Finding 2-Factors Closer to TSP in Cubic Graphs


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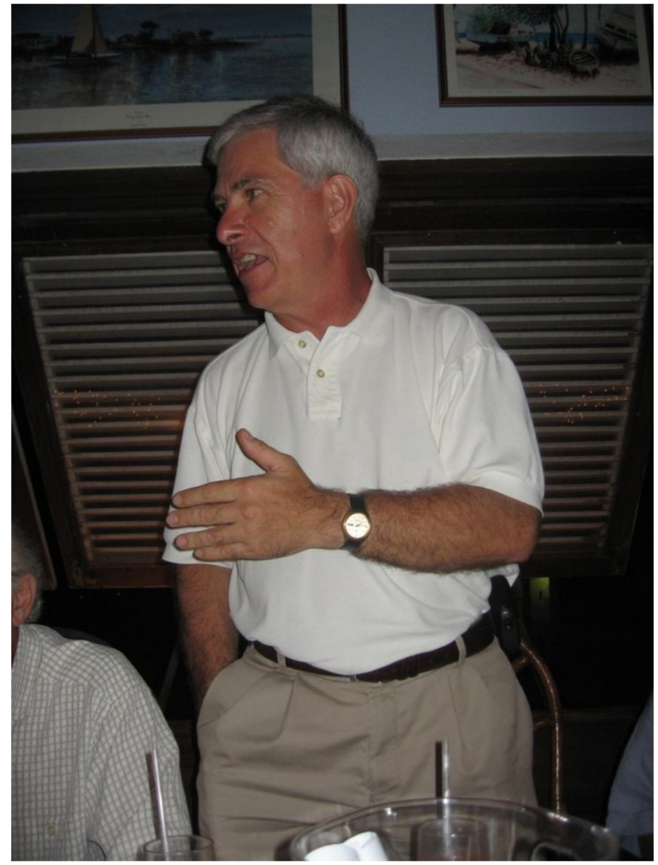
My History of Bills': Bill Cook

My History of Bills': Bill Pulleyblank



My History of Bills': Bill Cunningham

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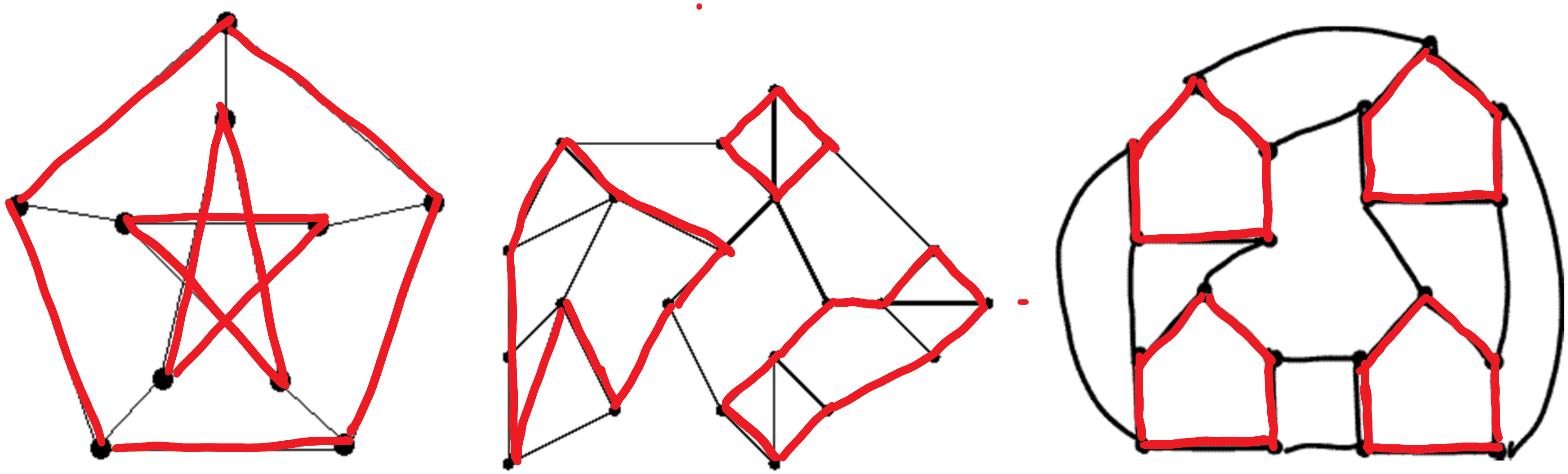
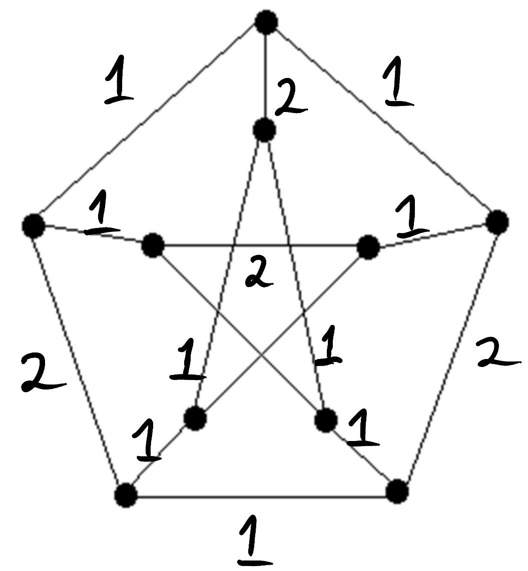
2-Factor

A subset of edges of the graph that forms a set of disjoint cycles, and covers all the nodes



A Minimum Weight 2-Factor

Given weights on the edges, a 2-factor of the graph for which the edges have minimum total weight.



Can be found in polynomial time.

The 2-Factor Polytope

Convex hull of the incidence vectors of all 2-factors of $G = (V,E)$

J. Edmonds (1965):

$x(\delta(v)) = 2$	for all $v \in V$
$0 \leq x_e \leq 1$, integer	for all $e \in E$

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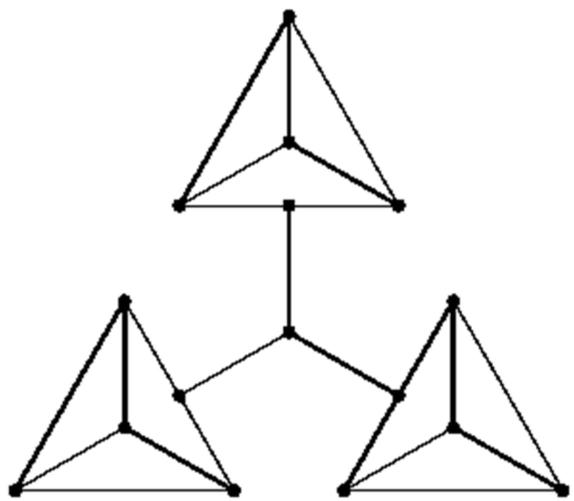
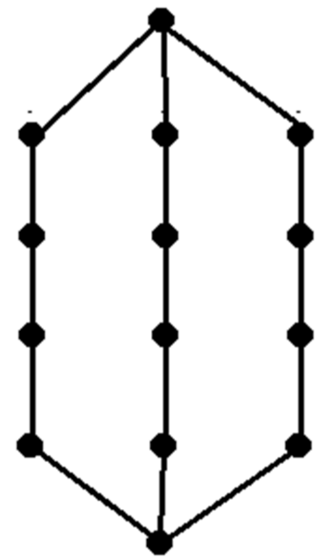
$0 \leq x_e \leq 1$, integer	for all $e \in E$
$x(Y) - x(\delta(S) \setminus Y) \leq \|Y\| - 1$	for all $S \subset V$, $Y \subseteq \delta(S)$, Y a matching, $\|Y\|$ odd

