A new class of low discrepancy sequences of partitions and points

Ingrid Carbone

Department of Mathematics
University of Calabria
i.carbone@unical.it
Italy

Fields Institute - Toronto
May 2, 2012

Outline

1. Uniformly distributed sequences of points
   - Uniform distribution
   - Discrepancy
2. *LS*-sequences of partitions
   - Kakutani sequences and Volčič ρ-refinements
   - *LS* - sequences of partitions
3. *LS*-sequences of points in the unit interval
   - van der Corput sequence
   - *LS*-sequences of points
4. *LS*-sequences of points in the unit square
   - van der Corput, Hammersley, Halton sequences
   - *LS*-sequences of points

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Given any real number x, let us denote by $[x]$ its integer part, as usual.

Theorem (Kroneker)

Given an irrational number θ, the sequence $\{\theta n - [\theta n]\}_n$ of the fractional parts of $\{\theta n\}$ is dense in $[0, 1]$.

Definition (Weyl, 1914)

A sequence of points $\{x_n\}$ of the interval $[0, 1[$ is said to be uniformly distributed if for all $0 \leq a < b \leq 1$ we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \chi_{[a,b[}(x_i) = b - a.$$

Theorem (Bohl, 1909 - Sierpiński, 1910 - Weyl, 1914)

Given an irrational number θ, the sequence $\{\theta n - [\theta n]\}_n$ is uniformly distributed in $[0, 1[$.

Theorem (Weyl, 1914)

A sequence of points $\{x_n\}$ of $[0, 1[$ is uniformly distributed if for any $f \in \mathcal{C}([0, 1])$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f(x_i) = \int_0^1 f(t)\, dt.$$

- Some extension: on curves, on surfaces, in higher dimension, in compact spaces, on fractals, in topological spaces

- Application: numerical integration - Quasi-Monte Carlo methods

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Definition (van der Corput and Pisot, 1939)

Given $X = \{x_n\}_n$ in $[0, 1[$, the discrepancy of X is defined as

$$D_N(X) = D(x_1, \ldots, x_N) = \sup_{0 \leq a < b \leq 1} \left| \frac{1}{N} \sum_{i=1}^{N} \chi_{[a,\, b[}(x_i) - (b - a) \right| ,$$

and the star-discrepancy as

$$D^*_N(X) = D^*(x_1, \ldots, x_N) = \sup_{0 < b \leq 1} \left| \frac{1}{N} \sum_{i=1}^{N} \chi_{[0,\, b[}(x_i) - b \right| .$$

- $D_N^*(X) \leq D_N(X) \leq 2D_N^*(X)$
- $X = \{x_n\}$ is u. d. <=> $D_N^*(X) \to 0$ as $N \to \infty$
- $D\left(0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}\right) = \frac{1}{N}$

Theorem (van der Corput-Pisot, 1939)

For any finite sequence $X = \{x_1, \ldots, x_N\}$ we have

$$\frac{1}{N} \leq D(x_1, \ldots, x_N) \leq 1.$$

Theorem (Schmidt, 1972)

Given $X = \{x_n\}_n$ in $[0, 1[$, there exists a positive constant c such that

$$N\, D^*_N(X) \geq c \log N$$

for infinitely many $N \in \mathbb{N}$.

Low discrepancy sequences of **points**:

$$D^*_N(X) \leq C \frac{\log N}{N} \quad \textit{for all} \;\; N \in \mathbb{N}$$

