# Discrete Legendre Duality in Matrix Pencils

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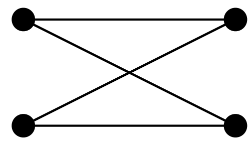
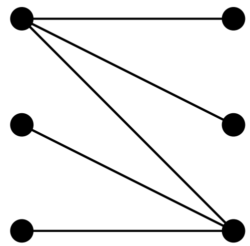
General interest 1

# Linear Algebra $\Longleftrightarrow$ Combinatorics

- rank: $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\leq$ term-rank: 

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$



max matching

- Tutte matrix − nonbipartite matching
- rigidity matrix − truss structure
- mixed matrix − electric circuit

**Combinatorial approach**

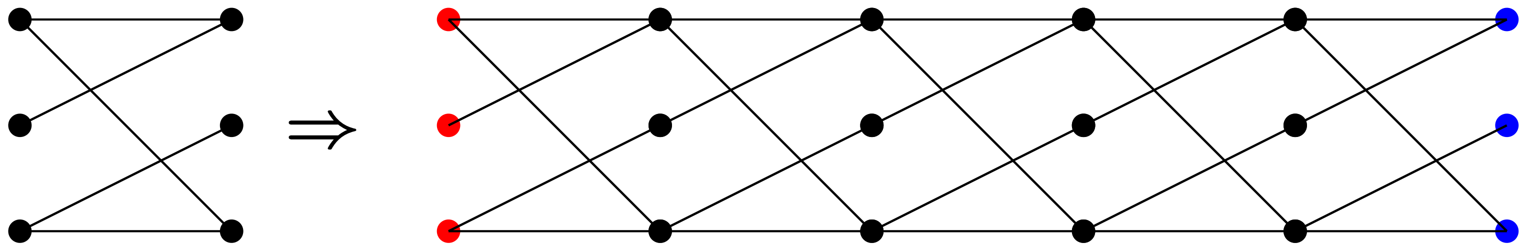
numerical − algebraic − combinatorial

computability, tightness: $=$ in $\leq$

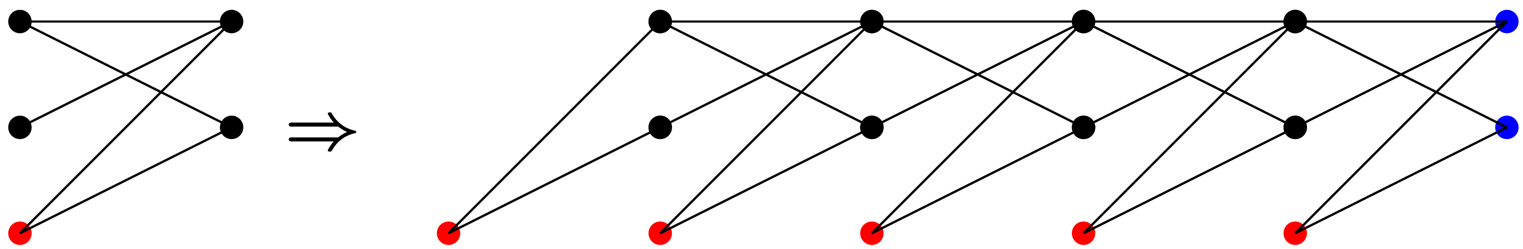
General interest 2

# Periodic Structures (graph/matroid)

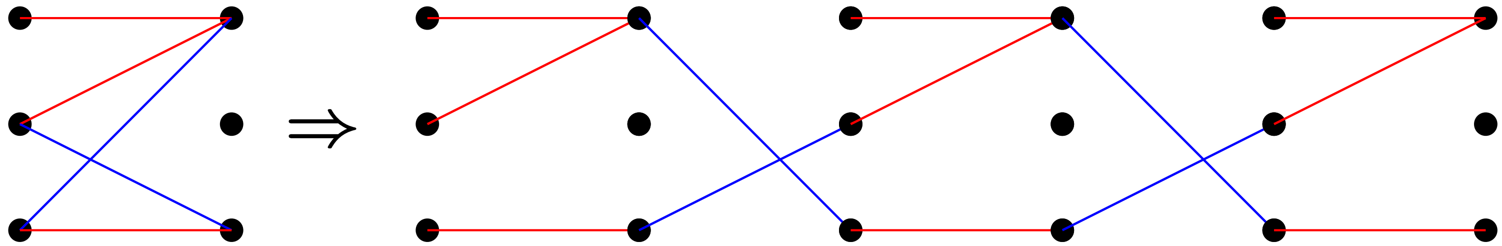
$A - A - A - A - A$ "eigenset" Murota (1990)



$(A, B) - (A, B) - (A, B)$ "controllability" Murota (1988)



$A - B - A - B - A$ Iwata-Shimizu (2007), Iwata (2007)



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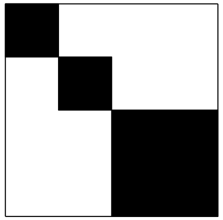
— Valuated Bimatroid

# Matrix Pencil
# — Kronecker Canonical Form

# Matrix Pencil and Equivalence

**matrix pencil:** $sA + B$ $\left(s \leftrightarrow \frac{d}{dt}\right)$

**Kronecker canonical form:**

block-diagonal form  by

$$U(sA + B)V$$

$U$, $V$: nonsingular (constant)

**strict equivalence**

# Kronecker Form

$$
\begin{bmatrix}
s+2 & 1 & & & & & & & & & & \\
0 & s+2 & & & & & & & & & & \\
 & & s & 1 & & & & & & & & \\
 & & 0 & s & & & & & & & & \\
 & & & & 1 & s & 0 & & & & & \\
 & & & & 0 & 1 & s & & & & & \\
 & & & & 0 & 0 & 1 & & & & & \\
 & & & & & & & s & 1 & 0 & & \\
 & & & & & & & 0 & s & 1 & & \\
 & & & & & & & & & & s & 0 \\
 & & & & & & & & & & 1 & s \\
 & & & & & & & & & & 0 & 1
\end{bmatrix}
$$

$N_3 \rightarrow$ (the 3×3 block) nilpotent

# Kronecker Form

$$\left[\begin{array}{cc|cc|ccc|ccc|cc}
s+2 & 1 & & & & & & & & & & \\
0 & s+2 & & & & & & & & & & \\
\hline
 & & s & 1 & & & & & & & & \\
 & & 0 & s & & & & & & & & \\
\hline
 & & & & 1 & s & 0 & & & & & \\
 & & & & 0 & 1 & s & & & & & \\
 & & & & 0 & 0 & 1 & & & & & \\
\hline
 & & & & & & & s & 1 & 0 & & \\
 & & & & & & & 0 & s & 1 & & \\
\hline
 & & & & & & & & & & s & 0 \\
 & & & & & & & & & & 1 & s \\
 & & & & & & & & & & 0 & 1
\end{array}\right]$$