Not every graph has a 2-factor



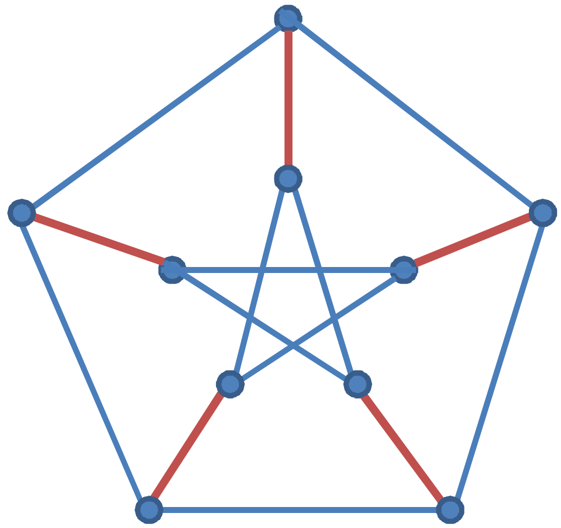


Petersen's Theorem

Every bridgeless cubic graph has a perfect matching.

↓

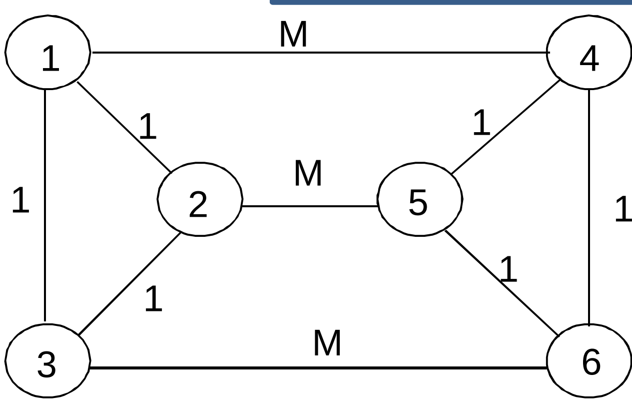
Every bridgeless cubic graph has a 2-factor.

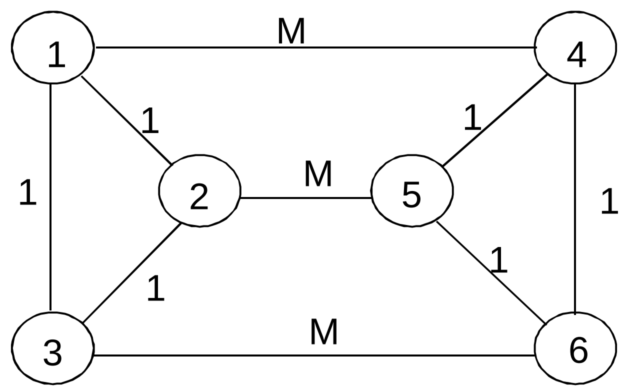


Relationship of 2-Factor and TSP

The Travelling Salesman Problem (TSP) solution is a min. cost 2-factor that consists of only one cycle.

$$OPT(\text{2-Factor}) \leq OPT(TSP)$$

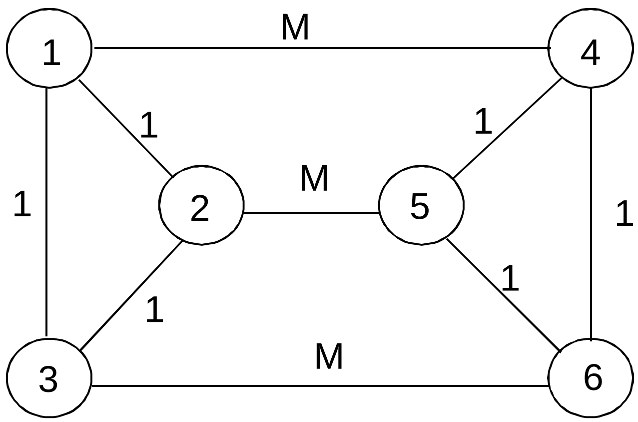




2-Factors with no cycles of length 3

Triangle-free 2-factor, or △-free 2-factor.

OPT(2-Factor) ≤ OPT(△-free 2-Factor) ≤ OPT(TSP)



C_k-Restricted 2-Factors

A 2-factor with no cycles of length k or less.

Hierarchy:

2-factor ⟷ TSP

in P

NP-hard, even for bridgeless cubic graphs)

Interesting, because:

What is known for △-free 2-factors

<u>General graphs</u>

- Unweighted: in P
  (Hartvigsen 1984)
- Weighted: complexity unknown

<u>Sububic graphs</u>

Weighted: in P (Vornberger 1980) (Hartvigsen and Li 2007)
(Kobayashi, 2010)

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General graphs

- Unweighted: in P (Hartvigsen 1984)
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Sububic graphs

Weighted: in P (Vornberger 1980) (Hartvigsen and Li 2007) (Kobayashi, 2010)

Polytope for △-free 2-factors

Cunningham and Wang, 2000:

Studied the polytopes for C_k-free 2-factors in complete graphs.

For △-free 2-factor polytope :

- Showed a subclass of the bipartition inequalities are facet-defining (and still not enough!)

Polytope for $\triangle$-free 2-factors

Sububic graphs $G=(V,E)$

Hartvigsen and Li (2007):

Give a complete linear description of the polytope

$x(\delta(v)) = 2$ for all $v \in V$

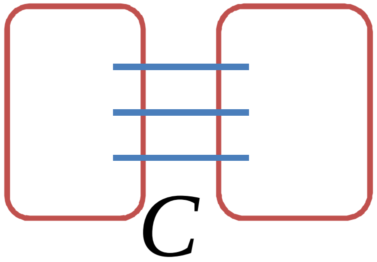
$x(Y) - x(\delta(S) \setminus Y) \leq |Y| - 1$ for all $S \subset V$, $Y \subseteq \delta(S)$, Y a matching, $|Y|$ odd

