COLORING SOME PERFECT GRAPHS

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$G(V, E)$

VERTICES EDGES

$\chi(G)$ CHROMATIC NUMBER

SMALLEST # OF COLORS NEEDED TO COLOR $V(G)$

$\omega(G)$ CLIQUE NUMBER

MAX SIZE OF A CLIQUE IN G

$$\chi(G) \geq \omega(G)$$

WHEN IS $\chi(G) = \omega(G)$?

BIPARTITE GRAPHS

COMPLEMENTS OF BIP

COMPARABILITY GRAPHS

⋮

≤ 99 VERT

G IS **PERFECT** IF

$\chi(H) = \omega(H)$ FOR EVERY

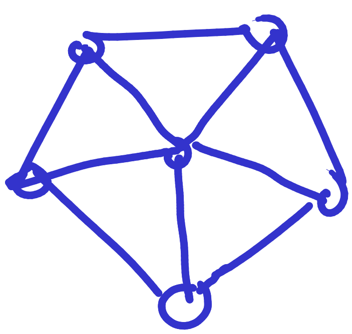
INDUCED SUBGRAPH H OF G

(BERGE, 1961)

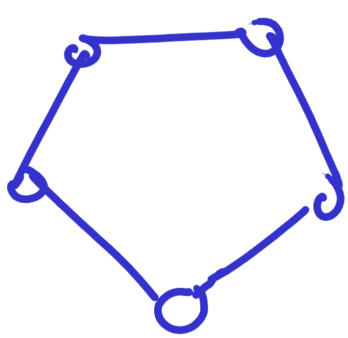
H IS AN **INDUCED SUBGRAPH** OF G IF

$V(H) \subseteq V(G)$, AND

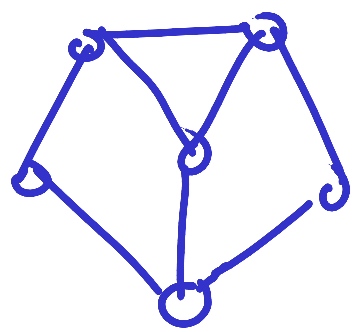
$uv \in E(H)$ IFF $uv \in E(G)$
& $u, v \in V(H)$



G



INDUCED S.G. OF G



NOT AN INDUCED S.G. OF G

THM (LOVASZ, 1972)

THE WEAK PERFECT GRAPH THM

G IS PERFECT IFF G^c IS PERFECT

THM (C., ROBERTSON, SEYMOUR, THOMAS, 2006)

THE STRONG PERFECT GRAPH THM

G IS PERFECT IFF NO INDUCED SUBGRAPH OF G IS ISOMORPHIC TO AN ODD CYCLE OF LENGTH ≥ 5, OR THE COMPLEMENT OF ONE

THM (C., CORNUEJOLS, LIU, SEYMOUR, VUŠKOVIĆ, 2005)

THERE IS A POLY-TIME ALGORITHM TO TEST IF A GRAPH IS PERFECT

OPEN QUESTION:

FIND A POLY-TIME COMBINATORIAL COLORING ALGORITHM FOR PERFECT GRAPHS

(THERE IS A POLY-TIME ALG USING THE ELLIPSOID METHOD)

REMARK (KRATOCHVIL & SEBO; TROTIGNON & VUSKOVIC)

LET $\mathcal{C}$ BE A CLASS OF PERFECT GRAPHS. TO FIND A COLORING ALG FOR $\mathcal{C}$, ENOUGH TO FIND $\omega(G)$ AND $\alpha(G)$ FOR EVERY WEIGHTED $G \in \mathcal{C}$.

COR IF $\mathcal{C}$ IS CLOSED UNDER COMPLEMENTATION, ENOUGH TO FIND $\omega(G)$ FOR EVERY WEIGHTED $G \in \mathcal{C}$.

A DECOMPOSITION THM FOR PERFECT GRAHS

(C., ROBERTSON, SEYMOUR, THOMAS)

IF G IS PERFECT, THEN EITHER

1. G OR G^c IS BIPARTITE, OR
2. G OR G^c IS THE LINE GRAPH OF A BIPARTITE GRAPH, OR
3. G IS A DOUBLE SPLIT GRAPH, OR
4. G ADMITS A BALANCED SKEW PARTITION, OR
5. G OR G^c ADMITS A 2-JOIN, OR
6. G ADMITS A HOMOGENEOUS PAIR (NOT NEEDED, C.)

1, 2, 3 BASIC

THE COMPLEMENT G^c OF G:

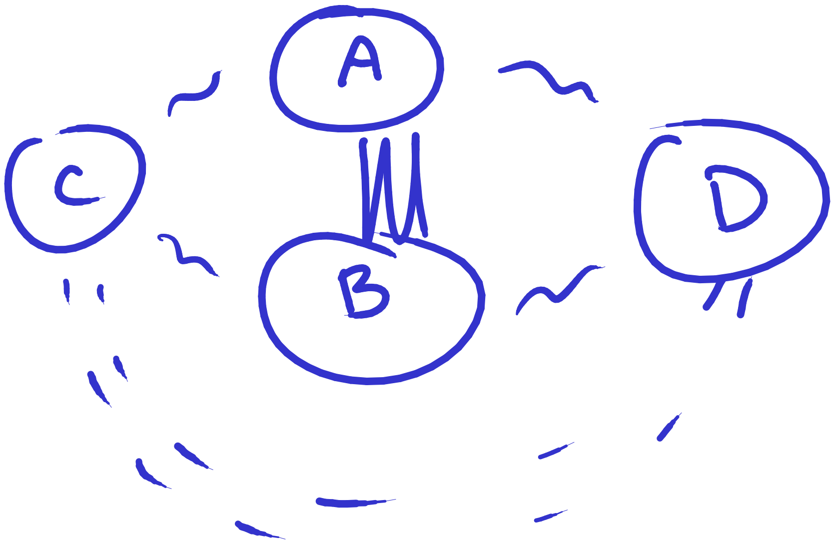
- $V(G^c) = V(G)$
- $uv \in E(G^c)$ IFF $uv \notin E(G)$

THE LINE GRAPH $E(G)$ OF G:

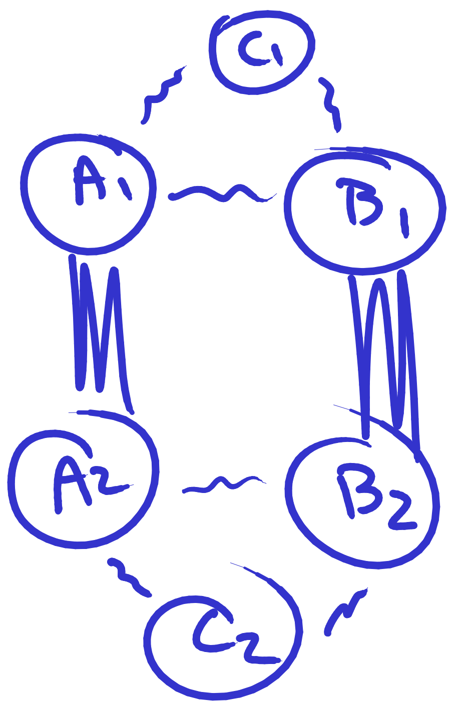
- $V(L(G)) = E(G)$
- $uv \in E(L(G))$ IFF

  u, v SHARE AN END IN G

SKEW PARTITION

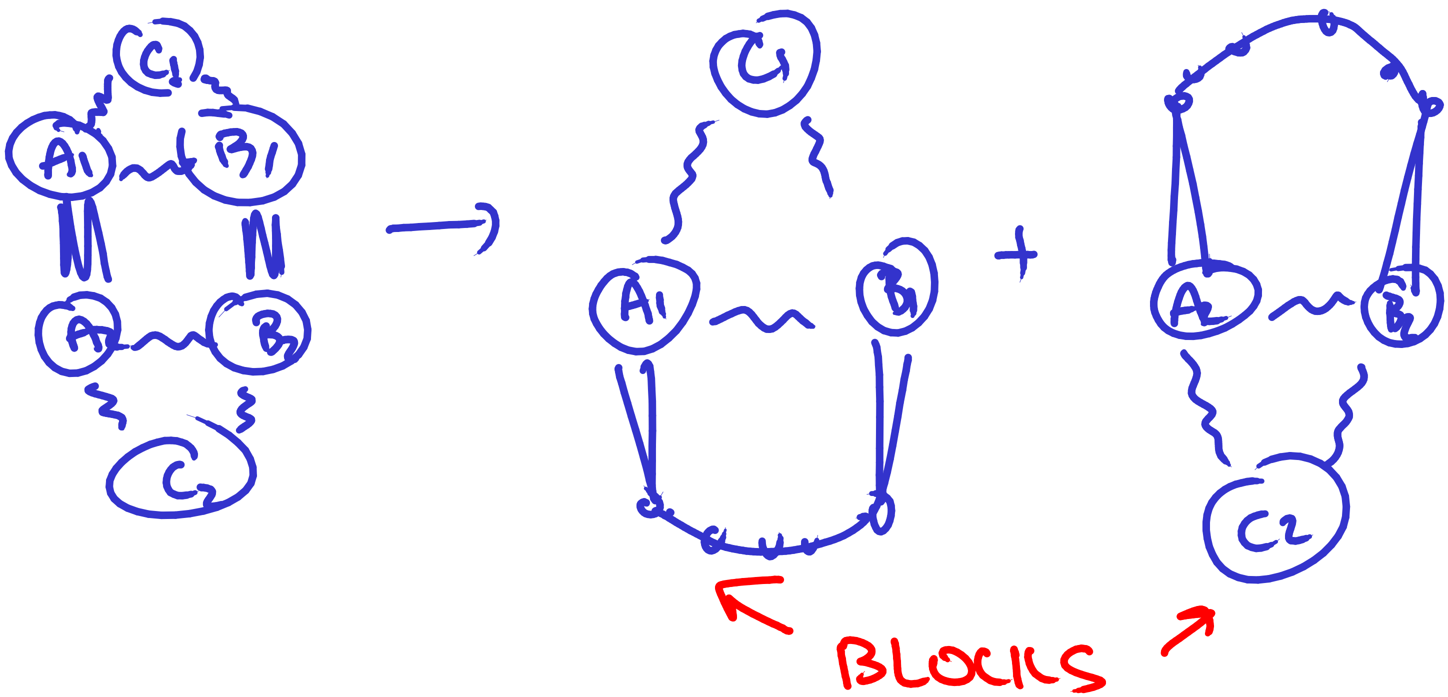


2-JOIN

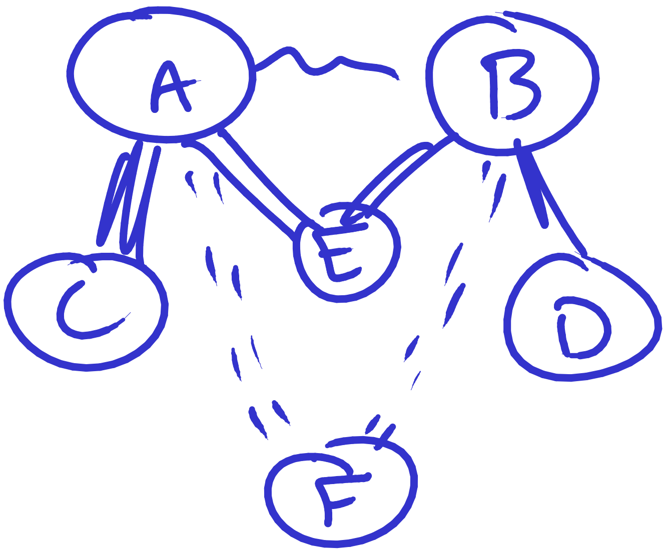


- EVEN
- ODD

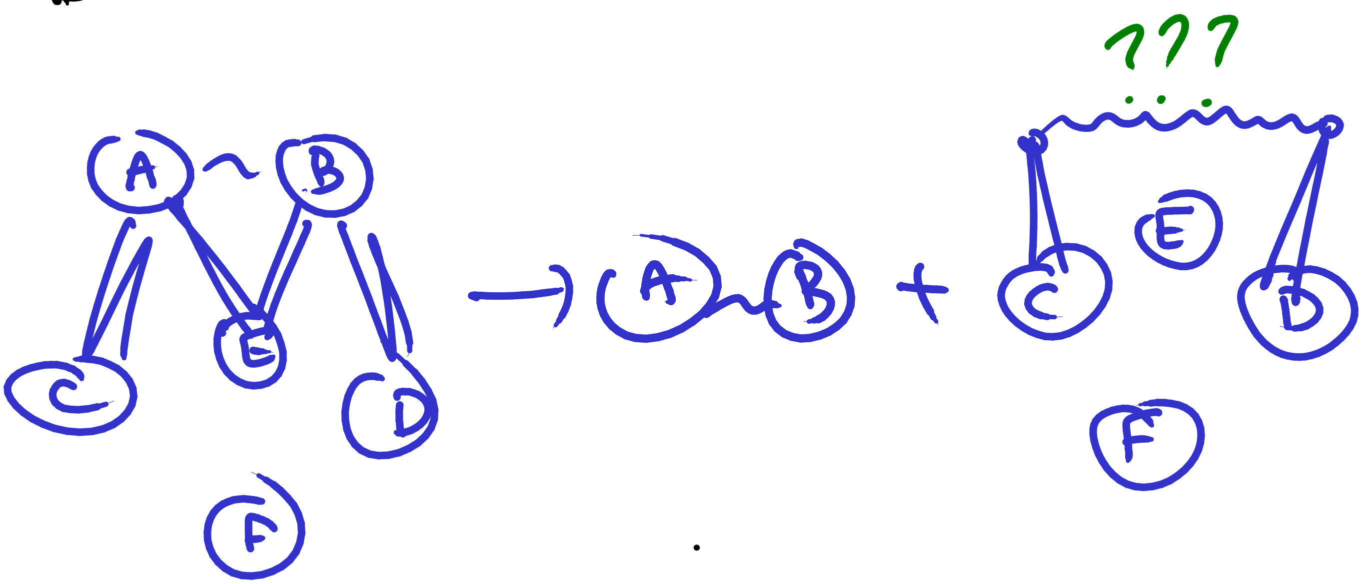
DECOMPOSING BY 2-JOINS:



HOMOGENEOUS PAIR



DECOMPOSING BY H. PAIRS



SKEW PARTITIONS ARE "BAD" FOR COLORING

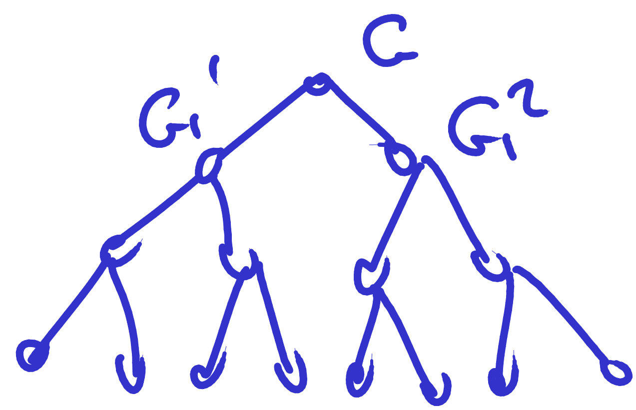
WHAT ABOUT PERFECT GRAPHS THAT DO NOT ADMIT BALANCED SKEW PARTITIONS? $\mathcal{C}$

THM THERE IS A POLY-TIME ALG THAT GIVEN A WEIGHTED GRAPH G WITH NO BALANCED SKEW PARTITION OUTPUTS $\omega(G)$

$G \in \mathcal{C}$ IFF $G^c \in \mathcal{C}$, SO CAN ALSO COMPUTE $\alpha(G)$