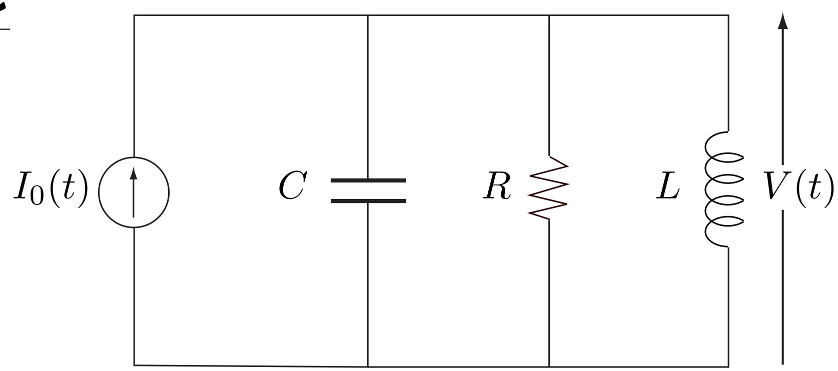
$N_3 \rightarrow$
nilpotent

$$N_\mu(s) = \begin{bmatrix} 1 & s & 0 & \cdots & 0 \\ 0 & 1 & s & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 & s \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_d \geq 1$$

indices of nilpotency

# Simple RLC circuit




| currents | | | voltages | | |
|---|---|---|---|---|---|
| 1 | 1 | 1 | | | |
| | | | 1 | −1 | 0 |
| | | | 0 | 1 | −1 |
| −1 | 0 | 0 | $sC$ | 0 | 0 |
| 0 | $R$ | 0 | 0 | −1 | 0 |
| 0 | 0 | $sL$ | 0 | 0 | −1 |

$$= sA + B$$

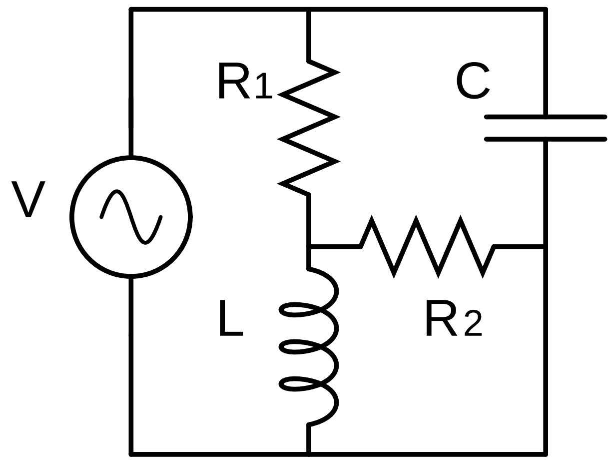
| | | | | | |
|---|---|---|---|---|---|
| $s$ | $-1/L$ | | | | |
| $1/C$ | $s + 1/(RC)$ | | | | |
| | | 1 | | | |
| | | | 1 | | |
| | | | | 1 | |
| | | | | | 1 |

**Kronecker form**

$$\text{diag}\,(H(s), 4 \times N_1(s))$$

$$N_\mu, \;\; \mu = 1, 1, 1, 1$$

# Another electric circuit




$$sA + B =$$

$$\left[\begin{array}{ccccc|ccccc}
1 & -1 & 0 & 0 & -1 & & & & & \\
-1 & 0 & 1 & 1 & 1 & & & & & \\
\hline
 & & & & & -1 & 0 & 0 & 0 & -1 \\
 & & & & & 0 & 1 & 1 & 0 & -1 \\
 & & & & & 0 & 0 & -1 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & R_1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & R_2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & sL & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & sC
\end{array}\right]$$

**Kronecker form:**

$$\operatorname{diag}\left(\left[\, s + \frac{R_1 R_2}{L(R_1 + R_2)} \,\right], \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}, [1], [1], \cdots, [1], [1]\right)$$

**nilpotent block** $N_\mu$, $\mu = 2, 1 \cdots, 1$

# Structural Indices

**pencil**

$sA + B$

**Kronecker form**

$U(sA + B)V$

nilpotency $\mu_1 \geq \cdots \geq \mu_d$

**degree: subdet**

$\delta_1, \delta_2, \delta_3, \ldots$

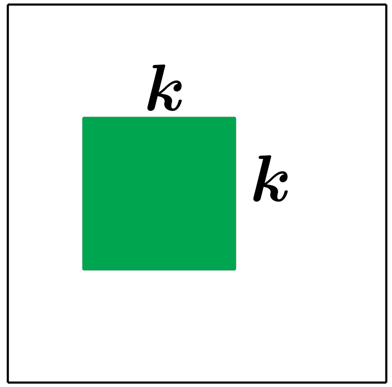
**rank: expanded mtx**

$\theta_1, \theta_2, \theta_3, \ldots$

# Degree $\delta_k$ and rank $\theta_k$

$\delta_k$ = max degree in $s$ of a $k \times k$ minor of $sA + B$

$$(k = 0, 1, \ldots, r = \mathrm{rank}(sA + B))$$




$$\theta_k = \mathrm{rank} \begin{bmatrix} A & & & & \\ B & A & & & \\ & B & \ddots & & \\ & & \ddots & A & \\ & & & B & A \end{bmatrix}$$

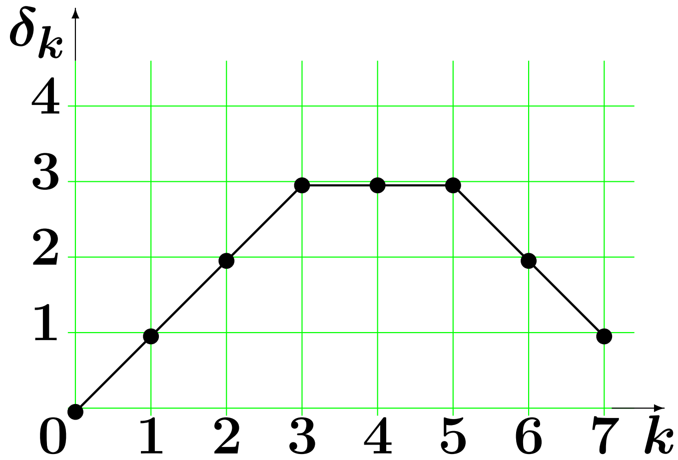
expanded matrix ($k$ blocks)