$0 \leq x_e \leq 1$ for all $e \in E$

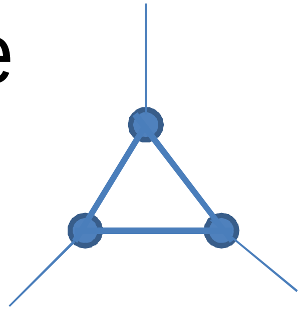
$x(E(T)) = 2$ for all triangles T in G

Covering 3-Edge Cuts in Cubic Graphs

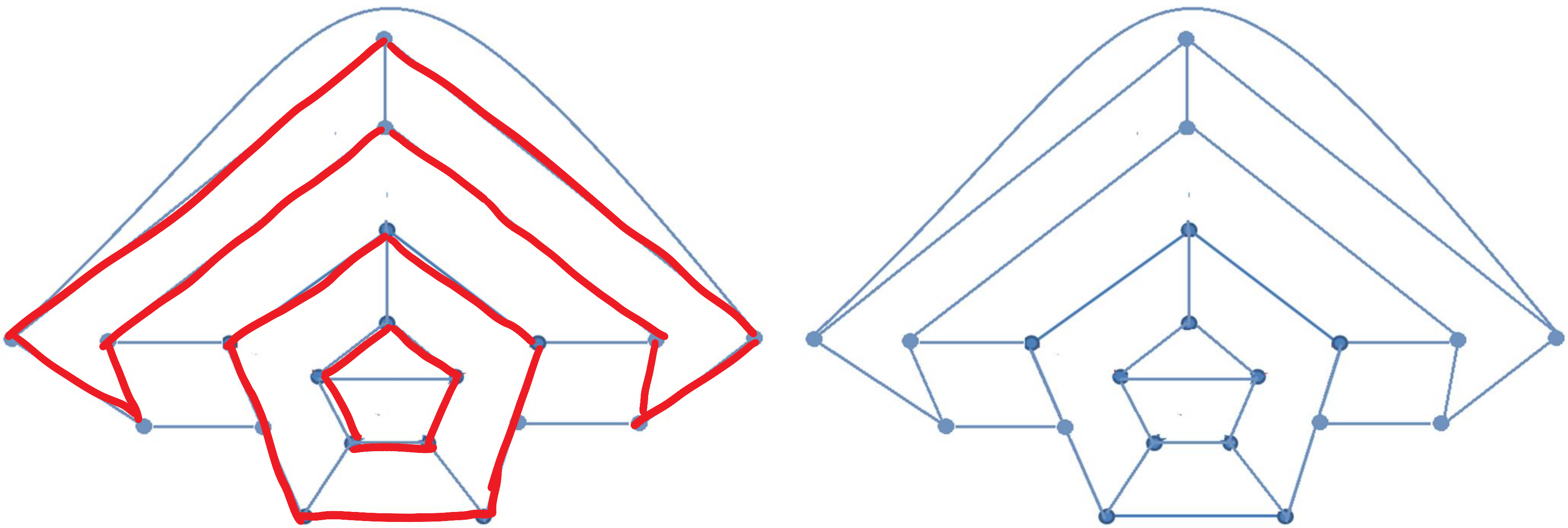
Consider a 2-factor that "covers" all the proper 3-edge cuts in a cubic bridgeless graph G



- call this a 3-cut 2-factor
- note it will be triangle-free



Covering 3-Edge Cuts in Cubic Graphs



OPT(2-factor) ≤OPT(Δ-free 2-factor)

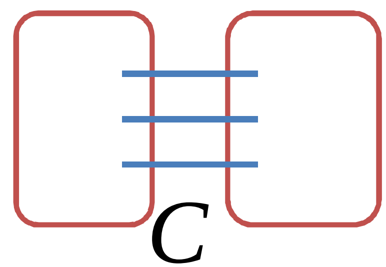
≤ OPT(3-cut 2-factor) ≤ OPT(TSP)

Covering 3-Edge Cuts in Cubic Graphs

Every bridgeless cubic graph has a 2-factor covering all 3-edge cuts which doesn't use a specified edge e*.

$$\frac{2}{3}\chi_E \in \text{conv}\{\chi_F \mid F : 2\text{-factor}\}$$

$$\frac{2}{3}\chi_E = \sum_{i \in I} \lambda_i \chi_{F_i} \qquad (\lambda_i \geq 0, \ \sum_{i \in I} \lambda_i = 1)$$



$$\frac{2}{3}\chi_E(C) = \sum_{i \in I} \lambda_i \chi_{F_i}(C) = \sum_{i \in I} \lambda_i |F_i \cap C|.$$

$$|F_i \cap C| = 2, \quad \forall i \in I.$$

Our New Results

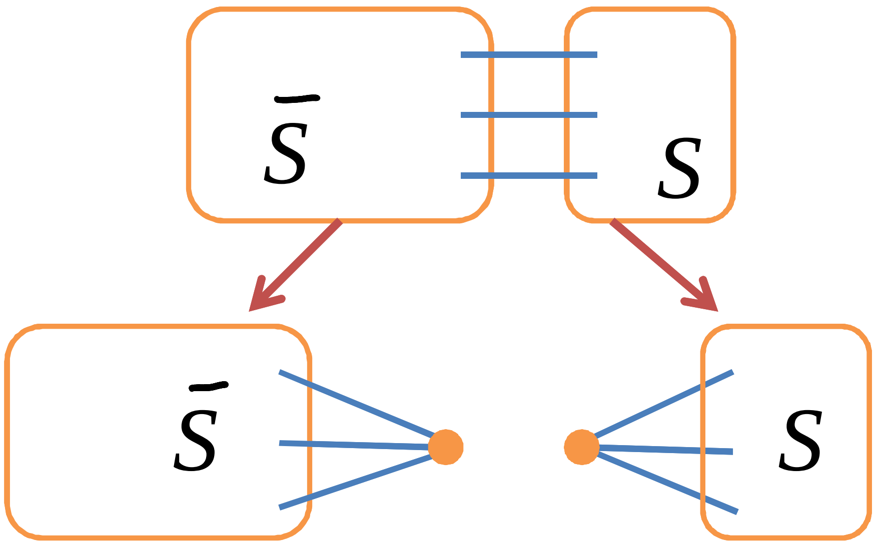
An Efficient Algorithm for Finding a Minimum Cost 2-Factor Covering 3-Edge Cuts in Bridgeless Cubic Graphs.

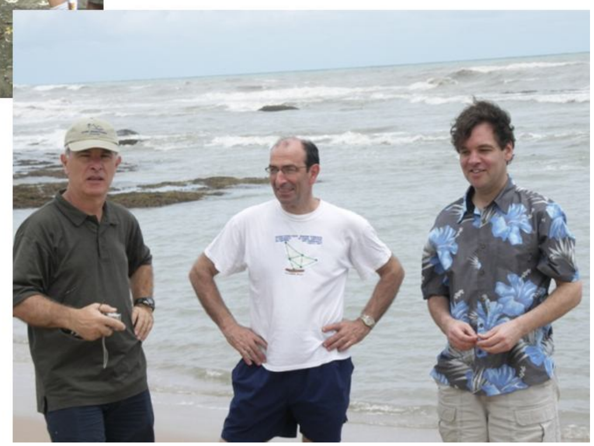
A Complete Linear Description of the Polytope of 3-Cut 2-factors For Bridgeless Cubic Graphs