$\Rightarrow$ CAN COLOR

PROOF IDEA: DECOMPOSE INTO BASIC GRAPHS BY 2-JOINS AND COMPLEMENT 2-JOINS



PROBLEMS:

1. TOO MANY NODES IN THE DECOMPOSITION TREE
2. MAY INTRODUCE BALANCED SKEW PARTITIONS

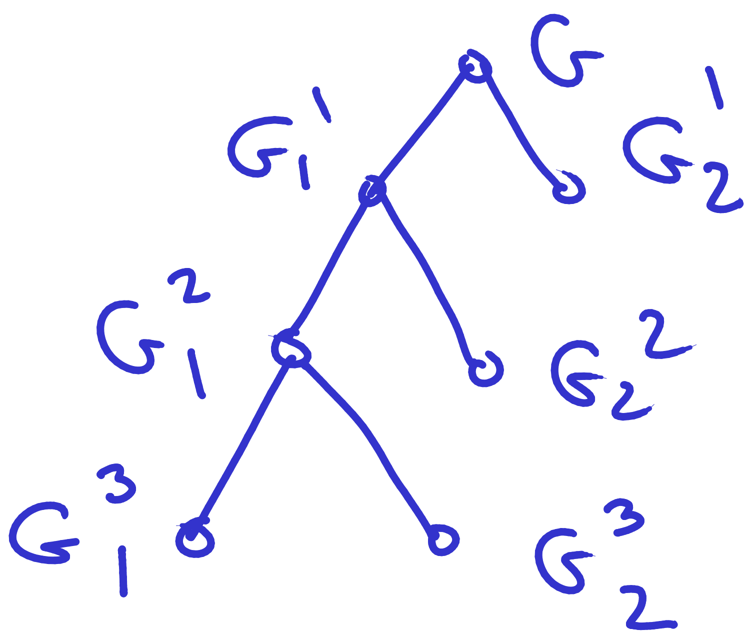
SOLUTIONS :

(1) EXTREME DECOMPOSITIONS

$$G \to G_1 + G_2$$

ONE OF G_1, G_2 IS BASIC

DECOMP TREE:



G_2^i BASIC

2 TRIGRAPHS

GRAPH → EDGES
→ NON-EDGES

TRIGRAPH → EDGES
→ NON-EDGES
→ UNDECIDED PAIRS
SWITCHABLE EDGES

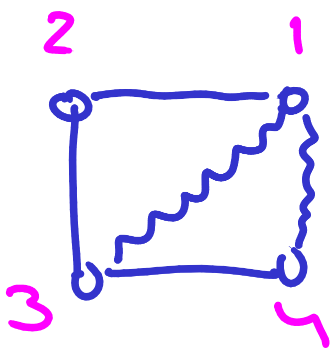
$T(V, E, N, S)$

E: EDGES
N: NON-EDGES
S: SWITCHABLE EDGES

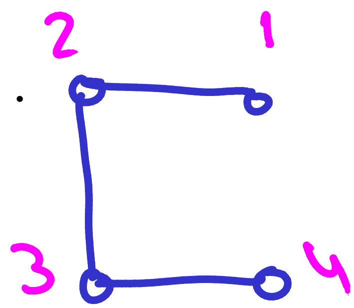
A REALIZATION G OF A TRIGRAPH T IS A GRAPH S.T.

