2-1}) = {\mathsf{O}}(\omega^{\lfloor(\nu-1)/2\rfloor-\nu})$ 



 $v(b) = O(\omega^{-\nu/2})$   $v'(b) = O(\omega^{-\nu/2+1})$ 

 $v^{(n)}(b) = O(\omega^{-\nu/2+n}), n = 0, 1, ..., \nu/2 - 1$ 

 $v^{(n)}(a) = O(\omega^{-\nu/2+n}), n = 0, 1, ..., \nu/2 - 1$ 

 $Q_{\nu}^{EF}[f] - I[f] = I[\nu]$  $=-\sum^{\infty}$ 1  $(-i\omega)^{m+1}$  $\left[e^{i\omega b}v^{(m)}(b)-e^{i\omega a}v^{(m)}(a)\right]$  $\sum$ 2−1 1  $\frac{1}{(-i\omega)^{m+1}}O(\omega^{-\nu/2+m})+O(\omega^{-\nu/2-1})$  $= {\mathsf{O}}(\omega^{-\nu/2-1}) = {\mathsf{O}}(\omega^{\lfloor(\nu-1)/2\rfloor-\nu})$ 



 $v(b) = O(\omega^{-\nu/2})$   $v'(b) = O(\omega^{-\nu/2+1})$  $v^{(n)}(b) = O(\omega^{-\nu/2+n}), n = 0, 1, ..., \nu/2 - 1$ 

 $v^{(n)}(a) = O(\omega^{-\nu/2+n}), n = 0, 1, ..., \nu/2 - 1$  $Q_{\nu}^{EF}[f] - I[f] = I[\nu]$  $=-\sum^{\infty}$ 1  $(-i\omega)^{m+1}$  $\left[e^{i\omega b}v^{(m)}(b)-e^{i\omega a}v^{(m)}(a)\right]$ 

$$
=-\sum_{m=0}^{\nu/2-1}\frac{1}{(-i\omega)^{m+1}}O(\omega^{-\nu/2+m})+O(\omega^{-\nu/2-1})
$$

 $= {\mathsf{O}}(\omega^{-\nu/2-1}) = {\mathsf{O}}(\omega^{\lfloor(\nu-1)/2\rfloor-\nu})$ 



 $v(b) = O(\omega^{-\nu/2})$   $v'(b) = O(\omega^{-\nu/2+1})$  $v^{(n)}(b) = O(\omega^{-\nu/2+n}), n = 0, 1, ..., \nu/2 - 1$  $v^{(n)}(a) = O(\omega^{-\nu/2+n}), n = 0, 1, ..., \nu/2 - 1$ 

 $Q_{\nu}^{EF}[f] - I[f] = I[\nu]$  $=-\sum^{\infty}$  $m=0$ 1  $(-i\omega)^{m+1}$  $\left[e^{i\omega b}v^{(m)}(b)-e^{i\omega a}v^{(m)}(a)\right]$  $\sum$ 2−1 1  $\frac{1}{(-i\omega)^{m+1}}O(\omega^{-\nu/2+m})+O(\omega^{-\nu/2-1})$  $= {\mathsf{O}}(\omega^{-\nu/2-1}) = {\mathsf{O}}(\omega^{\lfloor(\nu-1)/2\rfloor-\nu})$ 



 $v(b) = O(\omega^{-\nu/2})$   $v'(b) = O(\omega^{-\nu/2+1})$  $v^{(n)}(b) = O(\omega^{-\nu/2+n}), n = 0, 1, ..., \nu/2 - 1$  $v^{(n)}(a) = O(\omega^{-\nu/2+n}), n = 0, 1, ..., \nu/2 - 1$ 

$$
Q_{\nu}^{EF}[f] - I[f] = I[\nu]
$$
  
=  $-\sum_{m=0}^{\infty} \frac{1}{(-i\omega)^{m+1}} \left[ e^{i\omega b} v^{(m)}(b) - e^{i\omega a} v^{(m)}(a) \right]$   
=  $-\sum_{m=0}^{\nu/2-1} \frac{1}{(-i\omega)^{m+1}} O(\omega^{-\nu/2+m}) + O(\omega^{-\nu/2-1})$   
=  $O(\omega^{-\nu/2-1}) = O(\omega^{\lfloor (\nu-1)/2 \rfloor - \nu})$ 