Algorithm Outline

Borrowed an idea of Cornuejols, Naddef and Pulleyblank, '85

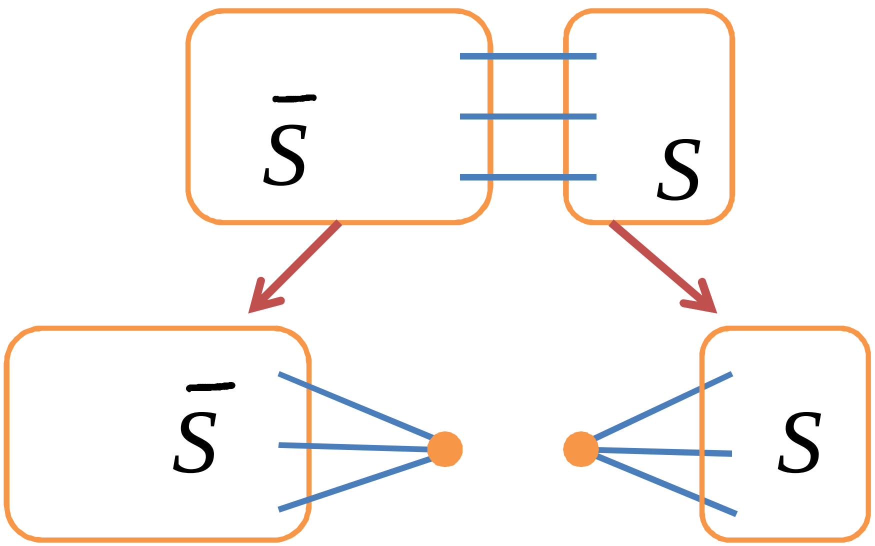
Find a proper 3-edge cut.





Algorithm Outline

1. Find a proper 3-edge cut $D=\delta(S)$ s.t. no proper set forms a 3-edge cut. Let $D = \{e_1, e_2, e_3\}$.
2. Find a min. cost 2-factor F_i in G_1 that doesn't use edge e_i, for i= 1, 2, 3. Let $L_i=c(F_i \cap \gamma(S))$ for i= 1, 2, 3.



3. In G_2, add extra weight α_i to each edge e_i, where $L_1 = \alpha_2 + \alpha_3$, $L_2 = \alpha_1 + \alpha_3$, $L_3 = \alpha_1 + \alpha_2$.
4. Solve the problem recursively for G_2 with new weights.

Polytope for 3-cut 2-factors

A complete linear description of the polytope for bridgeless cubic graphs $G = (V,E)$:

$x(\delta(v)) = 2$ for all $v \in V$

$x(Y) - x(\delta(S) \setminus Y) \leq |Y| - 1$ for all $S \subset V$, $Y \subseteq \delta(S)$, Y a matching, $|Y|$ odd

$0 \leq x_e \leq 1$ for all $e \in E$

$x(\delta(S)) = 2$ for all $S \subset V$, $\delta(S)$ a proper 3-edge cut of G

Linear Description for $P^{3\text{-cut2F}}$

Proof: Show (1) $P^{3\text{-cut2F}} \subseteq P$ and (2) $P \subseteq P^{3\text{-cut2F}}$

Proof that $P \subseteq P^{3\text{-cut2F}}$

let $x^* \in P$

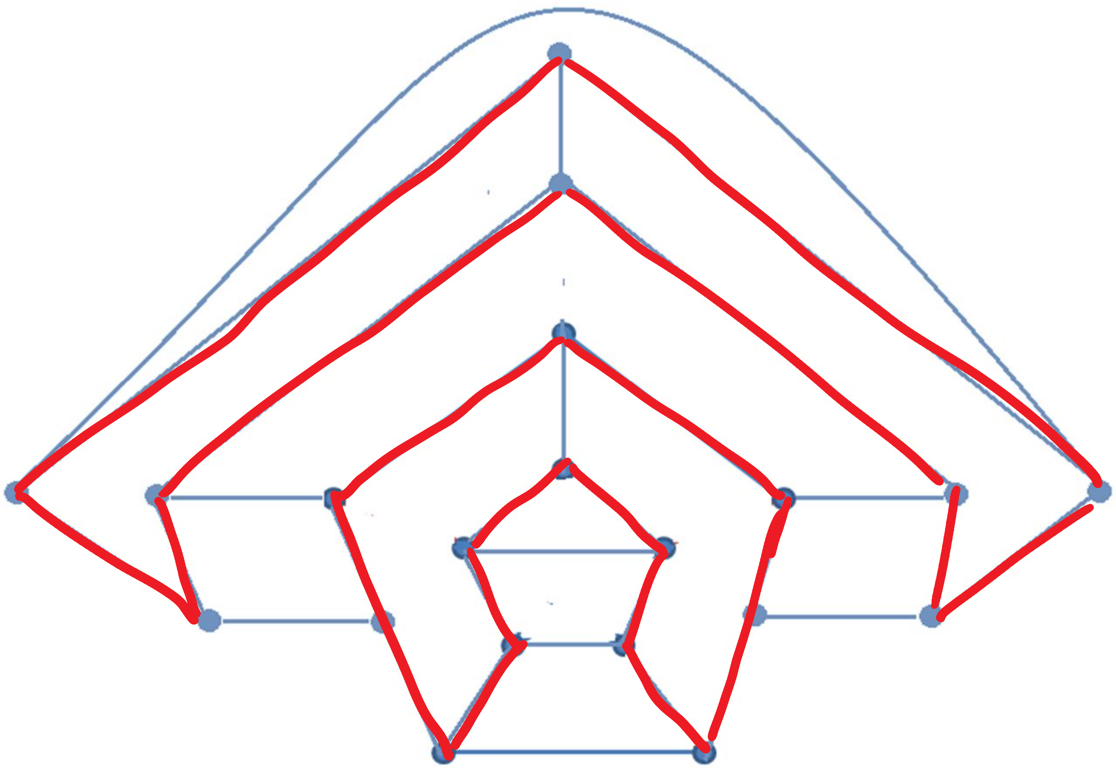
$$x^* \in \mathrm{conv}\{\chi_F \mid F : 2\text{-factor}\}$$

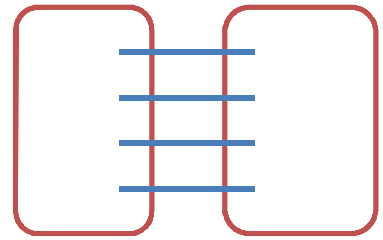
$$x^* = \sum_{i \in I} \lambda_i \chi_{F_i} \quad (\lambda_i \geq 0, \ \sum_{i \in I} \lambda_i = 1)$$

$$x^*(C) = \sum_{i \in I} \lambda_i \chi_{F_i}(C) = \sum_{i \in I} \lambda_i \, | F_i \cap C |.$$

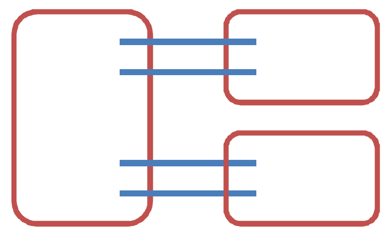
$$| F_i \cap C | = 2, \quad \forall i \in I.$$