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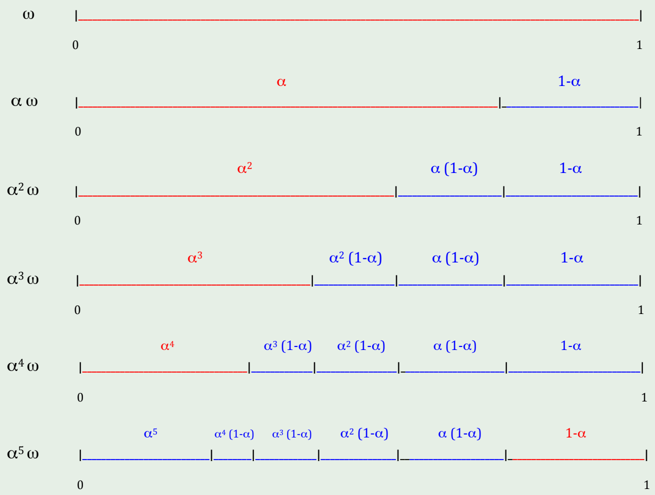
Definition (Kakutani, 1976)

Fix a real number $\alpha \in]0,1[$. If π is any partition of $[0,1[$, its α-refinement, denoted by $\alpha\pi$, is obtained subdividing the longest interval(s) of length ℓ into two intervals of lengths $\alpha\,\ell$ and $(1-\alpha)\ell$. By $\alpha^n\pi$ we denote the α-refinement of $\alpha^{n-1}\pi$. The sequence $\{\kappa_n\}$ of successive α-refinements of the trivial partition $\omega = \{[0,1[\}$ of $[0,1[$ is the Kakutani α-sequence.

$\alpha = \frac{1}{2}$: binary sequence of partitions $\left\{\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right[,\ 1 \le i \le 2^n\right\}$

α - Kakutani-sequence $\{\kappa_n\}$



Definition

We say that a sequence of partitions $\{\pi_n\}$ of $[0,1[$, where $\pi_n = \{[y_{i-1}^n, y_i^n[\,, 1 \leq i \leq t_n\}$, is uniformly distributed if for all $0 \leq a < b \leq 1$ we have

$$\lim_{n\to\infty} \frac{1}{t_n} \sum_{i=1}^{t_n} \chi_{[a,b[}(y_i^n) = b - a.$$

Theorem (Kakutani, 1976)

For any $\alpha \in]0,1[$ the Kakutani α-sequence $\{\kappa_n\}$ is uniformly distributed.

Theorem

The sequence of partitions $\{\pi_n\}$ *of* $[0,1[$, *with* $\pi_n = \{[y^n_{i-1}, y^n_i[\,,\ 1 \le i \le t_n\}$, *is uniformly distributed if for all* $f \in \mathcal{C}([0,1])$ *we have*

$$\lim_{n\to\infty} \frac{1}{t_n} \sum_{i=1}^{t_n} f(y^n_i) = \int_0^1 f(t)\, dt.$$

Definition

The discrepancy of the sequence of partitions $\{\pi_n\}$ of the interval $[0, 1[$, with $\pi_n = \{[y_{i-1}^n, y_i^n[, 1 \leq i \leq t_n\}$ is

$$D(\pi_n) = \sup_{0\leq a<b\leq 1} \left| \frac{1}{t_n} \sum_{i=1}^{t_n} \chi_{[a,\, b[}(y_i^n) - (b-a) \right|$$

and the star-discrepancy is

$$D^*(\pi_n) = \sup_{b\leq 1} \left| \frac{1}{t_n} \sum_{i=1}^{t_n} \chi_{[0,\, b[}(y_i^n) - \ b \right| .$$

- $D^*(\pi_n) \leq D(\pi_n) \leq 2D^*(\pi_n)$
- $\{\pi_n\}$ is u. d. <=> $D^*(\pi_n) \to 0$ when $n \to \infty$
- $\frac{1}{t_n} \leq D(\pi_n) \leq 1$
- Example: Knapowski sequence $\{[\frac{i-1}{n}, \frac{i}{n}[\ , 1 \leq i \leq n\})$
- Low discrepancy sequences of **partitions**:

$$D(\pi_n) \leq C\,\frac{1}{t_n} \quad for\ \ all\ \ n \in \mathbb{N}$$

Definition (A. Volčič)

For any non trivial finite partition ρ of $[0,1[$, the ρ-refinement of a partition π of $[0,1[$ (denoted by $\rho\pi$) is obtained by subdividing all the intervals of π having maximal length positively (or directly) homothetically to ρ. For any $n \in \mathbb{N}$, the ρ-refinement of $\rho^{n-1}\pi$ is indicated by $\rho^n\pi$.

