Eulerian and 2-Edge Connected Spanning Subgraphs via

Matchings, Matroids and Extensions

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min Eulerian spanning: 7/5-approximation
" Eulerian s,t – trail " : 3/2

" 2-Edge Connected " : 4/3

TSP: metrics & tours

K complete graph $c: E(K) \to \mathbb{R}_+$, minimize:

Travelling salesman tour

c is a metric

Conjecture. 4/3-approx

~

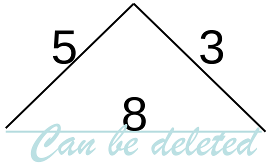
Spanning Eul. in 2G

c is arbitrary

Conjecture : 4/3-approx

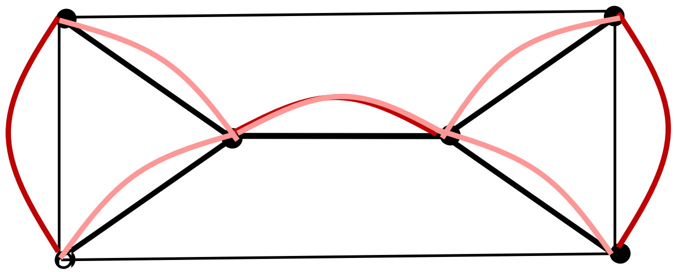
Degrees =2
Tour : equiv for c and its metric compl
Equivalence of the two if c is a metric,
- **putting two copies of every edge** -

Relaxation, advantage :



- no restriction on c
- equivalence with a sparser graph
- **has a cardinality case $c \equiv 1$.**

Matchings, Matroids and Extensions



$J \subseteq E(G)$ is a *T-join*, if T = set of odd degree vertices of J
Edmonds (1965)

Fact: G connected, $|T|$ even $\Rightarrow \exists$ T-join

Christofides' tour : c-min spanning tree F + c-min T_F-join
where T_F is the set of odd degree vertices of F.

tour of G: connected $\emptyset$ - join in 2G; *(s,t)-tour " " conn. {s,t}-join in 2G*

INPUT : G graph
OUTPUT : tour of min size

in G NP-hard
to approximate !
(HAM in 3-reg)

Min weight (connected) T-joins

$\tau(G,T,w)$ good characterization theorems via cut packings
Edmonds-Johnson (1973), Lovász (1975), Seymour (1981), ...

c: $E(G) \to \mathbb{R}_+$; opt (G,T,c):= opt value; OPT(G,T,c):= an optimal solution (in 2G)

Christofides' tour: min spanning tree F + min T_F-join

(s,t)-tour: min sp.tree F+min $T_F \Delta \{s,t\}$-join ≤ 5/3 OPT (Hoogeveen)

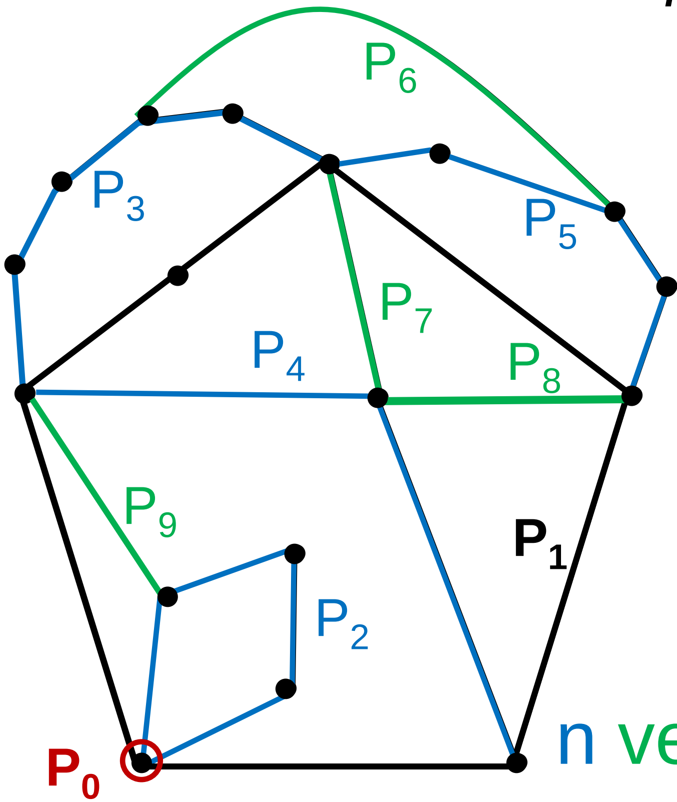
connected T-join: " + min $T_F \Delta T$ - join ≤ 5/3 OPT