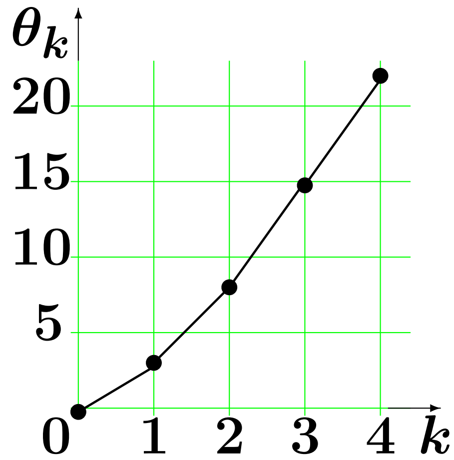
concave:

$\delta_{k-1} + \delta_{k+1} \le 2\delta_k$

convex:

$\theta_{k-1} + \theta_{k+1} \ge 2\theta_k$



# Fundamental Relations (linear algebra)

pencil

$sA + B$

Kronecker form

$U(sA + B)V$

nilpotency $\mu_1 \geq \cdots \geq \mu_d$

$$\delta_k = k - \sum_{i=r-k+1}^{d} \mu_i$$

$$\theta_k = kr - \sum_{i=1}^{d} \min(k, \mu_i)$$

degree: subdet

$\delta_1, \delta_2, \delta_3, \ldots$

rank: expanded mtx

$\theta_1, \theta_2, \theta_3, \ldots$

# Combinatorial Bounds — Genericity and Tightness

# Combinatorial Bounds and Tightness

| | linear algebraic | | combinatorial graph/matroid |
|---|---|---|---|
| degree | $\delta_k$ | $\leq$ | $\hat{\delta}_k$ |
| rank | $\theta_k$ | $\leq$ | $\hat{\theta}_k$ |
| index | $\mu_k$ | $\approx$ | $\hat{\mu}_k$ |

- What combinatorial bounds ?
- When "=" ? (tightness)
- Efficiently computable ?

$$\delta_k = k - \sum_{i=r-k+1}^{d} \mu_i, \qquad \theta_k = kr - \sum_{i=1}^{d} \min(k, \mu_i)$$

# Combinatorial Bounds: known results

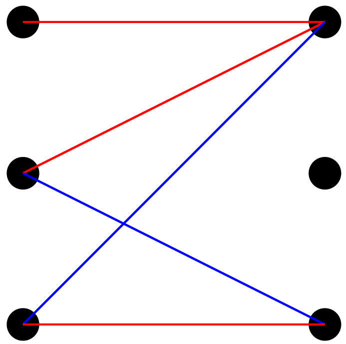
Thm: "=" holds under some genericity assumption

| | method | pencil | |
|---|---|---|---|
| $\delta_k \leq \hat{\delta}_k$ | weighted bipart. matching | generic | folklore Murota(1995) |
| (degree) | wtd./valuated matroid intersection | mixed | Iri, Recski (1970's) Murota (1999) |
| $\theta_k \leq \hat{\theta}_k$ | bipartite matching | generic | Iwata-Shimizu (2007) |
| (rank) | matroid intersection | mixed | Iwata-Takamatsu (2011) |

? Relation btwn "$\delta_k = \hat{\delta}_k$" and "$\theta_k = \hat{\theta}_k$"

# Graph Representation

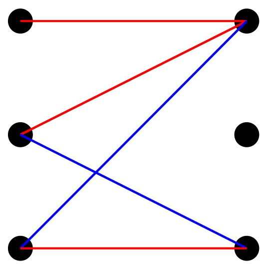
$$sA + B = \begin{bmatrix} sa_1 & 0 & 0 \\ sa_2 & 0 & b_1 \\ b_2 & 0 & sa_3 \end{bmatrix}$$



**edge weight:** $A \to 1 \quad B \to 0$

# Graph-theoretic Bounds for $sA + B$

$\hat{\delta}_k^{\mathrm{g}}$ = max-weight $k$-matching    Folklore, M.(1995)

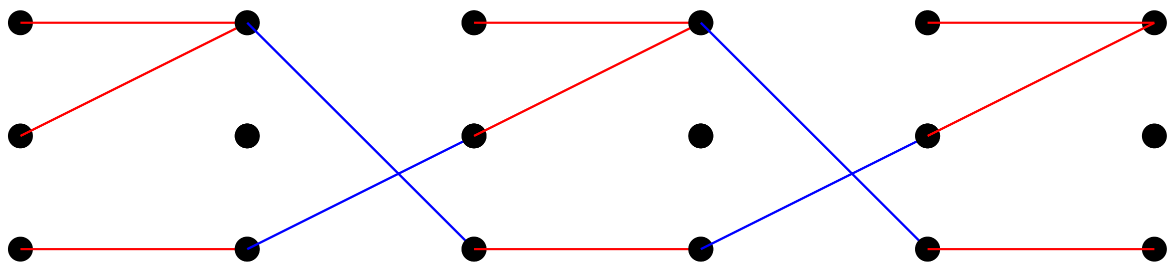


weight: $A \to 1 \quad B \to 0$

easy Thm: $\delta_k = \hat{\delta}_k^{\mathrm{g}}$ for generic pencils

$\hat{\theta}_k^{\mathrm{g}}$ = max-size matching    Iwata-Shimizu (2007)

$A - B - A - B - A - \cdots - B - A$   ($k \times A$'s)



Thm: $\theta_k = \hat{\theta}_k^{\mathrm{g}}$ for generic pencils

# Result (graph-theoretic bounds)

$$\begin{array}{|ccc|} \hline \delta_k & \leq & \hat{\delta}_k \\ \updownarrow & & \updownarrow \\ \theta_k & \leq & \hat{\theta}_k \\ \hline \end{array}$$

$\delta_k =$ max degree of $k \times k$ minor

$$\theta_k = \mathrm{rank} \begin{bmatrix} A & & & \\ B & A & & \\ & B & \ddots & \\ & & B & A \end{bmatrix}$$

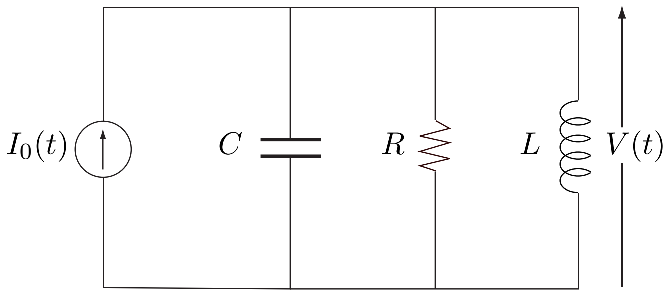
**Equivalence of tightness:**

For any $sA + B$ (without genericity)

$$\delta_k = \hat{\delta}_k^{\mathrm{g}} \ (\forall k) \iff \theta_k = \hat{\theta}_k^{\mathrm{g}} \ (\forall k)$$