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 $v(b) = O(\omega^{-\nu/2})$   $v'(b) = O(\omega^{-\nu/2+1})$  $v^{(n)}(b) = O(\omega^{-\nu/2+n}), n = 0, 1, ..., \nu/2 - 1$  $v^{(n)}(a) = O(\omega^{-\nu/2+n}), n = 0, 1, ..., \nu/2 - 1$ 

$$
Q_{\nu}^{EF}[f] - I[f] = I[\nu]
$$
  
=  $-\sum_{m=0}^{\infty} \frac{1}{(-i\omega)^{m+1}} \left[ e^{i\omega b} v^{(m)}(b) - e^{i\omega a} v^{(m)}(a) \right]$   
=  $-\sum_{m=0}^{\nu/2-1} \frac{1}{(-i\omega)^{m+1}} O(\omega^{-\nu/2+m}) + O(\omega^{-\nu/2-1})$   
=  $O(\omega^{-\nu/2-1}) = O(\omega^{\lfloor (\nu-1)/2 \rfloor-\nu})$ 

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# Filon-type

L. N. G FILON, On a quadrature formula for trigonometric integrals, Proc. Royal Soc. Edinburgh, 49 (1928), pp. 38–47.

Interpolate only the function  $f(x)$  at  $c_1h, \ldots, c_{\nu}h$  by a polynomial  $\bar{f}(x)$ 

$$
I[f] \approx Q_{\nu}^{F}[f] = \int_{0}^{h} \bar{f}(x)e^{i\omega x} dx = h \sum_{l=1}^{\nu} b_{l}(i\hbar\omega)f(c_{l}h)
$$

$$
b_l(\mathrm{i} h\omega)=\int_0^1 \ell_l(x) \mathrm{e}^{\mathrm{i} h\omega x} \mathrm{d}x
$$

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 $\ell_l$  is the *l*th cardinal polynomial of Lagrangian interpolation.



# Filon-type

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$$
I[f] \approx Q_{\nu}^{F}[f] = \int_{0}^{h} \overline{f}(x)e^{i\omega x} dx = h \sum_{l=1}^{\nu} b_{l}(ih\omega)f(c_{l}h)
$$

$$
b_{l}(ih\omega) = \int_{0}^{1} \ell_{l}(x)e^{ih\omega x} dx
$$

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 $\ell_l$  is the *l*th cardinal polynomial of Lagrangian interpolation.



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#### 1-node Filon-type rule

$$
I[f] = \int_0^h F(x)dx = \int_0^h f(x) \exp(i\omega x)dx
$$

$$
Q_1^F[f] = \frac{\exp(ih\omega) - 1}{i\omega}f(c_1 h)
$$

$$
Q_1^{EF}[F] = \frac{e^{ih\omega} - 1}{i\omega}f(h/2)
$$

 $Q_1^F[f] = Q_1^{EF}[F]$  iff  $c_1 = \frac{1}{2}$ 



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# 1-node Filon-type rule

$$
I[f] = \int_0^h F(x)dx = \int_0^h f(x) \exp(i\omega x)dx
$$

$$
Q_1^F[f] = \frac{\exp(ih\omega) - 1}{i\omega}f(c_1 h)
$$

$$
Q_1^{EF}[F] = \frac{e^{ih\omega} - 1}{i\omega}f(h/2)
$$

$$
Q_1^F[f] = Q_1^{EF}[F] \text{ iff } c_1 = \frac{1}{2}
$$

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# 1-node Filon-type rule

$$
I[f] = \int_0^h F(x)dx = \int_0^h f(x) \exp(i\omega x)dx
$$

$$
Q_1^F[f] = \frac{\exp(ih\omega) - 1}{i\omega}f(c_1 h)
$$

$$
Q_1^{EF}[F] = \frac{e^{ih\omega} - 1}{i\omega} f(h/2)
$$

$$
Q_1^F[f] = Q_1^{EF}[F] \text{ iff } c_1 = \frac{1}{2}
$$

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#### 2-node Filon-type rule

$$
I[f] = \int_0^h F(x) dx = \int_0^h f(x) \exp(i\omega x) dx
$$

If f is interpolated at  $c_1$  h and  $c_2$  h, then

$$
Q_2^F[f] = h \left[ \left( \frac{i ((e^{i\psi} - 1) c_2 - e^{i\psi})}{(c_1 - c_2)\psi} + \frac{e^{i\psi} - 1}{(c_1 - c_2)\psi^2} \right) f(c_1 h) + \left( \frac{i ((e^{i\psi} - 1) c_1 - e^{i\psi})}{(c_2 - c_1)\psi} + \frac{e^{i\psi} - 1}{(c_2 - c_1)\psi^2} \right) f(c_2 h) \right]
$$