Covering Proper 3- and 4-Edge Cuts in Bridgeless Cubic Graphs



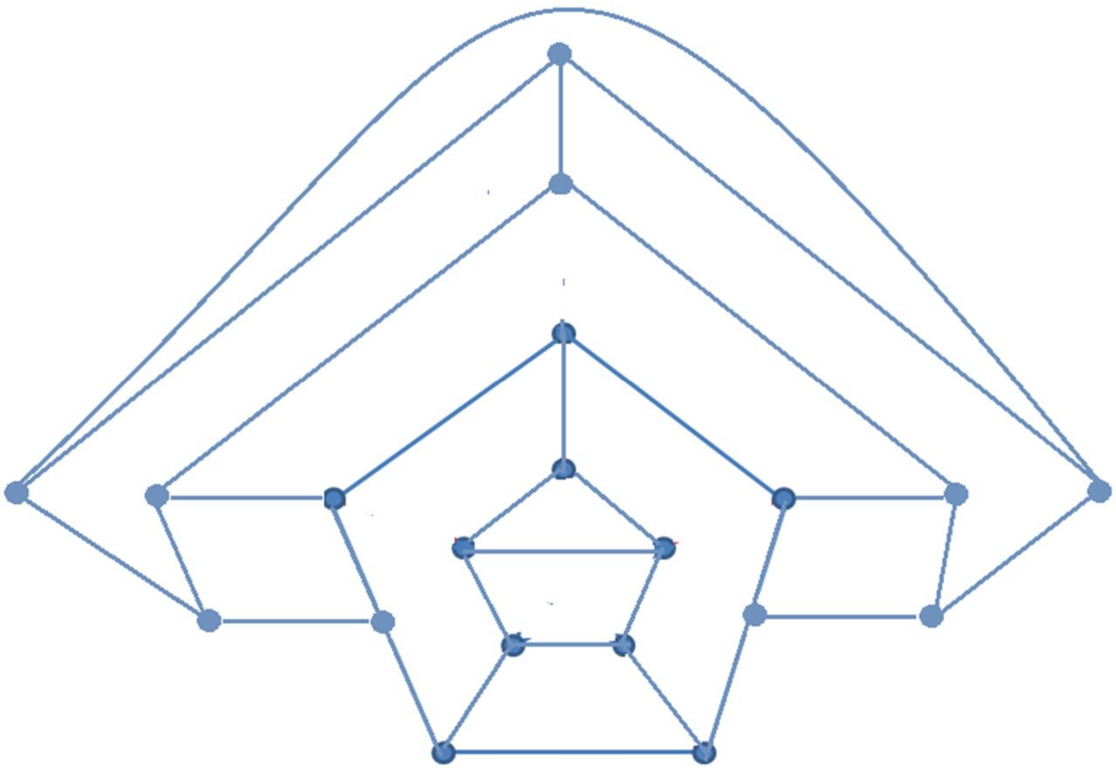


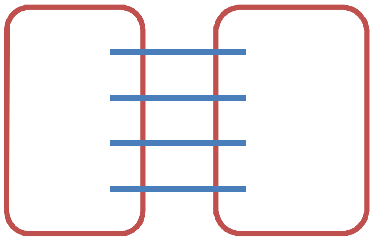
Proper 4-edge cut



Not a proper 4-edge cut

Covering Proper 3- and 4-Edge Cuts in Bridgeless Cubic Graphs





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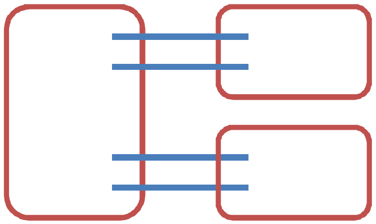


Not a proper 4-edge cut

•Covering 3- and 4-Edge Cuts

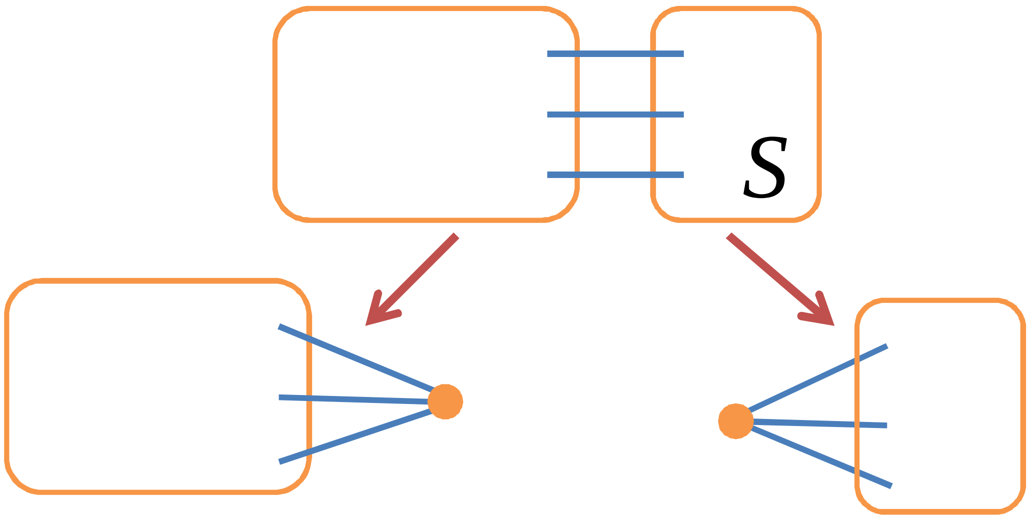
Kaiser and Škrekovski (2008)

Every bridgeless cubic graph has a 2-factor covering all 3- and proper 4-edge cuts.

An Efficient Algorithm for Finding a 2-Factor Covering 3- and 4-Edge Cuts in Bridgeless Cubic Graphs.

Algorithm Outline

Find a minimal proper 3-edge cut.