$$V(G) = V(T)$$

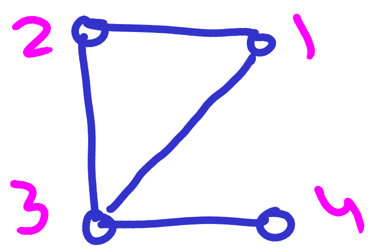
$$E(T) \subseteq E(G) \subseteq E(T) \cup E(S)$$



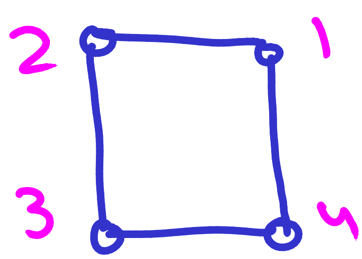
T



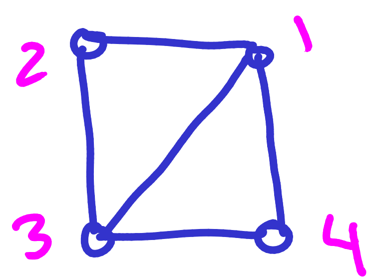
G_1



G_2



G_3



G_4

A TRIGRAPH T IS PERFECT IF EVERY REALIZATION OF T IS PERFECT

THE COMPLEMENT T^c OF T IS A TRIGRAPH

$$V(T^c) = V(T)$$

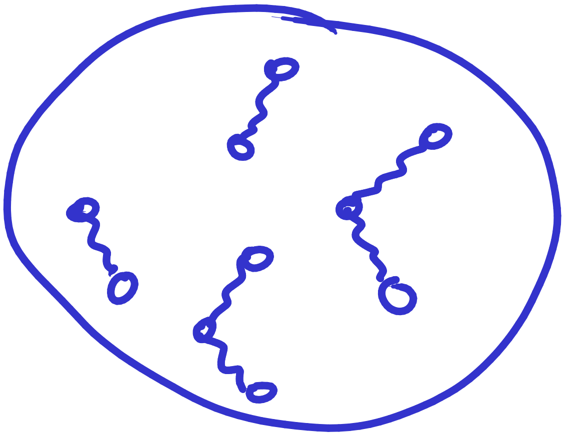
$$E(T^c) = N(T)$$

$$N(T^c) = E(T)$$

$$S(T^c) = S(T)$$

$$\deg_T(v) = \left|\{u : uv \in E(T) \cup S(T)\}\right|$$

T IS TAME IF

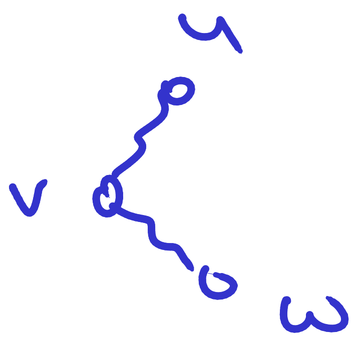


H: $V(H) = V(T)$

$E(H) = S(T)$

EVERY COMPONENT OF H IS AN EDGE OR A 2-EDGE PATH

AND IF $\deg_H(v) = 2$, THEN EITHER



$\deg_T(v) = 2$ & $uw \in N(T)$

OR

$\deg_{T^c}(v) = 2$ & $uw \in E(T)$

THM 1 LET T BE A TAME PERFECT TRIGRAPH. THEN EITHER

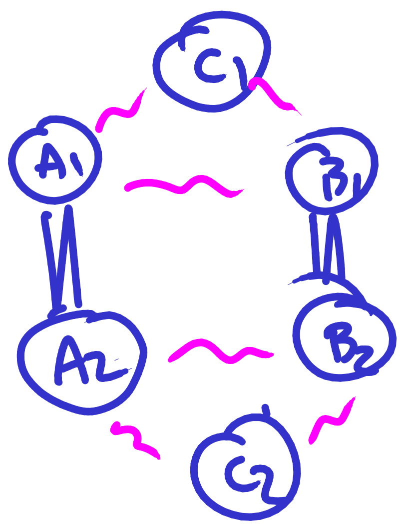
1. T IS BASIC, OR
2. T ADMITS A BALANCED SKEW PARTITION, OR
3. T OR T^c ADMITS A 2-JOIN, OR
4. T ADMITS A HOMOGENEOUS PAIR (NOT NEEDED)

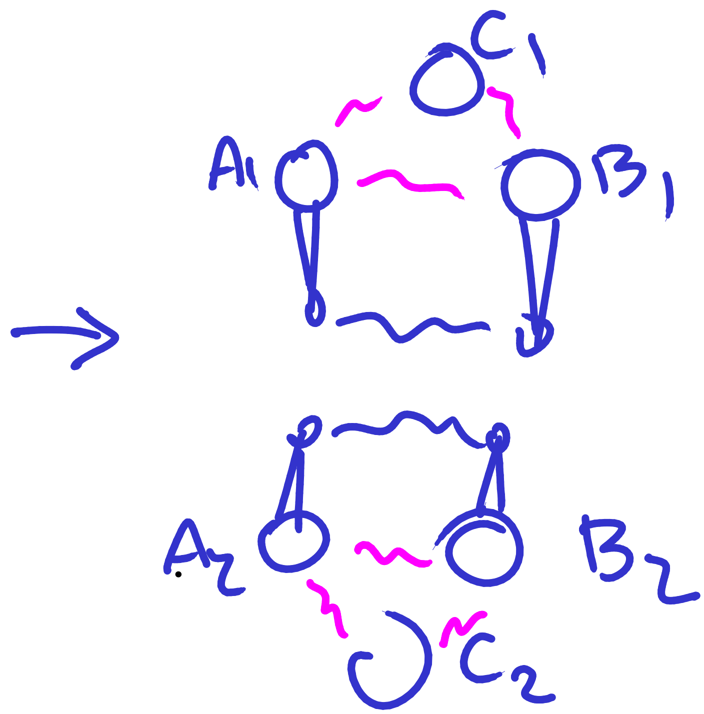
THM 2 LET T BE A TAME PERFECT TRIGRAPH. THEN EITHER

1. T IS BASIC, OR
2. T ADMITS A BALANCED SKEW PARTITION, OR
3. T OR T^c ADMITS AN EXTREME 2-JOIN, OR
4. T ADMITS AN EXTREME HOMOGENEOUS PAIR