 $Q_2^F[f] = Q_2^{EF}[F]$  iff the same nodes are used

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#### 2-node Filon-type rule

$$
I[f] = \int_0^h F(x) dx = \int_0^h f(x) \exp(i\omega x) dx
$$

If f is interpolated at  $c_1$  h and  $c_2$  h, then

$$
Q_2^F[f] = h \left[ \left( \frac{i ((e^{i\psi} - 1) c_2 - e^{i\psi})}{(c_1 - c_2)\psi} + \frac{e^{i\psi} - 1}{(c_1 - c_2)\psi^2} \right) f(c_1 h) + \left( \frac{i ((e^{i\psi} - 1) c_1 - e^{i\psi})}{(c_2 - c_1)\psi} + \frac{e^{i\psi} - 1}{(c_2 - c_1)\psi^2} \right) f(c_2 h) \right]
$$

 $Q_2^F[f] = Q_2^{EF}[F]$  iff the same nodes are used

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A. ISERLES, On the numerical quadrature of highly-oscillating integrals. I. Fourier transforms, IMA J. Numer. Anal., 24 (2004), pp. 365–391.

For small  $\omega$ , a Filon-type quadrature method has an order as if  $\omega = 0$ .

Legendre nodes : order  $2\nu$  Lobatto nodes : order  $2\nu$  – 2 For large  $\omega$  :

$$
Q_{\nu}^{F}[f] - I[f] \sim \begin{cases} O(\omega^{-1}) & c_1 > 0 \text{ or } c_{\nu} < 1 \\ O(\omega^{-2}) & c_1 = 0, c_{\nu} = 1 \end{cases}
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# Accuracy of Filon-type rules

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$$
  
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$$

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If  $(c_1, c_v) = (0, 1)$  then  $v(h) = v(0) = 0$  $\Rightarrow Q^F_\nu[f] - I[f] = O(\omega^{-2}).$ 



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• by using Hermite interpolation : asymptotic order  $p + 1$  can be reached where  $p$  is the number of derivatives at the endpoints:

 $\bar{f}^{(l)}(h) = f^{(l)}(h), \bar{f}^{(l)}(0) = f^{(l)}(0), l = 0, \ldots, p-1$ 

- by using adaptive Filon-type methods : allowing the interpolation points to depend on  $\omega$  (is discussed later)
- by using nodes in the complex plane (=method of steepest descent)



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# How to improve the accuracy of Filon-rules ?

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# Method of steepest descent

D. HUYBRECHS AND S. VANDEWALLE, On the evaluation of highly oscillatory integrals by analytic continuation, SIAM J. Numer. Anal., 44 (2007) pp 1026–1048.





#### Method of steepest descent

$$
\int_{a}^{b} f(x)e^{i\omega x} dx
$$
\n
$$
= e^{i\omega a} \int_{0}^{\infty} f(a + ip)e^{-\omega p} dp - e^{i\omega b} \int_{0}^{\infty} f(b + ip)e^{-\omega p} dp
$$
\n
$$
= \frac{e^{i\omega a}}{\omega} \int_{0}^{\infty} f(a + i\frac{q}{\omega})e^{-q} dq - \frac{e^{i\omega b}}{\omega} \int_{0}^{\infty} f(b + i\frac{q}{\omega})e^{-q} dq
$$

This leads to the numerical evaluation of the two resulting integrals with classical Gauss-Laguerre quadrature.

High asymptotic order is obtained : using  $\nu$  points for each integral, the error behaves as  $O(\omega^{-2\,\nu-1}).$ 



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$$

One ends up evaluating f at the points

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a + \mathrm{i} \frac{x_{nj}}{\omega}, \text{ and } b + \mathrm{i} \frac{x_{nj}}{\omega}, j = 1, ..., n,
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where  $x_{ni}$  are the *n* roots of the Laguerre polynomial of degree n.

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This approach is equivalent to using a Filon rule with the same interpolation points.