The sequence of ρ-refinements is the sequence $\{\rho^n\omega\}$ (briefly, $\{\rho^n\}$) of the successive ρ-refinements of ω.

If $\rho = \{[0,\alpha[, [\alpha,1[\}$, then $\{\rho^n\} = \{\kappa_n\}$.

Theorem (A. Volčič)

For any non trivial finite partition ρ of $[0, 1[$, the sequence $\{\rho^n\omega\}$ is uniformly distributed.

Drmota and Infusino (2012) gave upper and lower bounds for the discrepancy of the sequences of ρ-refinements $\{\rho^n\omega\}$.

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Definition (I. Carbone)

Let us fix two positive integers L and S and let $0 < \beta < 1$ be the real number such that $L\beta + S\beta^2 = 1$. Denote by $\rho_{L,S}$ the partition defined by L "long" intervals having length β followed by S "short" intervals having length β^2. The sequence of successive $\rho_{L,S}$-refinements of the trivial partition ω is denoted by $\{\rho^n_{L,S}\,\omega\}$ (or $\{\rho^n_{L,S}\}$ for short) and is called *LS*-sequences of partitions.

$\rho_{1,1} = \left\{ \left[0, \frac{\sqrt{5}-1}{2}\right[, \left[\frac{\sqrt{5}-1}{2}, 1\right[\right\}$, then $\{\rho^n_{1,1}\}$ is the Kakutani $\frac{\sqrt{5}-1}{2}$ - sequence.

- Each partition $\rho^n_{L,S}$ contains only two kinds of intervals: long and short intervals of length β^n and β^{n+1}, resp.

- $t_n = L\,t_{n-1} + S\,t_{n-2}$ with $t_0 = 1$ and $t_1 = L + S$

Example

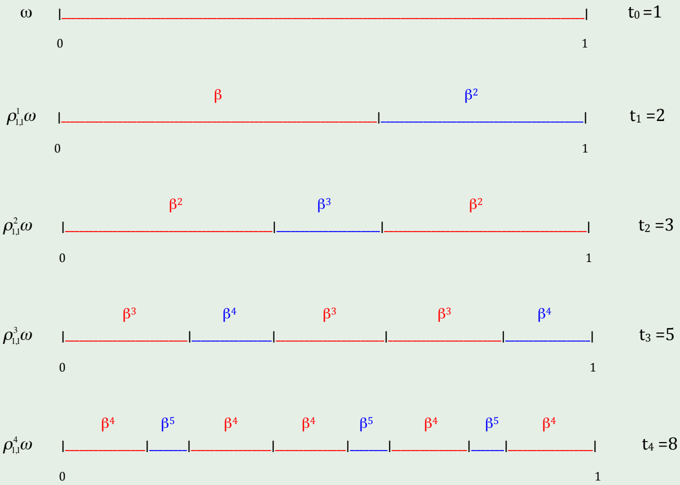
- $L = S = 1$

$$\beta + \beta^2 = 1 \quad \left(\beta = \frac{\sqrt{5}-1}{2}\right)$$

$$t_n = t_{n-1} + t_{n-2} \ \text{ with } \ t_0 = 1, \ t_1 = 2$$

Kakutani-Fibonacci sequence of partitions $\{\rho^n_{1,1}\}$

Kakutani - Fibonacci sequence of partitions $\{\rho^n_{1,1}\}$



Theorem (I. Carbone)

1. *$S \leq L$:*

$$C_1 \frac{1}{t_n} \leq D(\rho^n_{L,S}) \leq C_2 \frac{1}{t_n}.$$

2. *$S = L + 1$:*

$$C_3 \frac{\log t_n}{t_n} \leq D(\rho^n_{L,S}) \leq C_4 \frac{\log t_n}{t_n}.$$

3. *$S \geq L + 2$:*

$$C_5 \frac{1}{{t_n}^{\gamma}} \leq D(\rho^n_{L,S}) \leq C_6 \frac{1}{{t_n}^{\gamma}}$$

with $\gamma = 1 + \frac{\log(S\beta)}{\log \beta} < 1$.