# Mixed Matrix Pencil

$$sA + B = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & & & \\ \hline & & & 1 & -1 & 0 \\ & & & 0 & 1 & -1 \\ \hline -1 & 0 & 0 & sC & 0 & 0 \\ 0 & R & 0 & 0 & -1 & 0 \\ 0 & 0 & sL & 0 & 0 & -1 \end{array}\right]$$




$$A = Q_A + T_A$$

$$B = Q_B + T_B$$

**mixed matrix = const mtx + generic mtx**

rank → matroid intersection — Murota-Iri (1985)

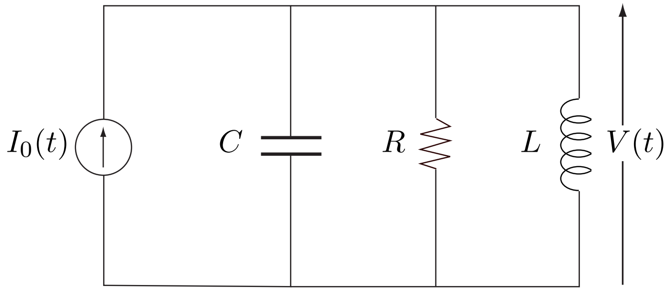
**mixed pencil = const penc + generic penc**

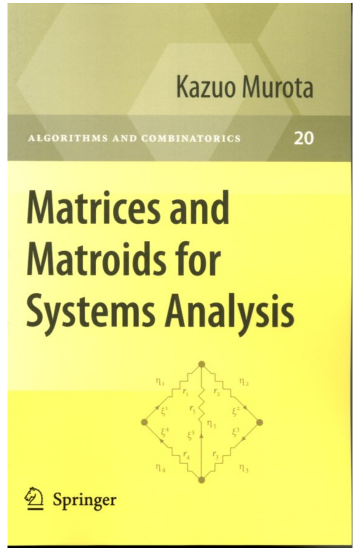
degree det → valuated matroid intersection

Murota (1999)

# Mixed Matrix Pencil

$$sA + B = \left[\begin{array}{ccc|ccc} \color{red}{1} & \color{red}{1} & \color{red}{1} & & & \\ \hline & & & \color{red}{1} & \color{red}{-1} & 0 \\ & & & 0 & \color{red}{1} & \color{red}{-1} \\ \hline \color{red}{-1} & 0 & 0 & s\color{blue}{C} & 0 & 0 \\ 0 & \color{blue}{R} & 0 & 0 & \color{red}{-1} & 0 \\ 0 & 0 & s\color{blue}{L} & 0 & 0 & \color{red}{-1} \end{array}\right]$$




$$A = \color{red}{Q_A} + \color{blue}{T_A}$$

$$B = \color{red}{Q_B} + \color{blue}{T_B}$$

**mixed matrix = const mtx + generic mtx**

rank → matroid intersection — Murota-Iri (1985)

**mixed pencil = const penc + generic penc**

degree det → valuated matroid intersection

Murota (1999)

# Matroid-theoretic Bounds for $sA + B$

$A = Q_A + T_A$ (no genericity in $T_A, T_B$)

$B = Q_B + T_B$ $\Rightarrow$ formal mixed matrix pencil

$$sA + B = (sQ_A + Q_B) + (sT_A + T_B)$$

$\hat{\delta}_k^{\mathrm{m}} \rightarrow$ valuated matroid intersection Murota (1999)

$\rightarrow$ weighted — (RLC circuit) Iri, Recski (1970's)

$$\hat{\theta}_k^{\mathrm{m}} = \mathrm{rank}\left( \begin{bmatrix} Q_A & & & \\ Q_B & Q_A & & \\ & Q_B & Q_A & \\ & & Q_B & Q_A \end{bmatrix} + \begin{bmatrix} T_A & & & \\ T_B & T'_A & & \\ & T'_B & T''_A & \\ & & T''_B & T'''_A \end{bmatrix} \right)$$

$\rightarrow$ matroid intersection Iwata-Takamatsu (2011)

# Result (matroid-theoretic bounds)

$$\begin{array}{|ccc|} \hline \delta_k & \leq & \hat{\delta}_k \\ \updownarrow & & \updownarrow \\ \theta_k & \leq & \hat{\theta}_k \\ \hline \end{array}$$

$\delta_k =$ max degree of $k \times k$ minor

$$\theta_k = \mathrm{rank} \begin{bmatrix} A & & & \\ B & A & & \\ & B & \ddots & \\ & & B & A \end{bmatrix}$$

**Equivalence of tightness:**

For a formal mixed matrix $sA + B$

$$\delta_k = \hat{\delta}_k^{\mathrm{m}} \ (\forall k) \quad \Longleftrightarrow \quad \theta_k = \hat{\theta}_k^{\mathrm{m}} \ (\forall k)$$