Proof 1.) tours: OPT conn, has a T_F-join F; OPT \ T_F-join = T_F-join
2.) In OPT + F there are 3 disjoint connected T-joins

tour : all degrees even & connected
relaxation : 2-edge-connected

2-Edge-Connected Spanning Subgraphs

Revise *ear-decompositions*: orderdered partition of E(G)



$G = P_0 + P_1 + P_2 + \ldots + P_k$

1-ears last ,

pendant ear : ‘last’ not 1-ear.

∃ ear-decomp ⇔ 2-edge-conn

(∃ open ear-decomp ⇔ 2-vertex-conn)

n vertices m edges: $k = m - n + 1$ ears

The longer the ears, the smaller the quotient n. of edges / vertices

π_i :=number of i-ears

Min n.of edges in 2ECSS ≤ 2n;
2-approx alg for 2ECSS: delete all 1-ears!

It is also $\leq 5/4(n-1)+\pi_2+ 1/2\pi_3$
2-ears and 3-ears are the worst.

Champions

conj:

Christofides 1976 3/2 **tour** ∀weight 4/3

Saberi, Singh 2011 3/2 - ε tour | |

Mömke, Svensson 2011 1,461 tour | |

Mucha 2011 1.444 =13/9

Hoogeveen 1991 5/3 **s,t tour** ∀weight 3/2

Easy : 5/3 **conn Tj** ∀weight 3/2

An, Kleinberg, Shmoys: ’11 1.619 s,t tour ∀weight

1.578 s,t tour | |

Khuller, Vishkin 1994 3/2 **2ECSS** | | 4/3

Cheriyan,S.,Szigeti 1998 17/12 2ECSS | |

S., Vygen: | | 2012 7/5 for tours, 3/2 for connTj, 4/3 for 2ECSS

The algorithms

For *opt size tours, for 2ECSS, for (s,t)-tours*

1. Minimize the n. of even ears : φ denotes this min.

$n-1+\varphi$ is (an LP) lower bound

$\frac{1}{2}(n-1+\varphi)$ edges are sufficient to correct parity.

2. Ear splicing: while keeping φ even ears,

make all short (2- and 3-) ears pendant, and without edges between them, i.e. forming an *eardrum,* i.e. the comps of an induced 

3. Optimize short ears (all are pendant !) for connecting

Use 2- and 3-ears for connecting with

matroid intersection lower bound and use same amount for part connectivity

FIX THE EAR DECOMPOSITION

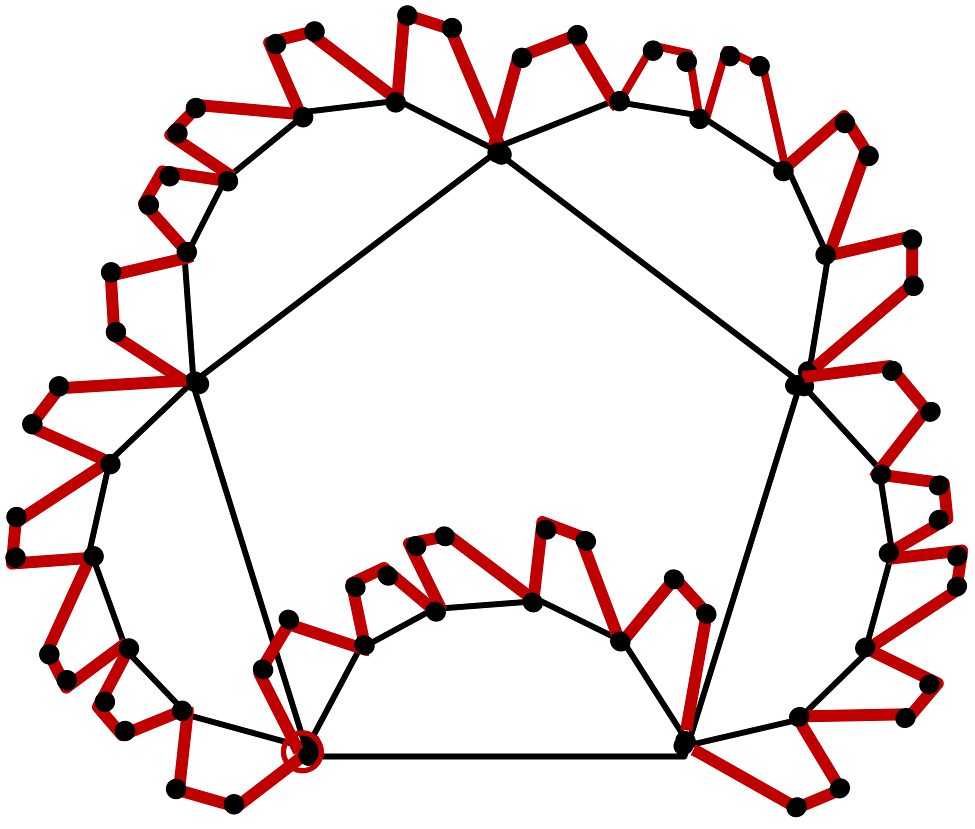
4. Optimize parity : find a tour or conn T-join or 2ECSS