- $S \leq L$: low discrepancy sequences of **partitions**

$$D(\rho^n_{L,S}) \leq C \, \frac{1}{t_n} \quad \textit{for all} \quad n \in \mathbb{N} \, .$$

- The Kakutani - Fibonacci sequences of partitions $\{\rho^n_{1,1}\}$ has low discrepancy.

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Definition (van der Corput, 1935)

Given any positive integer n, its dyadic expansion is $n = \sum_{i=0}^{M} n_i 2^i$, where $M = [\log_2 n]$, and its 2-radix notation is

$$[n]_2 = n_M n_{M-1} \ldots n_0 .$$

By reversing the order of the digits we get the number

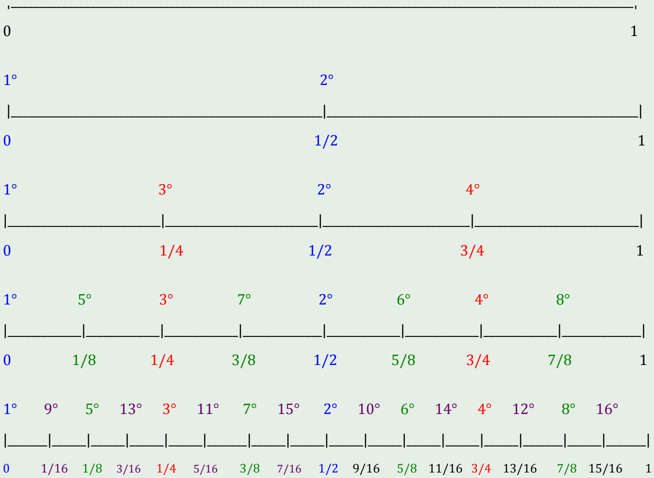
$$0.n_0 n_1 \ldots n_M ,$$

which is the 2*-radix notation* of the radical-inverse function Φ_2 in n:

$$\Phi_2(n) = \sum_{i=0}^{M} n_i 2^{-i-1} .$$

The sequence $\{\Phi_2(n)\}$ is the van der Corput sequence.

van der Corput sequence $\{\Phi_2(n)\}$



Theorem (van der Corput, 1935)

The van der Corput sequence $X = \{\Phi_2(n)\}_n$ *has low discrepancy, and satisfies*

$$D_N(X) \leq \frac{\log(N+1)}{N \log 2} \quad \textit{for} \quad \textit{any} \quad N \in \mathbb{N}.$$

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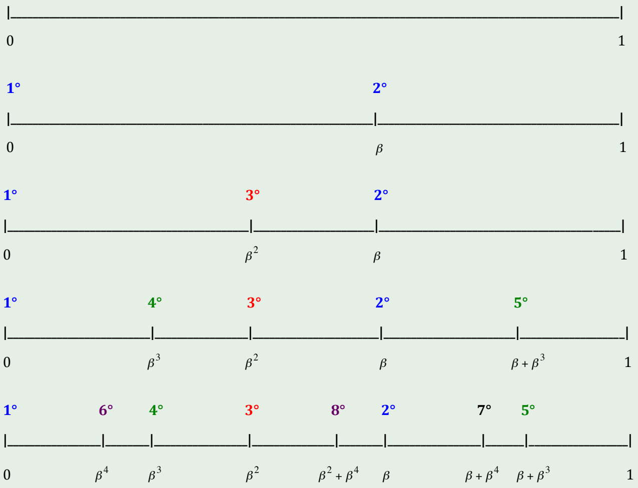
Definition (I. Carbone)

There exists an explicit algorithm “á la van der Corput" which reorders the left endpoints of the intervals of each partition $\rho^n_{L,S}$. The sequence of points $\{\xi^n_{L,S}\}$ obtained this way is called *LS*-sequence of points.

