Weighted Linear Matroid Parity

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Extensions of Matching and Matroids

- Matroid Parity [Matroid Matching]

  Lawler (1971), Lovász (1978)

- Jump Systems ← Delta-Matroids

  Bouchet & Cunningham (1995)

- Path-Matching → Even Factors

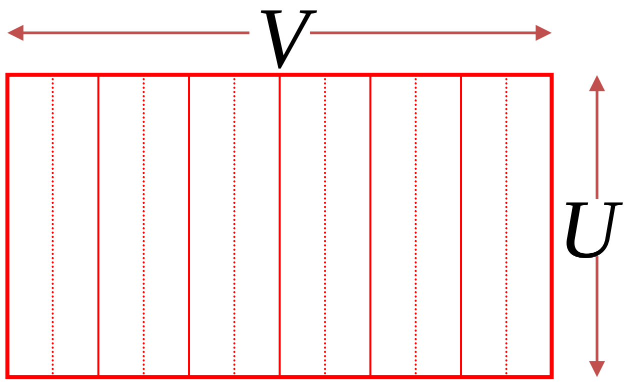
  Cunningham & Geelen (1997)

W. H. Cunningham: Matching, Matroids, and Extensions, MPB 91 (2002), 515-542.

Y. Kobayashi & K. Takazawa: Even Factors, Jump Systems, and Discrete Convexity, JCTB 99 (2009), 139-161.

Linear Matroid Parity

$A =$



Partitioned into Pairs

Line $\ell = \{u, \overline{u}\} \in L$

Parity Set: Union of Lines

Find an Independent Parity Set of Maximum Size

$\nu(A)$: Maximum Size of an Independent Parity Set

$$\mu(A) := \frac{1}{2}\nu(A)$$ Maximum Number of Lines in a Base

Linear Matroid Parity

Min-Max Theorem — Lovász (1978)

$$\mu(A) = \min\left\{ \dim K + \sum_{i=1}^{k} \left\lfloor \frac{\dim L_i / K}{2} \right\rfloor \right\}$$

K : Linear Subspace of $\operatorname{span} A$

$L_i := \operatorname{span} A[R, V_i]$

$V_1, \ldots, V_k$: Partition of V into Parity Sets

Linear Matroid Parity

Algorithms

Lovász (1978)	Polynomial
Gabow & Stallmann (1986)	$O(nr^3)$
Orlin & Vande Vate (1990)	$O(nr^4)$
Orlin (2008)	$O(nr^3)$
Cheung, Lau, Leung (2011)	$O(nr^2)$
	Randomized

$r = |U|, \quad n = |V|$

Applications of Linear Matroid Parity

- Unique Solvability of RCG Circuits

  Milic (1974)
- Pinning Down Planar Skeleton Structures

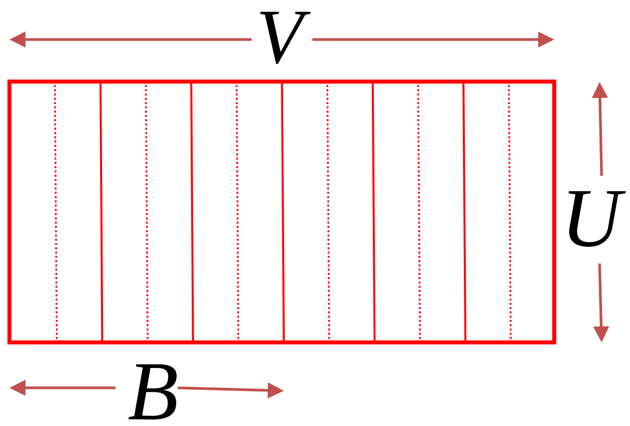
  Lovász (1980)
- Maximum Genus Cellular Embedding

  Furst, Gross, McGeoch (1988)
- Maximum Number of Disjoint $\mathscr{S}$-paths

  Lovász (1980), Schrijver (2003)