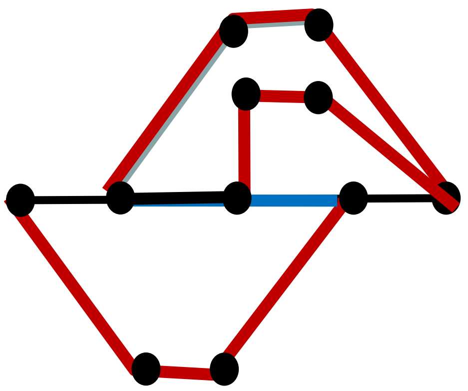
with « optimized Christofides » if many pendant or other methods if not many

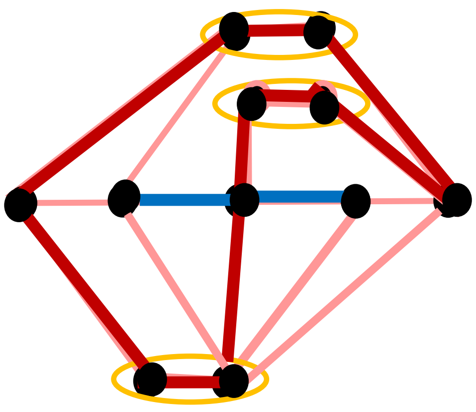
Examples for 2ECSS

Zoli's example in

Cheriyan, S., Szigeti (1998)



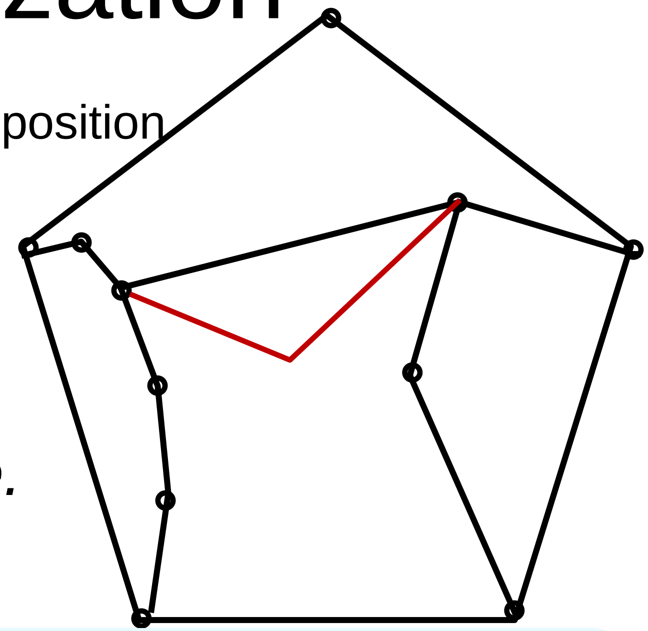




Idea: Let all short (2- and 3-) ears be all pendant &

changed s.t. useful for connecting.