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# Adaptive Filon-type rules

Idea : combine best properties of EF and Filon quadrature

- EF
	- accurate for small  $\omega$  h since the method reduces to Gauss-Legendre quadrature
	- + good results for large  $\omega$  h since the nodes tend to the endpoints (at a rate proportional to  $\omega^{-1}$ )
	- but : difficult to determine the nodes and weights for a given  $\omega$  h (iteration needed and ill-conditioned)
- Filon
	- + any set of nodes can be used
	- there is no optimal set of nodes for all  $\omega$  h
		- most accurate for small  $\omega$  h if the method is built on Legendre nodes
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# Adaptive Filon-type rules

Idea : create quadrature rules with  $\omega$ -dependent nodes that

- reduce to Legendre-nodes for small  $\omega$
- reduce to Lobatto-nodes for large  $\omega$
- for given value of  $\omega$  are easy to compute

To do so, we introduce S-shaped functions.



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#### Adaptive Filon-type methods

$$
S(\psi; r; n) = \frac{1 - \frac{\psi^{n} - r^{n}}{1 + |\psi^{n} - r^{n}|}}{1 + \frac{r^{n}}{1 + r^{n}}}
$$



Figure:  $S(x, r, 1)$  and  $S(x, r, 2)$  (dashed) for  $r = 5$  in [0, 20]

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Adaptive Filon-type methods •  $\nu = 2 : c_1(\psi) = \frac{3 - \sqrt{3}}{6}$  $\frac{1}{6}$ S( $\psi$ ; 2 $\pi$ ; 1);  $c_2(\psi) = 1 - c_1(\psi)$ 

• 
$$
\nu = 3
$$
:  $c_1(\psi) = \frac{10 - \sqrt{15}}{5} S(\psi; 3\pi; 1)$ ;  $c_3(\psi) = 1 - c_1(\psi)$ 



Figure:  $\mathbf{c}_2(\psi)$  of the adaptive Filon method  $\mathsf{Q}_2^{\mathsf{F-A}}$  and  $\mathbf{c}_3(\psi)$  of the adaptive Filon method  $Q_3^{F-A}$ . **A DIA K F A REIN A RIA K DIA K DIA R** 



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Asymptotic analysis for  $Q_2^{F-A}$ 

 $\tilde{c}_1 = c_1 h = \sigma_1(\omega)$  and  $\tilde{c}_2 = c_2 h = h + \sigma_2(\omega)$  with  $\sigma_{1,2}(\omega) \sim \omega^{-1}$ 

 $v(x) = s_h(x)(x - h - \sigma_2)$   $s_h(x) = \frac{f''(\xi_h(x))}{2}$  $\frac{n(x)}{2}(x-\sigma_1)$  $v'(x) = s_h(x) + s'_h(x)(x - h - \sigma_2)$  $v''(x) = 2s'_h(x) + s''_h(x)(x - h - \sigma_2)$ 

> $v(h) = -s_h(h)\sigma_2$  $v'(h) = s_h(h) - s'_h(h)\sigma_2$  $v''(h) = 2s'_h(h) - s''_h(h)\sigma_2$

Similar results for the other endpoint.

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> $v(h) = -s_h(h)\sigma_2$  $v'(h) = s_h(h) - s'_h(h)\sigma_2$  $v''(h) = 2s'_h(h) - s''_h(h)\sigma_2$

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v(x) = s_h(x)(x - h - \sigma_2) \qquad s_h(x) = \frac{f''(\xi_h(x))}{2}(x - \sigma_1)
$$
  
\n
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Q_2^{F-A}[f] - I[f] = I[v] \sim \sum_{m=0}^{\infty} \frac{1}{(-i\omega)^{m+1}} \left[ e^{i\omega h} v^{(m)}(h) - v^{(m)}(0) \right]
$$

Reordering for  $s_h(h)$ ,  $s'_h(h)$ , ...

$$
I[V] \sim s_h(h)e^{i\psi}\left[\frac{\sigma_2}{i\omega} - \frac{1}{\omega^2}\right] + s'_h(h)e^{i\psi}\left[\frac{\sigma_2}{\omega^2} + \frac{2}{i\omega^3}\right] + \dots
$$
  
+  $s_0(0)\left[\frac{\sigma_1}{i\omega} - \frac{1}{\omega^2}\right] + s'_0(0)\left[\frac{\sigma_1}{\omega^2} + \frac{2}{i\omega^3}\right] + \dots$ 

 $\sigma_2 = -\sigma_1$  with  $\sigma_{1,2}(\omega) \sim \psi^{-1} \Longleftrightarrow Q_2^{F-A}[f] - I[f] \sim O(\psi^{-2})$ 