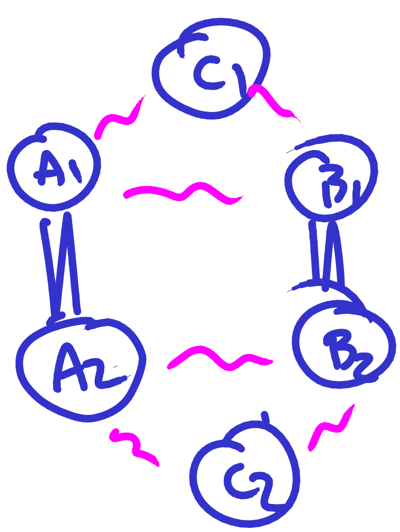
DECOMPOSITION BLOCKS

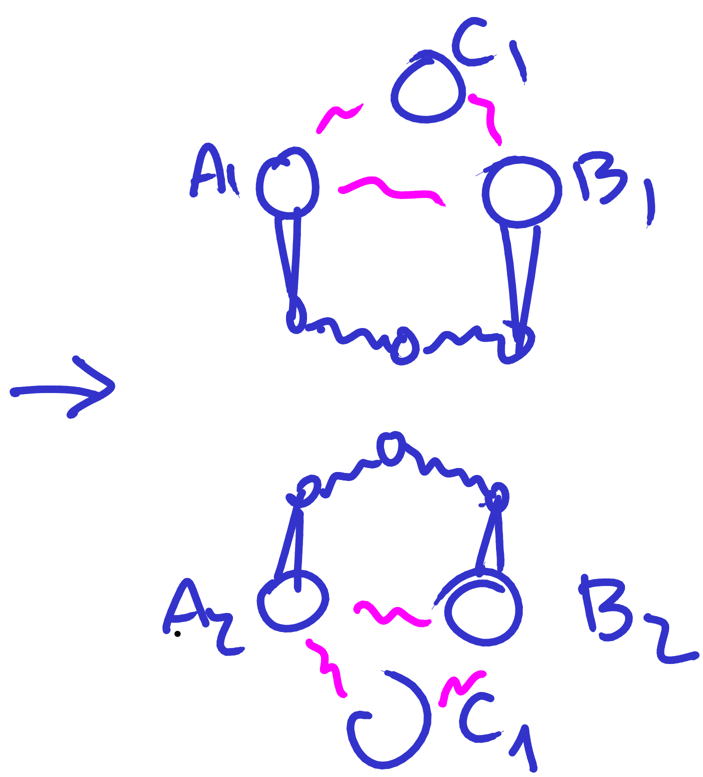
ODD 2-JOIN



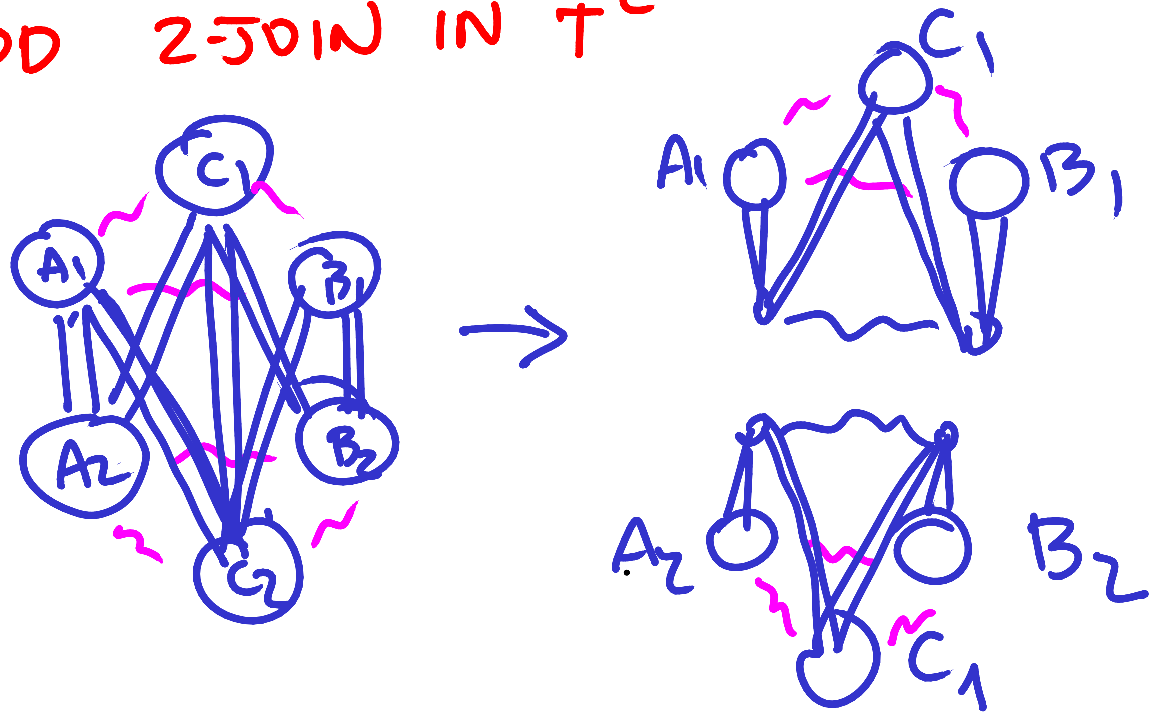


EVEN 2-JOIN

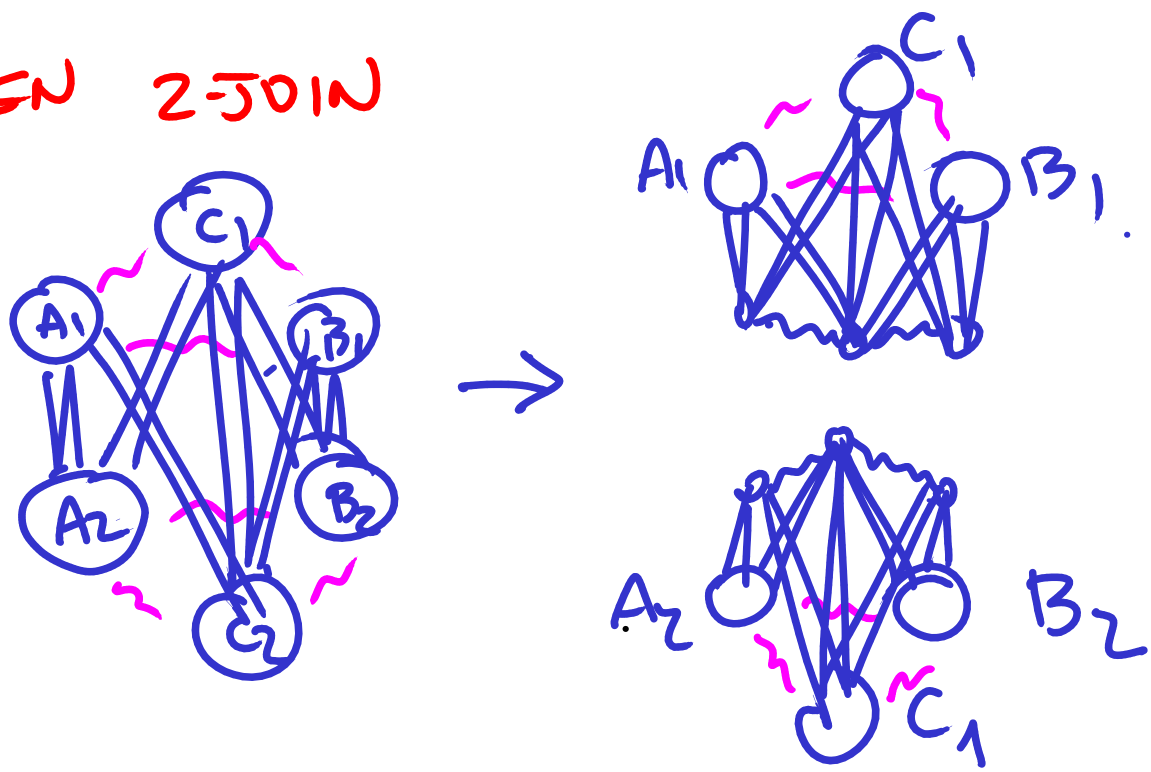