1 & 4 : Even ear minimization



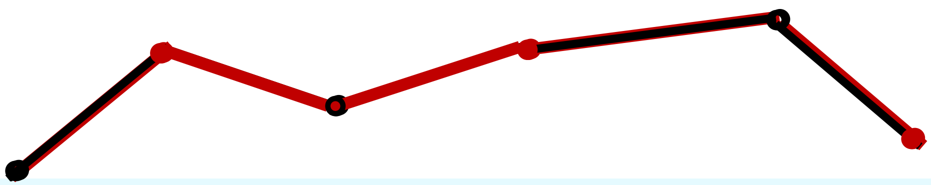
G 2-connected : φ(G) min n. of even ears in an ear-decomposition

Lovász : 'factor-critical' ⇔ ∃ odd ear-decomp (⇔ φ(G)=0)
∃ open odd ear-decomp ⇔ 2-vertex-conn factcrit

These ear-decomp can be found in polytime.

Frank's minmax thm: G 2-connected,T⊆ V(G), |T| even
τ(G,T) ≤1/2(n -1+φ) and *there exists T with equality*;
This T and φ can be found in polytime.

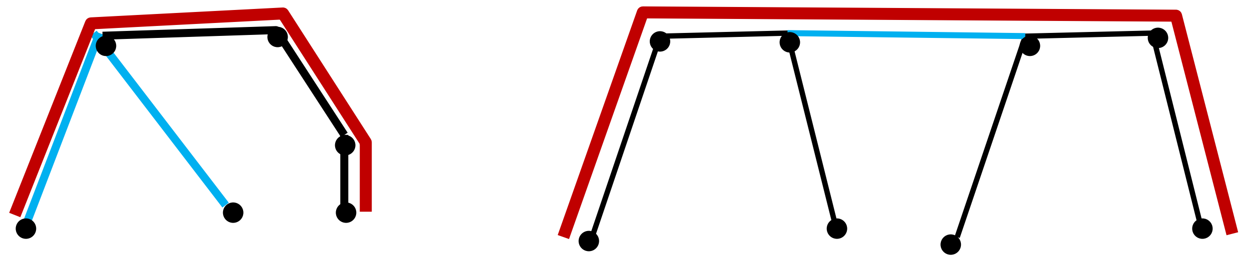
Lφ := n-1+φ



COROLLARY : ½ Lφ is an upper bound for parity correction, and at the same time Lφ is cut packing type LP lower bound for opt.

2. Nice ear-decomposition

Splicing:

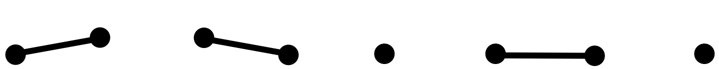


…

An ear-decompositon is *nice* if

- the number of even ears is min
- all *short* (2- and 3-) ears are *pendant* (‘last’ before 1-)
- Internal vertices of different short ears are non-adjacent