# Discrete Legendre Transform — Discrete Convexity

# Fundamental Relations (linear algebra)

**pencil**

$sA + B$

**Kronecker form**

$U(sA + B)V$

nilpotency $\mu_1 \geq \cdots \geq \mu_d$

$$\delta_k = k - \sum_{i=r-k+1}^{d} \mu_i$$

$$\theta_k = kr - \sum_{i=1}^{d} \min(k, \mu_i)$$

**degree: subdet**

$\delta_1, \delta_2, \delta_3, \ldots$

**rank: expanded mtx**

$\theta_1, \theta_2, \theta_3, \ldots$

# Fundamental Relations (linear algebra)

**pencil**

$sA + B$

**Kronecker form**

$U(sA + B)V$

nilpotency $\mu_1 \geq \cdots \geq \mu_d$

$\nearrow$ $\nwarrow$

$$\delta_k = k - \sum_{i=r-k+1}^{d} \mu_i$$

$$\theta_k = kr - \sum_{i=1}^{d} \min(k, \mu_i)$$

$\swarrow$ $\searrow$

**degree: subdet**

$\delta_1, \delta_2, \delta_3, \ldots$

**rank: expanded mtx**

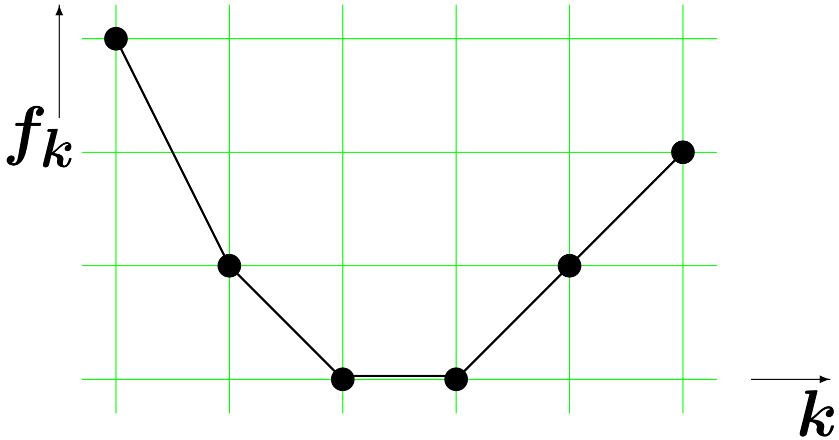
$\theta_1, \theta_2, \theta_3, \ldots$

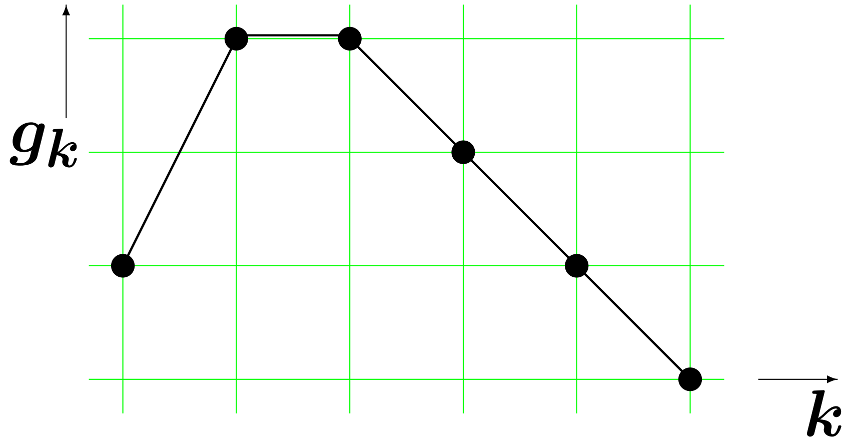
**concave** $\leftarrow$ **Legendre** $\rightarrow$ **convex**

# Discrete Legendre Transformation

**Convex seq.** $f_{k-1} + f_{k+1} \geq 2\, f_k$ **(int-valued)**

**Concave seq.** $g_{k-1} + g_{k+1} \leq 2\, g_k$ **(int-valued)**



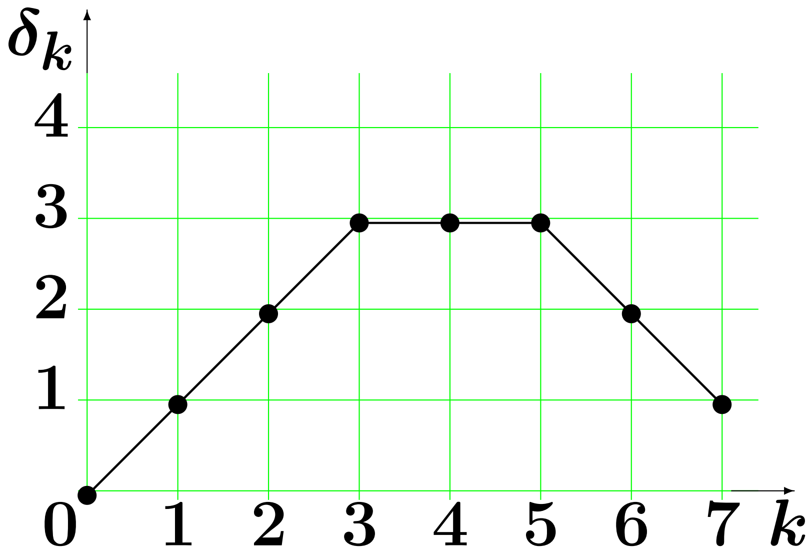



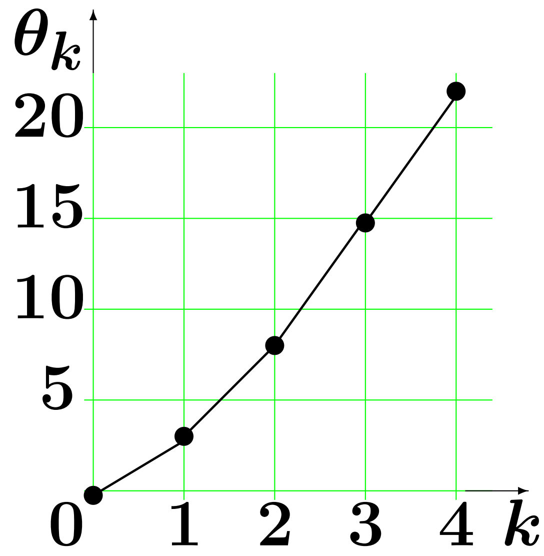

**Thm:** $g_k = \inf_l (f_l - kl) \Leftrightarrow f_k = \sup_l (g_l + kl)$

conjugate

# Conjugacy in Matrix Pencils

**degree $\delta_k$: concave** | **rank $\theta_k$: convex**




**Fact ($\delta$-$\theta$ conjugacy)**

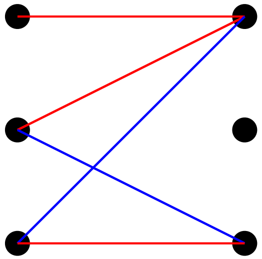
$$\delta_k = \min_l(\theta_{l+1} - kl), \qquad \theta_{k+1} = \max_l(\delta_l + kl)$$

**Proof** $\delta_k \longleftrightarrow \mu_i \longleftrightarrow \theta_k$ (explicit formulas)

Recall

# Graph-theoretic Bounds for $sA + B$

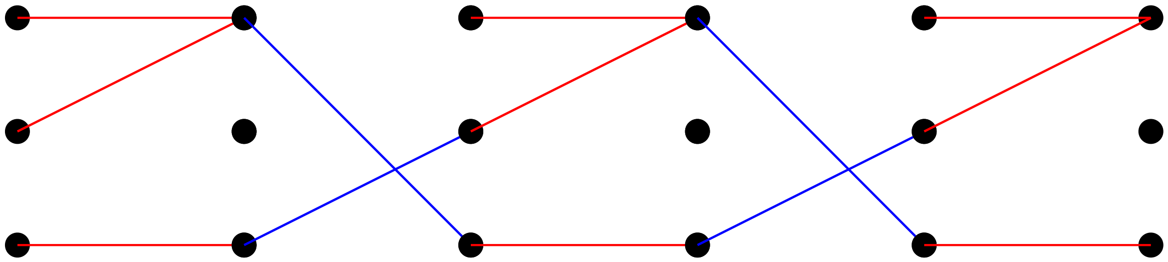
$\hat{\delta}_k^{\mathbf{g}}$ = max-weight $k$-matching    Folklore, M.(1995)