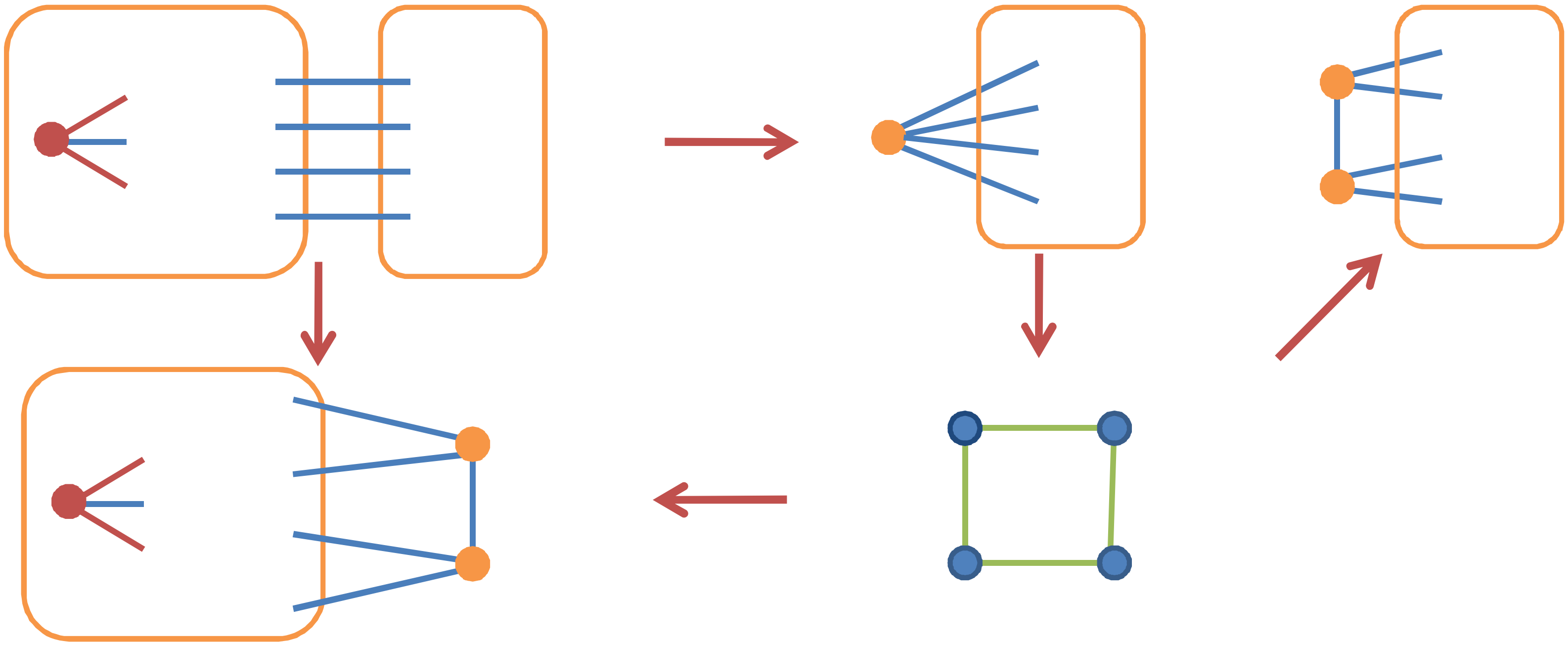


Find a 2-factor covering all proper 3- and 4-edge cuts recursively.

Find a 2-factor covering all proper 4-edge cuts and containing the two edges.

Covering 4-Edge Cuts

Find a minimal proper 4-edge cut.



Find a 2-factor covering all 4-edge cuts and containing the two edges recursively.

Complexity Analysis

- Finding a minimal proper 3-edge cut: $O(n^2)$
- Finding a minimal proper 4-edge cut when no proper 3-edge cuts exist: $O(n^2)$
- Finding a perfect matching containing a specified edge: $O(n \log^4 n)$

Biedl, Bose, Demaine, Lubiw (2001)

Total running time: $O(n^3)$

Question

An Efficient Algorithm for Minimum Weight Factor Covering 3- and 4-Edge Cuts in Bridgeless Cubic Graphs?

What is known for △ and □-free 2-factors ?

Unweighted:

Subcubic graphs: in P (Hartvigsen, Li 2009)
(Bercziand, Vegh 2010)

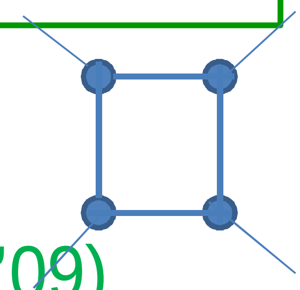
General graphs: unknown

Weighted:

NP-hard, even if G is cubic (Vornberger, 1980)

even if G is cubic, bipartite and planar (Berczi and Kobayashi, 2009)

What is known for △ and □-free 2-factors ?
With special weights: Vertex-induced



Subcubic graphs:

Polynomial-time algorithm (Berczi and Kobayashi,'09)

Bipartite graphs:

Polynomial-time algorithm (Takazawa,2009)

Polyhedral result: There always exists an integer optimal for

$$\min wx \quad \text{s. t.} \quad x(\delta(v)) = 2 \text{ for all } v \in V$$
$$x(E(S)) \leq 3 \text{ for all squares } S$$
$$0 \leq x_e \leq 1 \quad \text{for all } e \in E$$

Using these ideas to get approximation algorithms

For graph TSP on cubic graphs:

Find a triangle-square free 2-factor, join with a doubled MST

Get an Eulerian graph with at most $n + 2(n/5 - 1) = 7/5\, n - 2$

Using these ideas to get approximation algorithms

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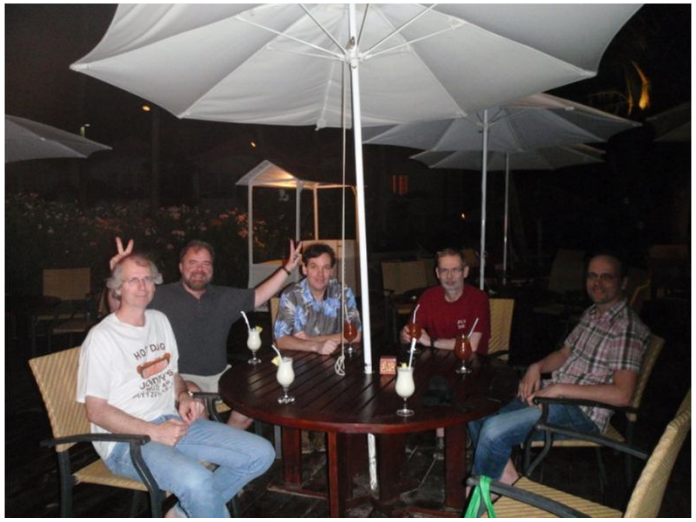
7/5-approximation algorithm for general graph TSP

(Sebo and Vygen, 2012)



New Result

A 6/5-Approximation Algorithm for the Minimum 2-Edge-Connected Spanning Subgraph Problem in 3-Edge-Connected Cubic Graphs.

Minimum 2-Edge-Connected Subgraph

Approximation Algorithms for General Graphs

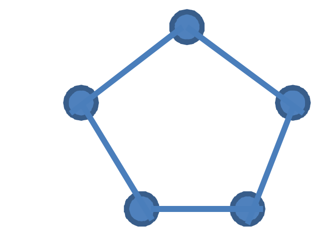
• Khuller, Vishkin (1994)	3/2
• Cheriyan, Sebő, Szigeti (1998)	17/12
• Vempala, Vetta (2000)	4/3
• Sebo, Vygen (2012)	4/3
• Jothi, Ranghavachari, Varadarajan (2003)	5/4