Kakutani - Fibonacci sequence of points $\{\xi^n_{1,1}\}$



$$
\begin{array}{llrlllll}
[0]_2 & = & 0 \to & 0.0 & = [\Phi_2(0)]_2 & \to & 0 & = \xi^1_{1,1} \\
[1]_2 & = & 1 \to & 0.1 & = [\Phi_2(1)]_2 & \to & \beta & = \xi^2_{1,1} \\
[2]_2 & = & 10 \to & 0.01 & = [\Phi_2(2)]_2 & \to & \beta^2 & = \xi^3_{1,1} \\
[3]_2 & = & 11 \to & 0.11 & & & & \\
[4]_2 & = & 100 \to & 0.001 & = [\Phi_2(4)]_2 & \to & \beta^3 & = \xi^4_{1,1} \\
[5]_2 & = & 101 \to & 0.101 & = [\Phi_2(5)]_2 & \to & \beta + \beta^3 & = \xi^5_{1,1} \\
[6]_2 & = & 110 \to & 0.011 & & & & \\
[7]_2 & = & 111 \to & 0.111 & & & & \\
[8]_2 & = & 1000 \to & 0.0001 & = [\Phi_2(8)]_2 & \to & \beta^4 & = \xi^6_{1,1} \\
[9]_2 & = & 1001 \to & 0.1001 & = [\Phi_2(9)]_2 & \to & \beta + \beta^4 & = \xi^7_{1,1} \\
[10]_2 & = & 1010 \to & 0.0101 & = [\Phi_2(10)]_2 & \to & \beta^2 + \beta^4 & = \xi^8_{1,1} \\
[11]_2 & = & 1011 \to & 0.1101 & & & & \\
[12]_2 & = & 1100 \to & 0.0011 & & & &
\end{array}
$$

Definition

Given a positive integer b, any $n \in \mathbb{N}$ has a b-adic expansion of the type $n = \sum_{i=0}^{M} n_i\, b^i$ (where $M = [\log_b n]$). The b-radix notation of n is

$$[n]_b = n_M n_{M-1} \dots n_0 .$$

By reversing the order of the digits we get the number

$$0.n_0 n_1 \dots n_M ,$$

which is the *b-radix notation* of the radical-inverse function Φ_b in n:

$$\Phi_b(n) = \sum_{i=0}^{M} n_i b^{-i-1} .$$

For each $0 \leq i \leq L-1$ we define the functions

$$\psi_i(x) = \beta x + i\beta \quad \text{restricted to} \quad 0 \leq x < 1,$$

and for every $L \leq i \leq L+S-1$ the functions

$$\psi_i(x) = \beta x + L\beta + (i-L)\beta^2 \quad \text{restricted to} \quad 0 \leq x < \beta.$$

We have

$$\Lambda^1_{L,S} = \left\{\xi^1_{L,S}, \xi^2_{L,S}, \ldots, \xi^{L+S}_{L,S}\right\} = \{\psi_0(0), \psi_1(0), \ldots, \psi_{L+S-1}(0)\}.$$

The compositions $\psi_{i,j} = \psi_i \circ \psi_j$ are not defined whenever

$$(i,j) \in E_{L,S} = \{L, L+1, \ldots, L+S-1\} \times \{1, \ldots, L+S-1\}.$$

Proposition

For any $n \in \mathbb{N}$ we have

$$\psi_{i_1, i_2, \ldots, i_n}(x) = \beta^n x + \sum_{k=1}^{n} \tilde{i}_k \, \beta^k,$$

where $(i_n, \ldots, i_1)$ is the ordered n-tuple of elements of the set $\{0, 1, \ldots, L+S-1\}$ such that $(i_{k+1}, i_k) \notin E_{L,S}$ for any $1 \leq k \leq n-1$ and

$\tilde{i}_k = i_k$ if $i_k \in \{0, 1, \ldots, L\}$,

$\tilde{i}_k = L + k\beta$ if $i_k \in \{L+1, \ldots, L+S-1\}$ with $i_k = L + k$.