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I[V] \sim s_h(h)e^{i\psi}\left[\frac{\sigma_2}{i\omega} - \frac{1}{\omega^2}\right] + s'_h(h)e^{i\psi}\left[\frac{\sigma_2}{\omega^2} + \frac{2}{i\omega^3}\right] + \dots
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 $\sigma_2 = -\sigma_1$  with  $\sigma_{1,2}(\omega) \sim \psi^{-1} \Longleftrightarrow Q_2^{F-A}[f] - I[f] \sim O(\psi^{-2})$ 



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Asymptotic analysis for  $Q_2^{F-A}$ 

$$
Q_2^{F-A}[f] - I[f] = I[v] \sim \sum_{m=0}^{\infty} \frac{1}{(-i\omega)^{m+1}} \left[ e^{i\omega h} v^{(m)}(h) - v^{(m)}(0) \right]
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[Exponentially fitted rules](#page-5-0) [Rules of Filon-type](#page-17-0) [Adaptive Filon rules](#page-26-0) [Conclusions](#page-37-0)<br>
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A complex adaptive Filon-rule :  $Q_2^{F-C}$ 

Are there better options than choosing  $\sigma_2 = -\sigma_1$ ?

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\mathsf{Q}_2^{\mathsf{F}\text{-}\mathsf{C}} = \frac{\mathsf{i} \mathsf{h}\left[\mathsf{f}(\mathsf{i} \mathsf{h}/\psi) - \mathsf{e}^{\mathsf{i} \psi}\mathsf{f}\left((\mathsf{i} + \psi)\mathsf{h}/\psi\right)\right]}{\psi}, \quad \psi = \omega \mathsf{h}
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Yes : Suppose  $\sigma_1 = \sigma_2 = i/\omega \implies Q_2^{F-G}[f] - I[f] \sim O(\psi^{-3}).$ 

$$
Q_2^{F\text{-}C} = \frac{\mathrm{i}h \left[ f(\mathrm{i}h/\psi) - e^{\mathrm{i}\psi} f((\mathrm{i}+\psi)h/\psi) \right]}{\psi}, \quad \psi = \omega h
$$

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#### **Illustration**



Figure: The normalised errors in some  $\nu = 2$  Filon-type schemes for  $f(x) = e^x$ ,  $h = 1/10$  and different values of  $\omega$ .

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Error control for  $Q_2^{F-C}$ 

$$
Q_2^{F\text{-}C}=\frac{\mathrm{i}h\left[f(\mathrm{i}h/\psi)-e^{\mathrm{i}\psi}f((\mathrm{i}+\psi)h/\psi)\right]}{\psi},\quad \psi=\omega h.
$$

Obtained by replacing f by interpolating polynomial  $\bar{f}$  in nodes  $\sinh(\omega \textrm{ and } h+ \textrm{i}\,h/\omega$  (for large  $\psi$  :  $\sim \psi^{-3})$ 

Similarly :  $Q_3^{F-C}$  by replacing f by interpolating polynomial  $\tilde{f}$  in nodes i  $h/\omega$ ,  $h/2$  and  $h+{\rm i}\, h/\omega$  (for large  $\psi$  : also  $\sim \psi^{-3}$  but about 100 times more accurate)

$$
I[f] - I[\bar{f}] \approx I[\tilde{f}] - I[\bar{f}] = \frac{(1 - e^{i\psi})2h}{\psi^2(4 + \psi^2)} \times \left( (2 - i\psi) f(\frac{i}{\omega}) - (2 + i\psi) f(h + \frac{i}{\omega}) + (2i\psi) f(\frac{h}{2}) \right)
$$

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**Illustration** 





Figure: Error estimations for the  $Q_2^{F-A}$  and  $Q_2^{F-C}$  method applied on the problem with  $f(x) = e^x$ ,  $h = 2$ .

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#### **Illustration**



Figure: Error estimations for the  $Q_2^{F-A}$  and  $Q_2^{F-C}$  method applied on the problem with  $f(x) = e^x$ ,  $h = 2$ .

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- Filon rules, EF rules, and steepest descent rules are built up starting from different points of view, the basic underlying idea is the same :  $f(x)$  is interpolated by a polynomial.
- Different choices can be made for the interpolation nodes.
- A choice of the (complex) interpolation nodes can improve the asymptotic behaviour of the quadrature rule.
- Even better asymptotic behaviour is obtained if the nodes are frequency dependent.
- Cheap error estimation is possible.



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## **Conclusions**

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