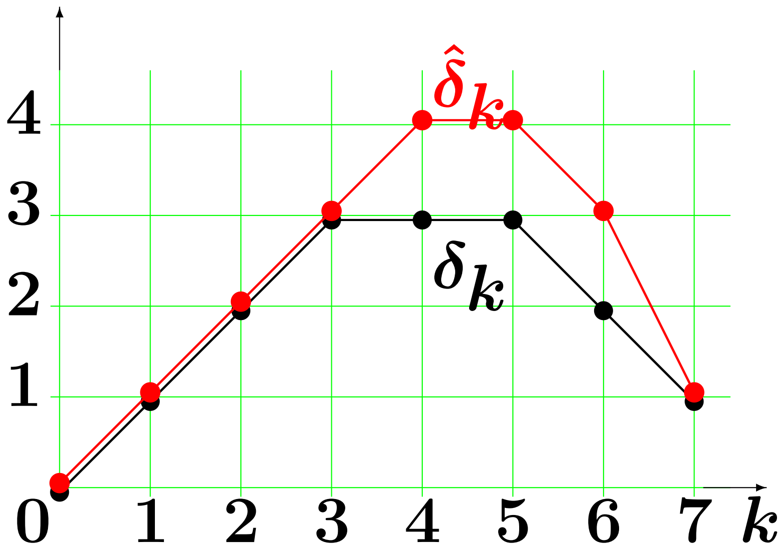
weight: $A \to 1 \quad B \to 0$

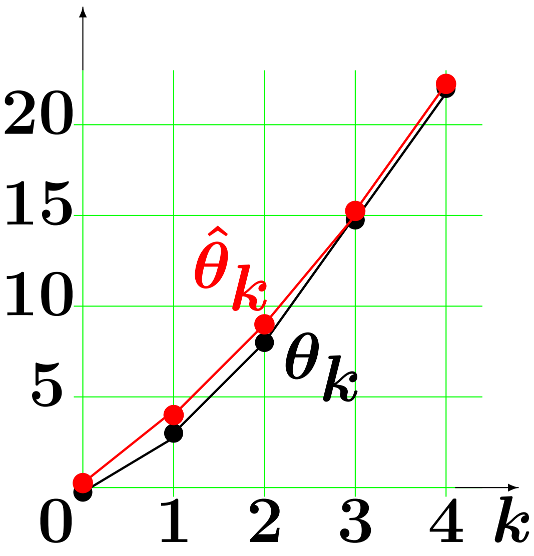
$\hat{\theta}_k^{\mathbf{g}}$ = max-size matching    Iwata-Shimizu (2007)

$A - B - A - B - A - \cdots - B - A \quad (k \times A\text{'s})$



# Thm 1: conjugacy of comb. bounds



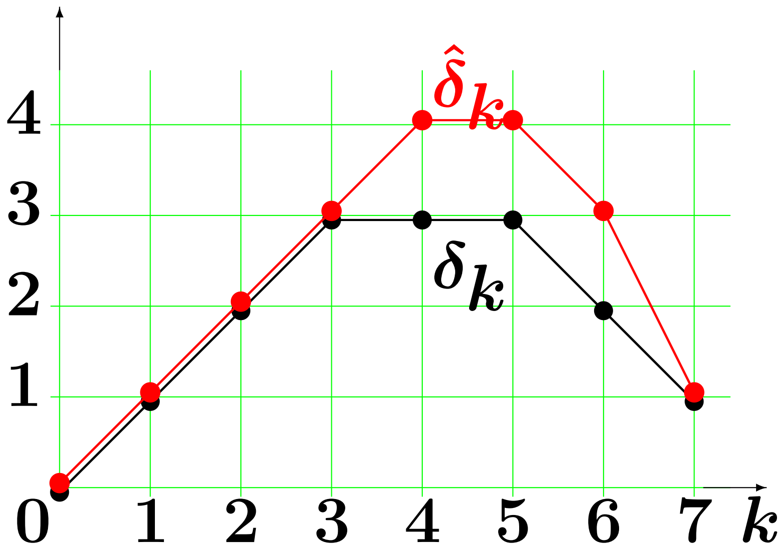



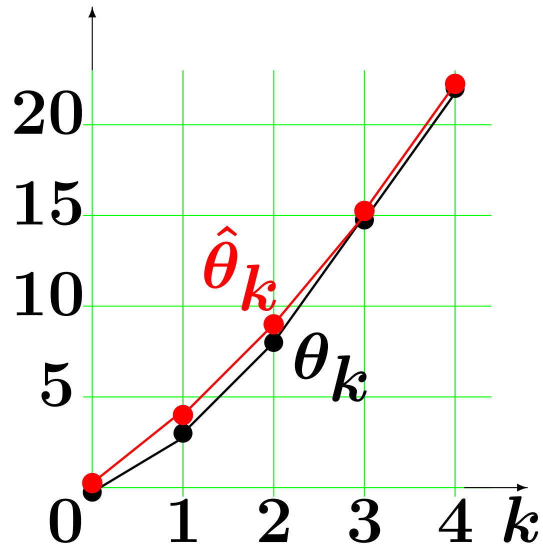

$$\hat{\delta}_k = \min_l (\hat{\theta}_{l+1} - kl), \qquad \hat{\theta}_{k+1} = \max_l (\hat{\delta}_l + kl)$$

$\hat{\delta}_k^{\mathbf{g}}, \hat{\delta}_k^{\mathbf{m}}$ $\hat{\theta}_{k+1}^{\mathbf{g}}, \hat{\theta}_{k+1}^{\mathbf{m}}$

| | | | | | |
|---|---|---|---|---|---|
| combinatorial | $\hat{\delta}_k$ | $\leftarrow$ | conj | $\rightarrow$ | $\hat{\theta}_k$ |
| | $\vee\vert$ | | | | $\vee\vert$ |
| linear algebraic | $\delta_k$ | $\leftarrow$ | conj | $\rightarrow$ | $\theta_k$ |

# Thm 2: equivalence of tightness




$$\delta_k = \hat{\delta}_k \quad (\forall k) \quad \Longleftrightarrow \quad \theta_k = \hat{\theta}_k \quad (\forall k)$$