Definition

For any $n \in \mathbb{N}$ whose $(L+S)$-radix notation is $[n]_{L+S} = n_M n_{M-1} \dots n_0$, we define its *LS*-radical inverse function

$$\Phi_{L,S}(n) = \sum_{j=0}^{M} \tilde{n}_j \, \beta^{j+1} \, ,$$

where $\tilde{n}_j = n_j$ if $0 \le n_j \le L$ and $\tilde{n}_j = L + j\beta$ if $L+1 \le n_j \le L+$ $+S-1$ with $n_j = L+j$.

For all n such that $(n_j \, , n_{j+1}) \notin E_{L,S}$, from

$$\psi_{n_0,n_1,\dots,n_M}(x) = \beta^n x + \sum_{j=0}^{M} \tilde{n}_j \, \beta^{j+1}$$

it follows that $\Phi_{L,S}(n) = \psi_{n_0,n_1,\dots,n_M}(0)$.

Definition

For any positive integer n, written in its $(L+S)$-radix notation $[n]_{L+S} = n_M n_{M-1} \dots n_0$, we denote by $\{n_{L,S}(n)\}$ the sequence of all positive integers such that $(n_j\,, n_{j+1}) \notin E_{L,S}$.

Theorem (I. Carbone)

For any $L, S \in \mathbb{N}$ and $0 < \beta < 1$ such that $L\beta + S\beta^2 = 1$, the LS-sequence of points $\{\xi^n_{L,S}\}$ is obtained as follows:

$$\{\xi^n_{L,S}\} = \{\Phi_{L,S}(n_{L,S}(n))\}.$$

Theorem (I.Carbone)

1) If $S \leq L$ we have

$$D\left(\xi^1_{L,S}, \xi^2_{L,S}, \dots, \xi^N_{L,S}\right) \leq k_1 \frac{\log N}{N} \quad \textit{for all} \quad n \in \mathbb{N}.$$

2) If $S = L + 1$ we have

$$D\left(\xi^1_{L,S}, \xi^2_{L,S}, \dots, \xi^N_{L,S}\right) \leq k_2 \frac{\log^2 N}{N} \quad \textit{for all} \quad n \in \mathbb{N}.$$

3) If $S \geq L + 2$ we have

$$D\left(\xi^1_{L,S}, \xi^2_{L,S}, \dots, \xi^N_{L,S}\right) \leq k_3 \frac{\log N}{N^\gamma} \quad \textit{for all} \quad n \in \mathbb{N}$$

with $\gamma = 1 + \frac{\log(S\beta)}{\log \beta} < 1$.

- $S \leq L$: low discrepancy sequences of **points**

$$D\left(\xi_{L,S}^1, \xi_{L,S}^2, \ldots, \xi_{L,S}^N\right) \leq \frac{\log N}{N} \quad \textit{for all} \;\; N \in \mathbb{N}\,.$$

- To each low discrepancy *LS*-sequence of partitions $\{\rho_{L,S}^n\}$ corresponds a low discrepancy *LS*- sequence of points $\{\xi_{L,S}^n\}$.