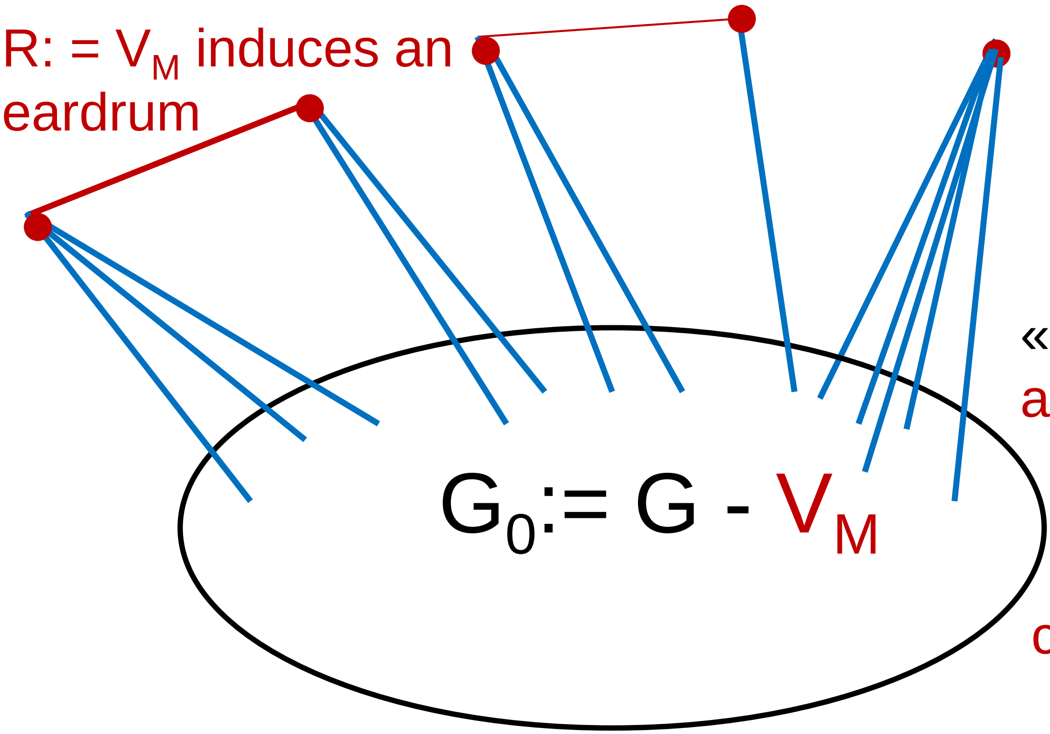
i.e. they form an eardrum



Proposition: If G 2-connected, ∃ nice ear-decomp.

3. Rerout short pendant ears

R: = V_M induces an eardrum



Short ears are bad, but easy to change for other short ears

« nice » ⇒ middle vertices induce an eardrum M on vertex-set V_M

Change so that the graph consisting of the short ears has min n. of components

SHORT EAR OPTIMIZATION

Minimize | F | so that

- all degrees of vertices in V_M are even in F
- (V(G), F) is connected

Parity and connectivity

Input : G graph, R $\subseteq$ V(G)

Output : Find a min size subgraph of 2G where

- all degrees of vertices in R are even
- which is connected on V(G)

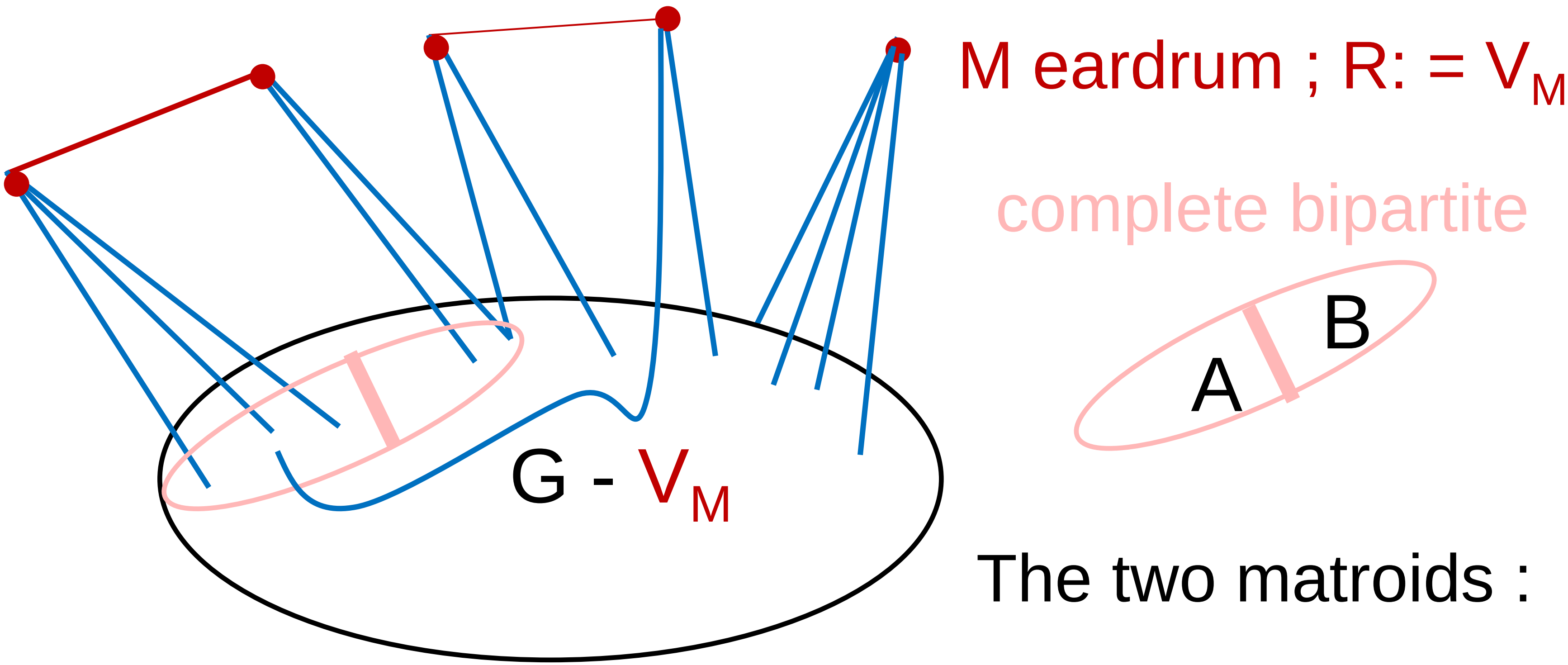
NP –hard : for R=V(G) it is the min tour problem