$\hat{\delta}_k^{\mathrm{g}}, \hat{\delta}_k^{\mathrm{m}}$ $\hat{\theta}_k^{\mathrm{g}}, \hat{\theta}_k^{\mathrm{m}}$

| | |
|---|---|
| combinatorial | $\hat{\delta}_k \leftarrow \text{conj} \rightarrow \hat{\theta}_k$ |
| | $\vee\!\mid \qquad\qquad \vee\!\mid$ |
| linear algebraic | $\delta_k \leftarrow \text{conj} \rightarrow \theta_k$ |

# Results (summary)

$\delta_k$ = max degree of $k \times k$ minor

$$\theta_k = \mathrm{rank} \begin{bmatrix} A & & & \\ B & A & & \\ & B & \ddots & \\ & & B & A \end{bmatrix}$$

Legendre

$$\begin{array}{ccc} \delta_k & \le & \hat{\delta}_k \\ \updownarrow & & \updownarrow \\ \theta_k & \le & \hat{\theta}_k \end{array}$$

**Equivalence of tightness:**

For $sA + B$ (without genericity)

- $\delta_k = \hat{\delta}_k^{\mathrm{g}} \ (\forall k) \iff \theta_k = \hat{\theta}_k^{\mathrm{g}} \ (\forall k)$
- $\delta_k = \hat{\delta}_k^{\mathrm{m}} \ (\forall k) \iff \theta_k = \hat{\theta}_k^{\mathrm{m}} \ (\forall k)$

**Abstraction** of $\updownarrow$ (conj) to matroid pencil

# Abstraction to Matroid Pencil
# — Valuated Bimatroid

# Linking System / Bimatroid

$(S, T; A)$ Schrijver (1979), Kung (1978)

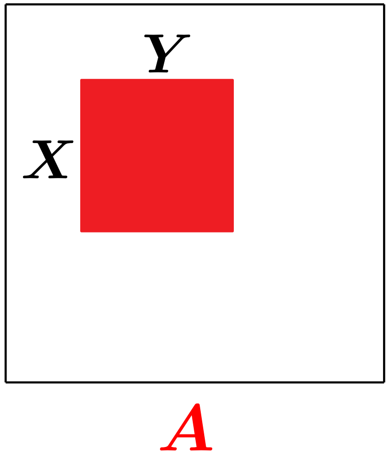
**(L1)** $(X, Y) \in A$, $x \in X \Rightarrow \exists\, y \in Y$: $(X - x, Y - y) \in A$;

**(L2)** $(X, Y) \in A$, $y \in Y \Rightarrow \exists\, x \in X$: $(X - x, Y - y) \in A$;

**(L3)** $(X_i, Y_i) \in A \ (i = 1, 2) \Rightarrow \exists\, X \subseteq S$, $Y \subseteq T$:

$(X, Y) \in A$, $X_1 \subseteq X \subseteq X_1 \cup X_2$, $Y_2 \subseteq Y \subseteq Y_1 \cup Y_2$.

Ex: $A = \{(X, Y) \mid$ nonsingular minors$\}$



# Linking System / Bimatroid

$(S, T; A)$ Schrijver (1979), Kung (1978)

**(L1)** $(X, Y) \in A$, $x \in X \Rightarrow \exists\, y \in Y$: $(X - x, Y - y) \in A$;

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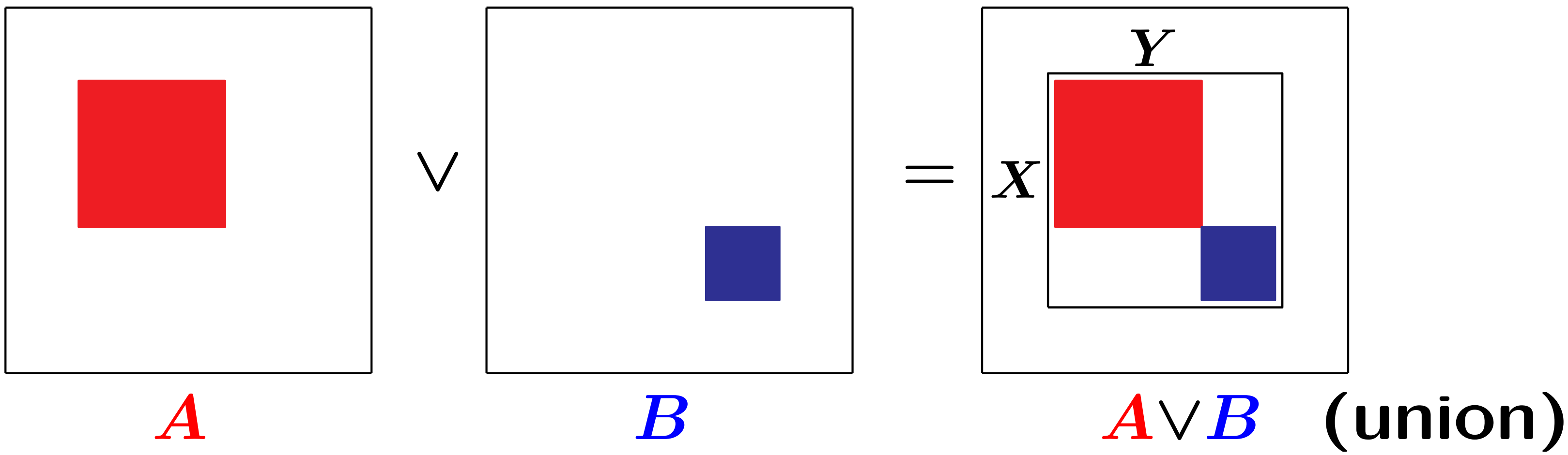
**(L3)** $(X_i, Y_i) \in A \ (i = 1, 2) \Rightarrow \exists\, X \subseteq S$, $Y \subseteq T$:

$$(X, Y) \in A, \ \ X_1 \subseteq X \subseteq X_1 \cup X_2, \ \ Y_2 \subseteq Y \subseteq Y_1 \cup Y_2.$$