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Definition (Weyl, 1914-1916)

Given $s \geq 2$, a sequence of points $\{\mathbf{x_n}\}$ in $I^s = [0,1[^s$ is said to be uniformly distributed if for all $[\mathbf{a},\mathbf{b}[$ of I^s we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \chi_{[\mathbf{a},\mathbf{b}[}(\mathbf{x}_n) = \lambda([\mathbf{a},\mathbf{b}[).$$

Theorem (Weyl)

A sequence of points $\{\mathbf{x_n}\}$ in I^s is uniformly distributed if for any $f \in \mathcal{C}(\bar{I}^s)$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f(\mathbf{x}_i) = \int_{\bar{I}^s} f(\mathbf{t})\, d\mathbf{t}.$$

Definition

Given $X = \{\mathbf{x_n}\}$ in $[0,1[^s$, the discrepancy of X is defined as

$$D_N(X) = D(\{\mathbf{x}_1, \dots, \mathbf{x}_N\}) = \sup_{[\mathbf{a},\mathbf{b}[} \left| \frac{1}{N} \sum_{i=1}^{N} \chi_{[\mathbf{a},\mathbf{b}[}(x_i) - \lambda([\mathbf{a},\mathbf{b}[) \right|$$

and the star-discrepancy as

$$D_N^*(X) = D^*(\{\mathbf{x}_1, \dots, \mathbf{x}_N\}) = \sup_{[\mathbf{0},\mathbf{b}[} \left| \frac{1}{N} \sum_{i=1}^{N} \chi_{[\mathbf{0},\mathbf{b}[}(x_i) - \lambda([\mathbf{0},\mathbf{b}[) \right| .$$

Theorem (Roth, 1955)

For any finite sequence $X = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ in I^s, $s \geq 2$, we have

$$N\, D_N^*(X) \geq C \;(\log N)^{\frac{s-1}{2}}.$$

For any sequence $X = \{\mathbf{x_n}\}$ in I^s, $s \geq 1$, we have

$$N\, D_N^*(X) \geq C \;(\log N)^{\frac{s}{2}}.$$

Theorem (Schmidt, 1972)

For any finite sequence $X = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ in I^2 we have

$$N\, D_N^*(X) \geq C \log N.$$

- Low discrepancy sequences of N **points** in I^2:

$$D_N^*(X) \leq C\,\frac{\log N}{N}$$

- Low discrepancy sequences of N **points** in I^s, $s > 2$:

$$D_N^*(X) \leq C\,\frac{(\log N)^{s-1}}{N}$$

- Low discrepancy sequences of **points** in I^s, $s \geq 2$:

$$D_N^*(X) \leq C\,\frac{(\log N)^{s}}{N} \quad \text{for all } N \in \mathbb{N}$$

Definition

The van der Corput sequence of order N in I^2 is

$$\left(\frac{n}{N}, \Phi_2(n)\right), \quad n = 0, 1, \ldots, N-1.$$

The Hammersley sequence of order N in I^s is

$$\left(\frac{n}{N}, \Phi_{p_1}(n), \ldots, \Phi_{p_{s-1}}(n)\right), \quad n = 0, 1, \ldots, N-1,$$

where $p_1, \ldots, p_{s-1}$ are the first $s-1$ prime numbers.
The Halton sequence in I^s is

$$\left\{\left(\Phi_{b_1}(n), \ldots, \Phi_{b_s}(n)\right)\right\},$$

where $b_1, \ldots, b_s$ are pairwise relatively prime.

Theorem (van der Corput, 1935)

The van der Corput sequence has low discrepancy:

$$D_N\left(\frac{n}{N}, \Phi_2(n)\right) \leq C\,\frac{\log N}{N}\,.$$

Theorem (Halton, 1960)

The Hammersley sequence in I^s and the Halton sequence in I^s, for any $s \geq 2$, have low discrepancy:

$$D_N^*\left(\frac{n}{N}, \Phi_{p_1}(n), \ldots, \Phi_{p_{s-1}}(n)\right) \leq C\,\frac{(\log N)^{s-1}}{N}\,,$$

$$D_N^*\left(\left\{\left(\Phi_{b_1}(n), \ldots, \Phi_{b_s}(n)\right)\right\}\right) \leq C\,\frac{(\log N)^{s}}{N}\,.$$

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Definition (I. Carbone)

1. For each *LS*-sequence of points $\{\xi^n_{L,S}\}$, the sequence

$$\left\{\left(\xi^n_{L,S}, \frac{n}{N}\right)\right\}, n = 1, \ldots, N-1,$$

is called *LS*-sequence of points à la van der Corput - Hammersley of order N in the unit square.