For 3-Edge-Connected Cubic Graphs

• Huh (2004)	5/4
• This work	6/5

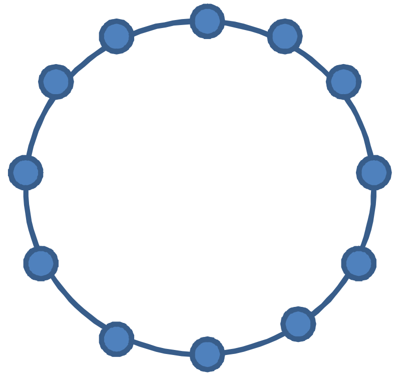
6/5-Approximation Algorithm

A 2-Factor Covering 3- and 4-Edge Cuts



Small Cycles

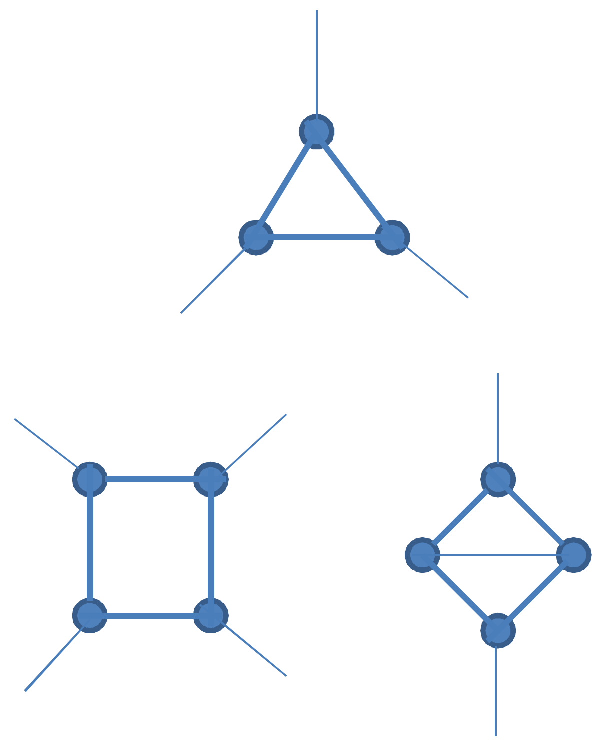
$5 \leq \text{length} \leq 9$



Large Cycles

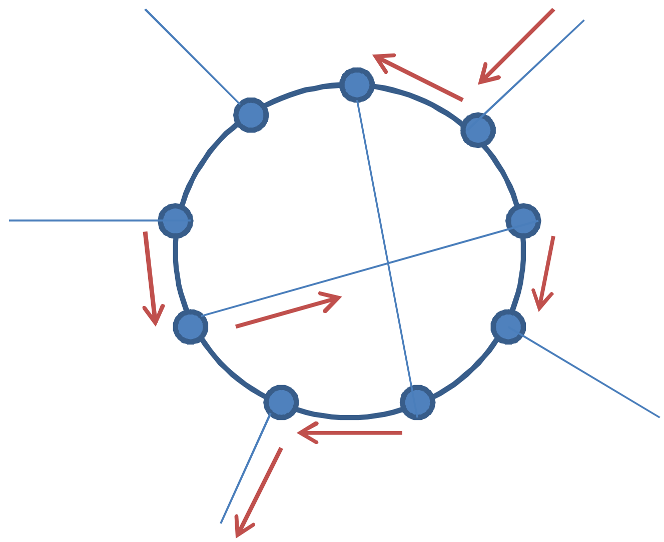
$\text{length} \geq 10$

Forbidden Cycles



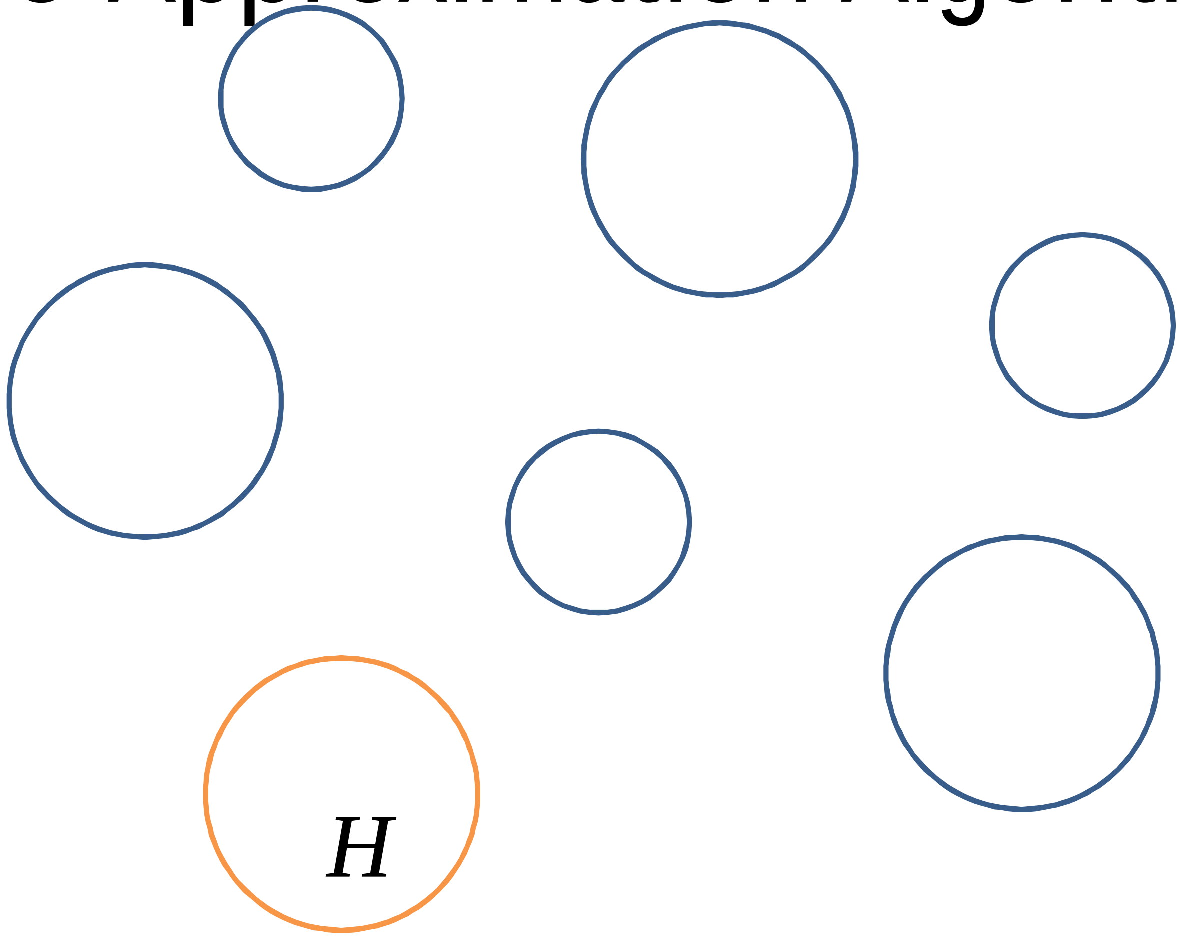
6/5-Approximation Algorithm

A small cycle has at most two chords.