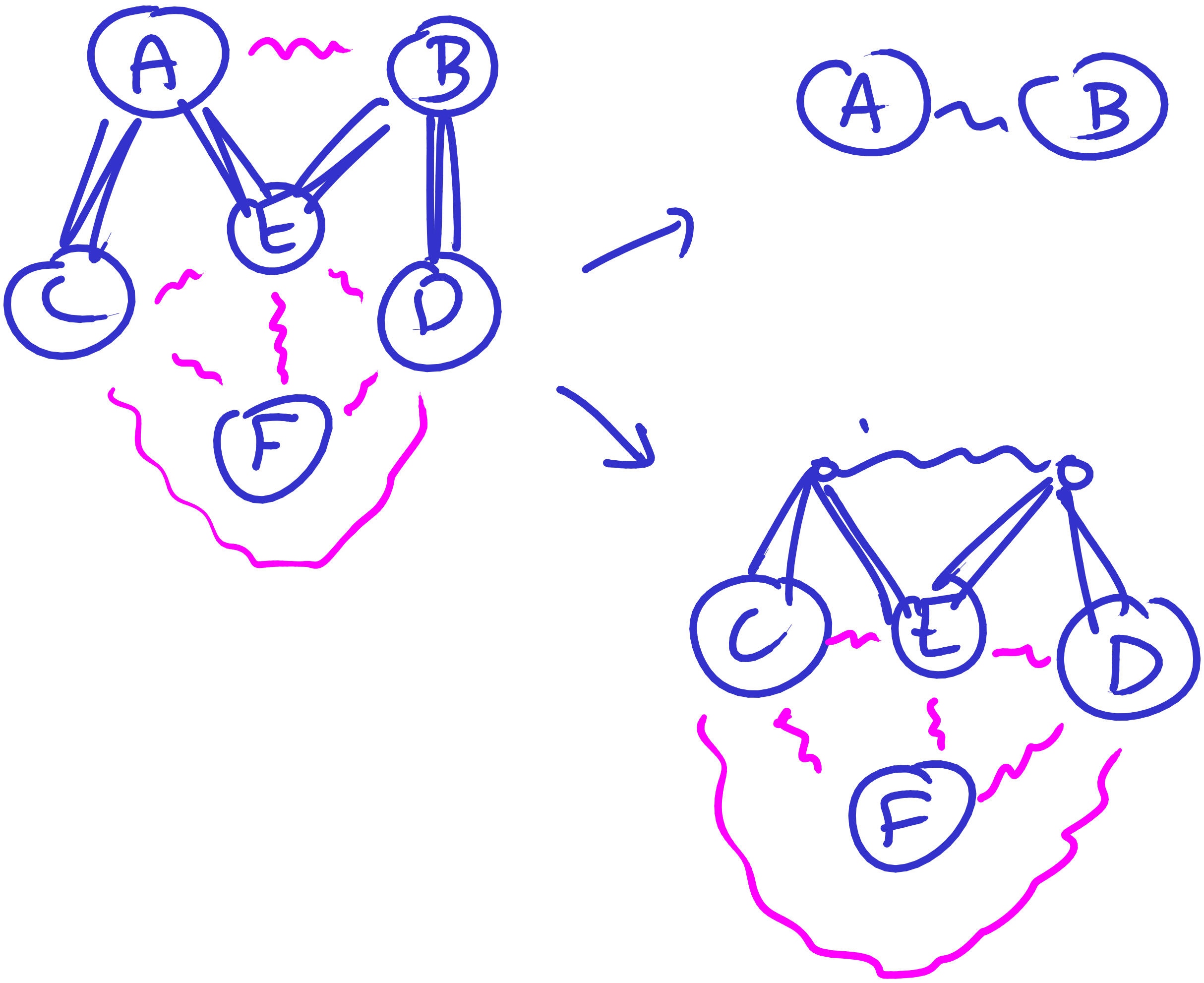
ODD 2-JOIN IN T^c



EVEN 2-JOIN



HOMOGENEOUS PAIR



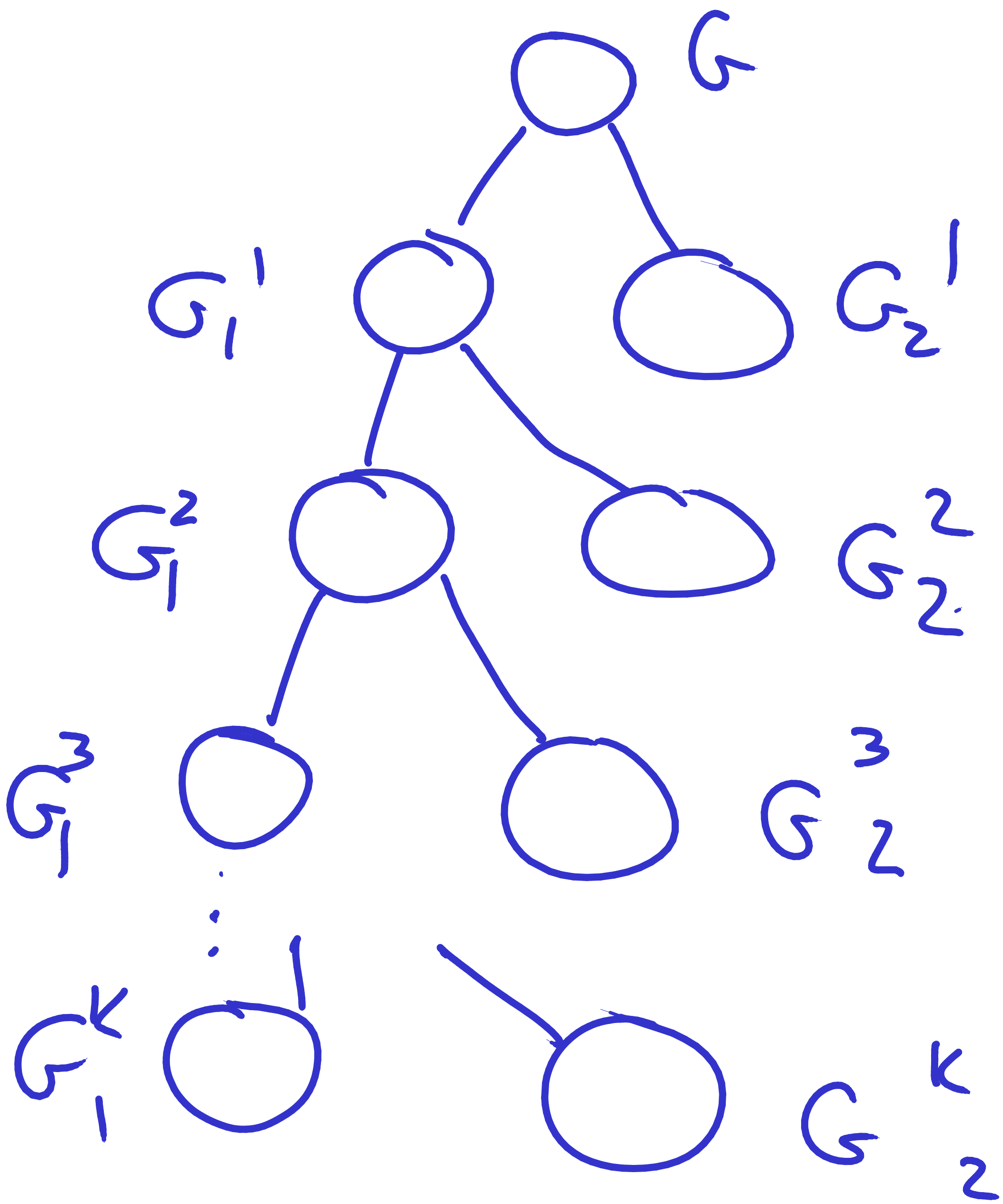
THE H. PAIR IS EXTREME IF $T|(A \cup B)$ IS BASIC

THM3 IF G DOES NOT ADMIT A BALANCED SKEW-PARTITION, THEN NEITHER DO THE BLOCKS

THM 4 LET T BE A TAME PERFECT TRIGRAPH WITH NO BALANCED SKEW PARTITION. THEN T CAN BE DECOMPOSED INTO BASIC TRIGRAPHS BY
EXTREME 2-JOINS IN T,
EXTREME 2-JOINS IN T^c, AND
EXTREME H. PAIRS