2. For each pair of *LS*-sequences of points $\{\xi^n_{L_1,S_1}\}$ and $\{\xi^n_{L_2,S_2}\}$, the sequence

$$\{(\xi^n_{L_1,S_1}, \xi^n_{L_2,S_2})\}$$

is called *LS*-sequences of points à la Halton in the unit square.

Theorem

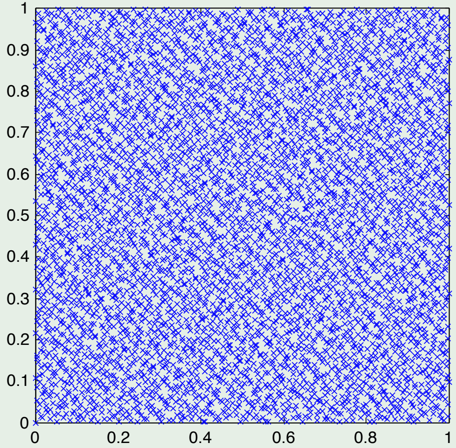
The discrepancy of any LS-sequence of points à la van der Corput-Hammersley $\{(\xi^n_{L,S}, \frac{n}{N})\}$ of order N in the unit square coincides with the discrepancy of $\{\xi^n_{L,S}\}$.

A consequence of the previous theorem is the following

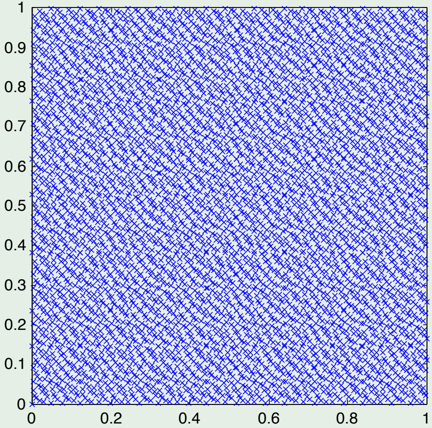
Theorem (I.Carbone)

The LS-sequence of points à la van der Corput-Hammersley $\{(\xi^n_{L,S}, \frac{n}{N})\}$ of order N has **low discrepancy** *whenever $L \geq S$.*

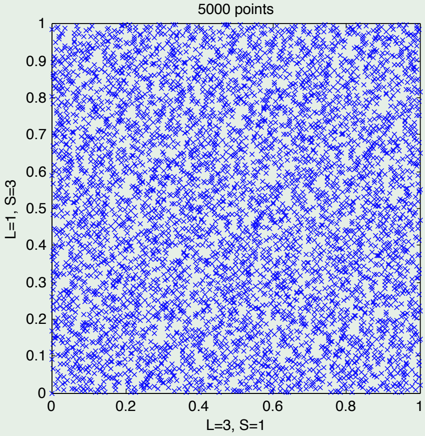
Halton sequence $\{(\Phi_n(2), \Phi_n(3))\}$ for $n \leq 5000$



LS-sequence à la van der Corput-Hammersley $\left\{\left(\xi_{1,1}^n, \frac{n}{5000}\right)\right\}$



LS-sequence à la Halton $\left\{(\xi^n_{1,3}, \xi^n_{3,1})\right\}$ for $n \leq 5000$



LS-sequence à la Halton $\left\{\left(\xi^n_{1,1}, \xi^n_{4,1}\right)\right\}$ for $n \leq 5000$