Special case in P: R is an eardrum 

i.e. comp. of R are vertices or edges & G-R connected

opt ≥ opt_R := the opt of this problem

Solution: $\cap$ of 2 matroids Edmonds(1970)



The two matroids :

1. An edge of each complete bipartite (partition matroid)

2. Cycle-matroid on $V \setminus V_M$

μ:= max of $\cap$; L_μ :=n-1+ (M-μ)

Theorem : $opt_R = L_\mu$; L_μ is an LP lower bound for opt.

Proof : Use Edmonds' $\cap$ thm & alg for proving the theorem & a pol alg.

Where did we get ?

π := number of pendant ears

Theorem 1: ∃ conn Tj of size $L_\mu + ½ L_\varphi - \pi$; ∈ P

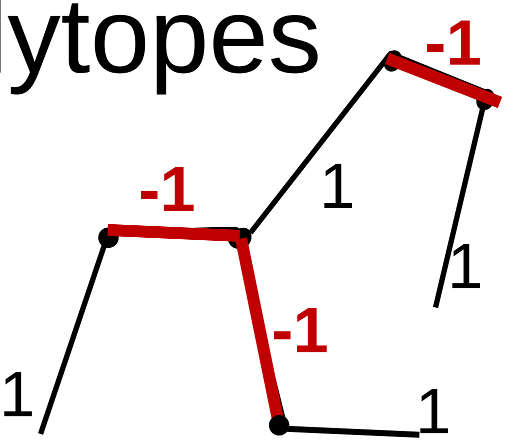
Proof. Optimized Christofides' : connection $=L_\mu$; parity correction ~½ $L\varphi$;
A credit of 1 edge is included in these for each short and each even ear ;
Long odd pendant ears are ≥ 5, 5-ears increase opt with ≥ 4, 3/2 OPT by 6=5+1.

Theorem 2.1: ∃ conn Tj of size 3/2 OPT + π_2 - ½ φ
(ear-induction)

Theorem 2.2: ∃ 2ECSS 5/4 OPT + $\pi/2$ (we saw it)

Theorem 2.3: ∃ tour 4/3 OPT + $2\pi/3$
(corollary of 'Mömke & Svensson's lemma')

Mömke-Svensson Lemma with Ear-theorems, T-join polytopes



Disjoint pairs of incident edges in v, d(v)≠2 s.t.
Deleting 1 from each pair: connected

π := number of pendant ears

Lemma: G 2-vertex-connected $\Rightarrow$ there exists a tour of size 4/3 (n-1) + 2/3 π & comb-poly

Proof. Face of the T-join **polytope** :

$x(F) - x(C \setminus F) \leq |F| - 1$ for any cut C and $F \subseteq C$

$x(\delta(v)) = 1$ for $v \in$ a subset of V(G) ; with a slight modification G' this expresses $x(P) \leq 1$ for all pairs.

G' 2-edge-connected $\Rightarrow$ $x \equiv$ **1/3** is in the **polytope**.

Problems

- 7/5 → 4/3 for min size tours ?

- Other tractable special cases of the « parity and connectivity » problem.

- improve the ratios for weighted problems

- Weighted 2ECSS in 2G and G ?

- Special cases of (weighted) ‘P & C’

- Questions about the integrality gap by spanning tree polytope ∩ T-join polytope