TO GET EXTREME DECOMPOSITIONS CHOOSE $A_1 \cup B_1 \cup C_1$ AND $A \cup B$ MINIMAL

DECOMPOSITION TREE



$G_1^k, \; G_2^1, \dots, G_2^k$ ARE BASIC

$$|V(G_1^i)| < |V(G_1^{i-1})|$$

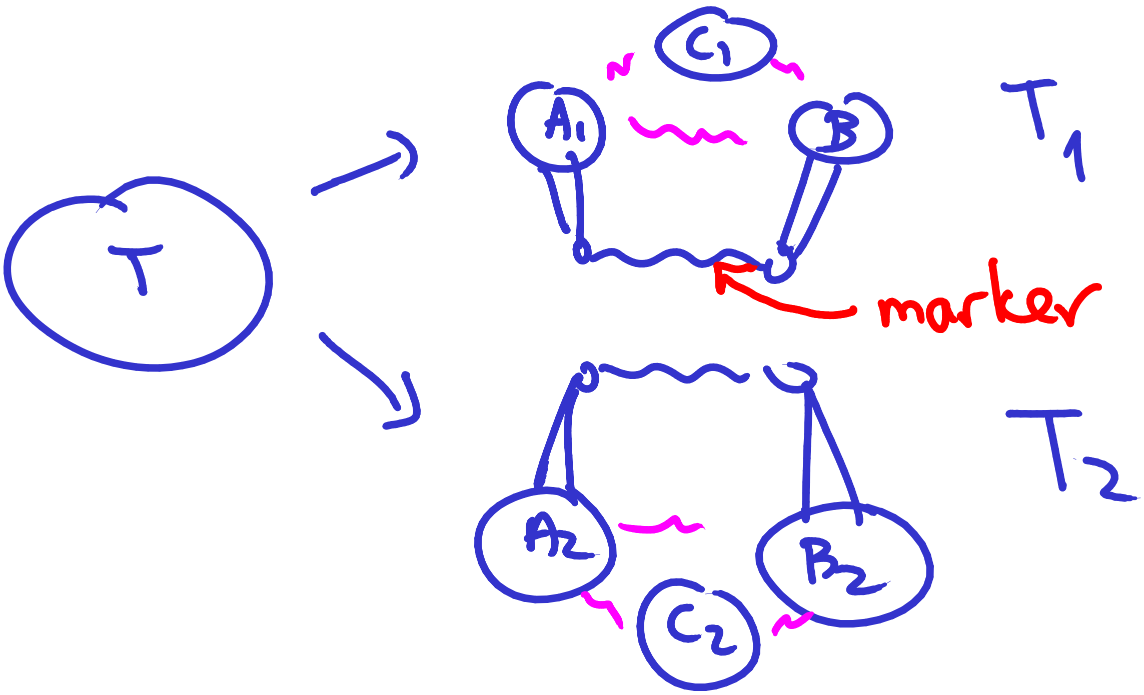
COMPUTING $\omega(T)$

T TRIGRAPH WITH WEIGHTS ON VERTICES

$S \subseteq V(T)$ IS A STRONG CLIQUE IF

$\forall u,v \in S, \quad uv \in E(T)$

$\omega(T)$ MAX WEIGHT OF A STRONG CLIQUE



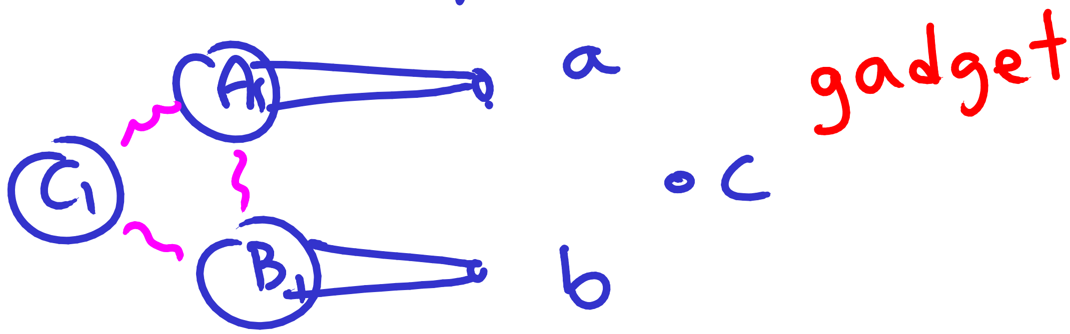
T_2 BASIC

COMPUTE: $\omega(T_2 | A_2) = a$

$\omega(T_2 | B_2) = b$

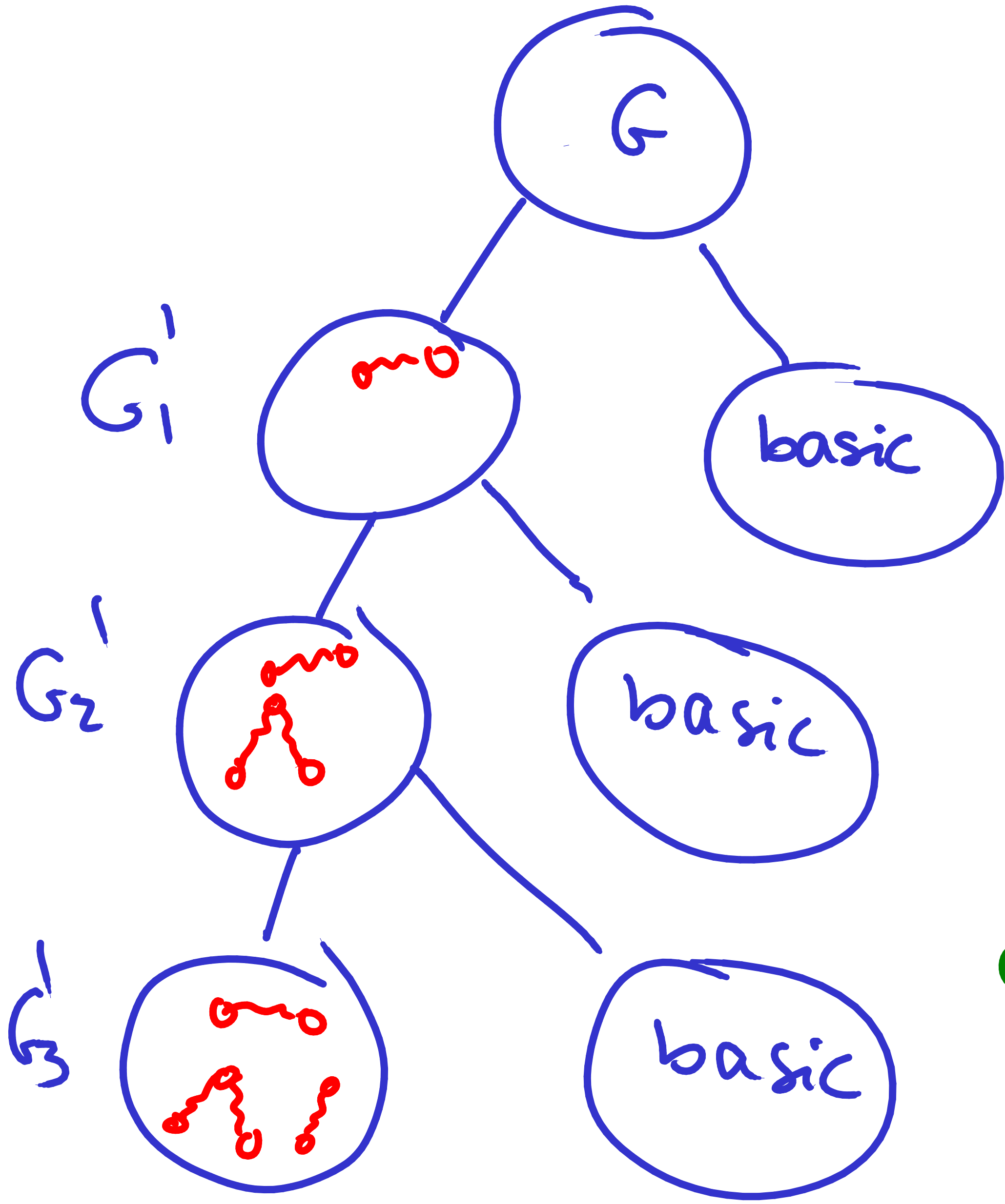
$\omega(T_2 | (A_2 \cup B_2 \cup C_2)) = c$

COMPUTE ω OF:



SIMILAR FOR OTHER DECOMPOSITIONS

DECOMPOSITION TREE



G_k' BASIC

IN G_k' REPLACE ALL markers BY APPROPRIATE gadgets, GET $\tilde{G}$.

THM $\tilde{G}$ IS BASIC.

$$\omega(\tilde{G}) = \omega(G)$$