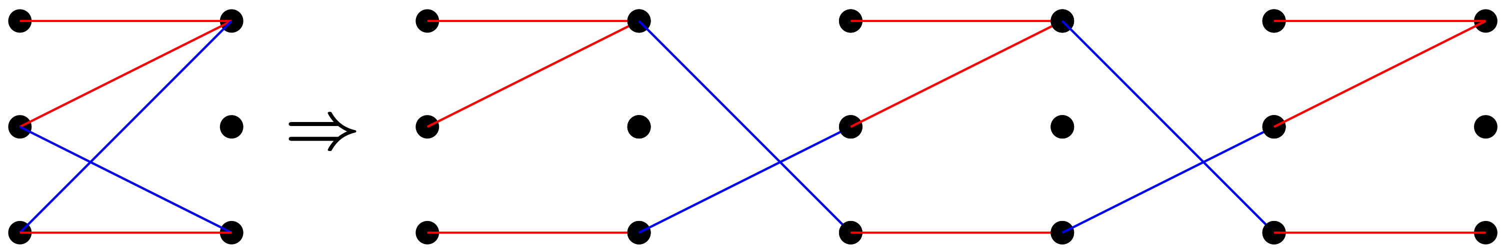
**matrix sum** $A + B \longleftrightarrow$ **bimatroid union** $A \vee B$



# Matroid Pencil

$(A, B)$: pair of bimatroids

$$A - B - A - B - A \quad (k \times A\text{'s})$$



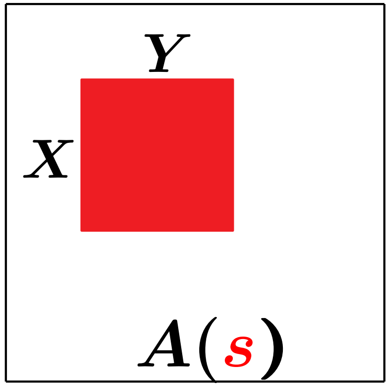
$\theta_k$ = max-size matching (linked pair)

**Thm** Iwata (2007)

(1) $\theta_k$: convex

(2) $$\theta_k = \max\{k|X| + (k-1)|\tilde{X}| \mid A \ni (X, Y) \perp (\tilde{X}, \tilde{Y}) \in B\}$$

# Valuated Bimatroid




$$f(X, Y) = \deg_s \det A(s)[X, Y]$$

$(S, T; A)$ bimatroid

equivalent to valuated matroid

$f : A \to \mathrm{R}$ valuated bimatroid

Murota (1995)

- For any $x' \in X' \setminus X$, (a) or (b) holds:

(a) $\exists\ y' \in Y' \setminus Y$:

$$f(X, Y) + f(X', Y') \leq f(X + x', Y + y') + f(X' - x', Y' - y')$$

(b) $\exists\ x \in X \setminus X'$:

$$f(X, Y) + f(X', Y') \leq f(X - x + x', Y) + f(X' - x' + x, Y')$$

- Symmetrically: For any $y \in Y \setminus Y' \cdots$

# Valuated Bimatroids for Matroid Pencils

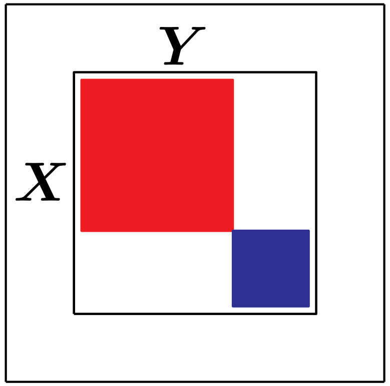
**Given** $(A, B)$: bimatroids

$f(X, Y) =$ max-size of $A$-part ■ in $(X, Y) \in A \vee B$

$g(X, Y) =$ max-size of $B$-part ■ in $(X, Y) \in A \vee B$

**Thm** $f, g$: valuated bimatroids

$A \vee B$




**Given** $(f, g)$: valuated bimatroids

s.t. $f(X, Y) \leq |X|, \;\; g(X, Y) \leq |X|$

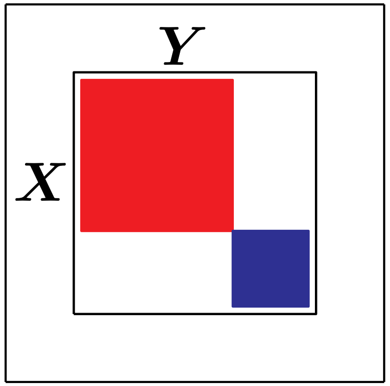
$A_f = \{(X, Y) \mid f(X, Y) = |X|\}, \;\; B_g = \{g(X, Y) = |X|\}$

**Thm** $(A, B) \to (f, g) \to (A_f, B_g) = (A, B)$

# Conjugacy in Matroid Pencils

$(A, B)$: matroid pencil

$f(X, Y) =$ max-size of ■ in $(X, Y) \in A \vee B$




$A \vee B$

**Def:**

$$\delta_k = \max\{f(X, Y) \mid |X| = k,\ (X, Y) \in A \vee B\}$$

**Thm (δ–θ conjugacy)**

(1) $\delta_k$: concave  cf. $\theta_k$: convex

(2) $\delta_k = \min_l(\theta_{l+1} - kl)$,  $\theta_{k+1} = \max_l(\delta_l + kl)$

# Concluding Remarks

- **Extension to polynomial matrices**
- **Smith-McMillan form at infinity**

**(Moriyama-Murota)**

# References (1)

This paper K. Murota: Legendre duality in combinatorial study of matrix pencils, Japan J. Indust. Appl. Math., to appear.

Degree of subdeterminant $\delta_k$

• K. Murota: Combinatorial relaxation algorithm for the maximum degree of subdeterminants — Computing Smith-McMillan form at infinity and structural indices in Kronecker form, Appl. Algebra Engin. Comm. Comput., 6 (1995), 251–273.

• K. Murota: Finding optimal minors of valuated bimatroids, Appl. Math. Lett., 8 (1995), 37–42.

• K. Murota: On the degree of mixed polynomial matrices, SIAM J. Matrix Anal. Appl., 20 (1999), 196–227.

• S. Iwata and K. Murota: Combinatorial relaxation algorithm for mixed polynomial matrices, Math. Progr., A90 (2001), 353–371.

• S. Iwata and M. Takamatsu: Computing the maximum degree of minors in mixed polynomial matrices via combinatorial relaxation, IPCO 2011.

# References (2)

Rank of expanded matrix $\theta_k$


• S. Iwata and R. Shimizu: Combinatorial analysis of singular matrix pencils, SIAM J. Matrix Anal. Appl., 29 (2007), 245–259.

• S. Iwata and M. Takamatsu: On the Kronecker canonical form of mixed matrix pencils, SIAM J. Matrix Anal. Appl., 32 (2011), 44–71.


Discrete Legendre conjugacy


• K. Murota: Discrete Convex Analysis, SIAM, 2003.


Matroid pencil and valuated matroid


• K. Murota: Matrices and Matroids for Systems Analysis, 2000.

• S. Iwata: Linking systems and matroid pencils, Journal of the Operations Research Society of Japan, 50 (2007), 315–324.