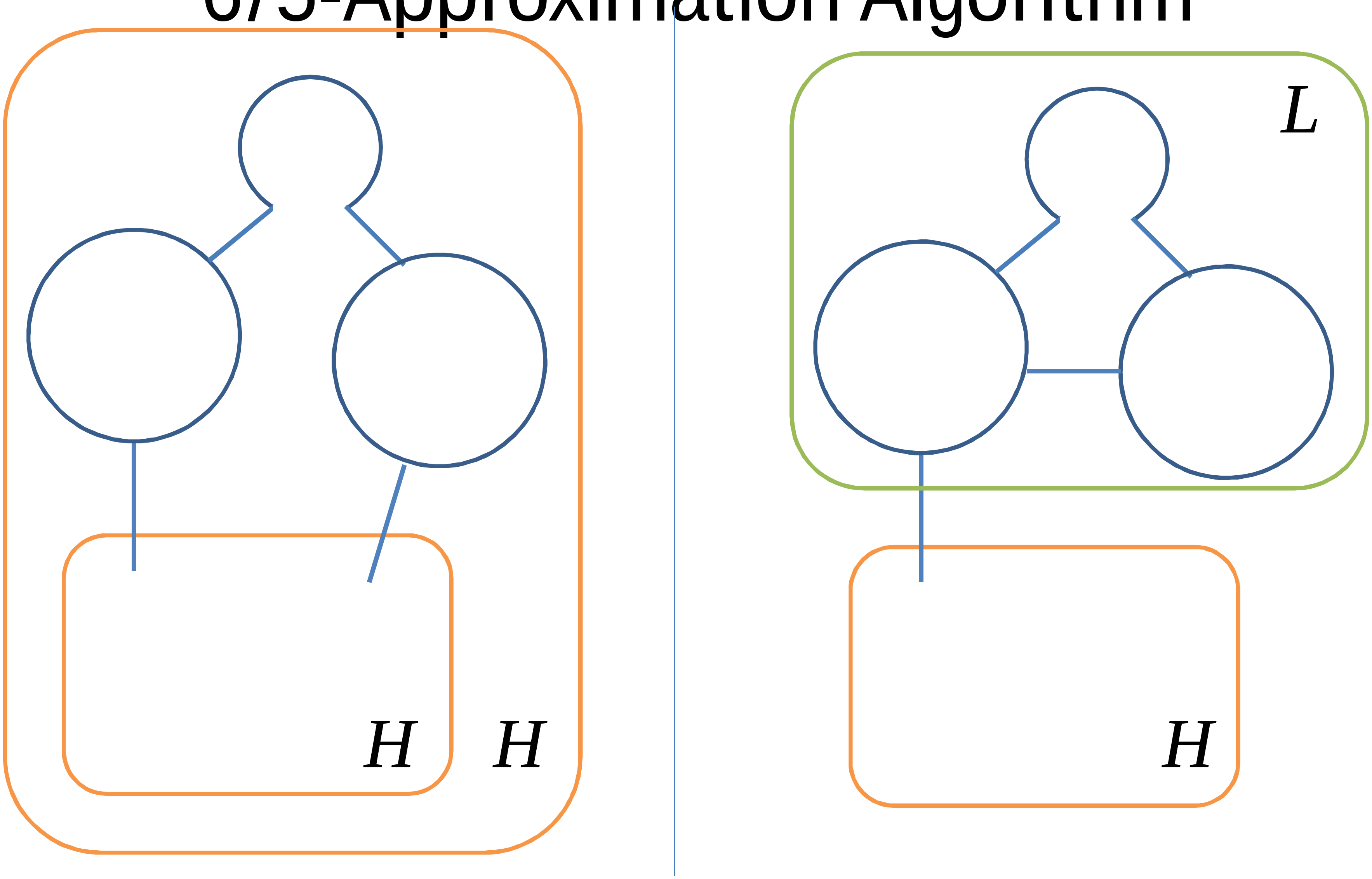


Starting from any vertex adjacent to the outside, one can get out of the small cycle after visiting every vertex exactly once.

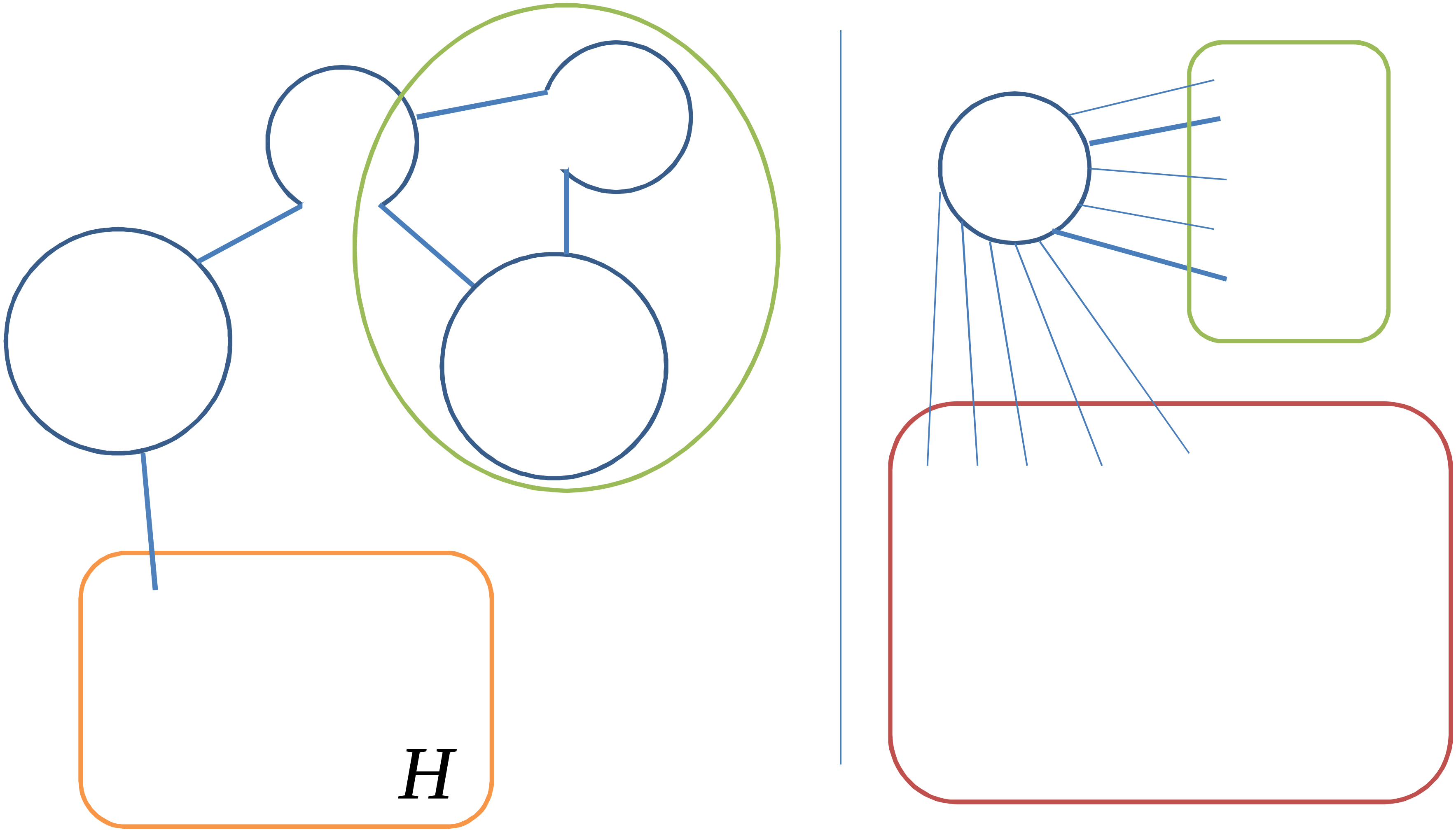
6/5-Approximation Algorithm



6/5-Approximation Algorithm



6/5-Approximation Algorithm



6/5-Approximation Algorithm

At termination,

$$|E(H)| \le \frac{6}{5}n - 1.$$

$$\left.\begin{array}{l} k : \#\,\text{small cycles} \\ \ell : \#\,\text{large cycles} \end{array}\right\} \quad n \ge 5k + 10\ell$$

$$|E(H)| \le n - (k-1) + 2(k + \ell - 1)$$

$$= n + k + 2\ell - 1 \le \frac{6}{5}n - 1.$$

Happy Birthday

Bill!