Weighted Linear Matroid Parity

$A =$



$$w : L \to \mathbf{R}$$

$$w(B) := \sum_{\ell \subseteq B} w(\ell)$$

Find a Parity Base of Minimum Weight

Minimum Weight Perfect Matching in General Graphs

Minimum Weight Common Base of Two Linear Matroids

Main Result and Tools

A Combinatorial, Deterministic, Strongly Polynomial Algorithm

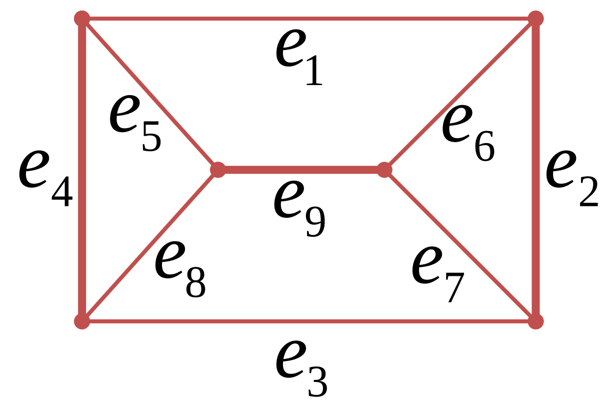
- Polynomial Matrix Formulation.
- Combinatorial Relaxation by Murota (1990).
- Augmenting Path Algorithm of Gabow and Stallmann (1986).

Running Time Bound: $O(nr^3)$

Alternating Matrix

$$T = \begin{bmatrix} 0 & t_1 & 0 & -t_4 & -t_5 & 0 \\ -t_1 & 0 & t_2 & 0 & 0 & -t_6 \\ 0 & -t_2 & 0 & t_3 & 0 & -t_7 \\ t_4 & 0 & -t_3 & 0 & t_8 & 0 \\ t_5 & 0 & 0 & -t_8 & 0 & t_9 \\ 0 & t_6 & t_7 & 0 & -t_9 & 0 \end{bmatrix}$$

Graph $G(T)$



$$\mathrm{Pf}\, T := \sum_M \mathrm{sgn}(M) \prod_{(u,v)\in M} T_{uv}$$

$$\det T = (\mathrm{Pf}\, T)^2$$

$$\mathrm{rank}\, T \leq 2\mu(G(T))$$

$\mu(\cdot)$: Maximum Matching Size

Matrix Formulation

Linear Matroid Parity

Geelen & I. (2005)

$$\hat{A} = \begin{pmatrix} O & A \\ -A^{\mathrm{T}} & \begin{matrix} D_1 & & & O \\ & D_2 & & \\ & & \ddots & \\ O & & & D_{n/2} \end{matrix} \end{pmatrix}$$

$$D_\ell = \begin{bmatrix} 0 & \alpha_\ell \\ -\alpha_\ell & 0 \end{bmatrix}$$

α_ℓ : Indeterminate

$\nu(A)$: Maximum Size of an Independent Parity Set

$$\nu(A) = \mathrm{rank}\,\hat{A} - n$$

Polynomial Matrix Formulation

Weighted Linear Matroid Parity

$$\hat{A}(\sigma) = \begin{bmatrix} O & A \\ -A^{\mathrm{T}} & \begin{matrix} D_1 & & O \\ & D_2 & \\ O & & \ddots \quad D_{n/2} \end{matrix} \end{bmatrix}$$

$$D_\ell = \begin{bmatrix} 0 & \alpha_\ell \sigma^{w(\ell)} \\ -\alpha_\ell \sigma^{w(\ell)} & 0 \end{bmatrix}$$

α_ℓ : Indeterminate

$\zeta(A)$: Minimum Weight of a Parity Base

$$\zeta(A) = \sum_{\ell \in L} w(\ell) - \deg_\sigma \mathrm{Pf}\, \hat{A}(\sigma)$$

Augmenting Path Algorithm

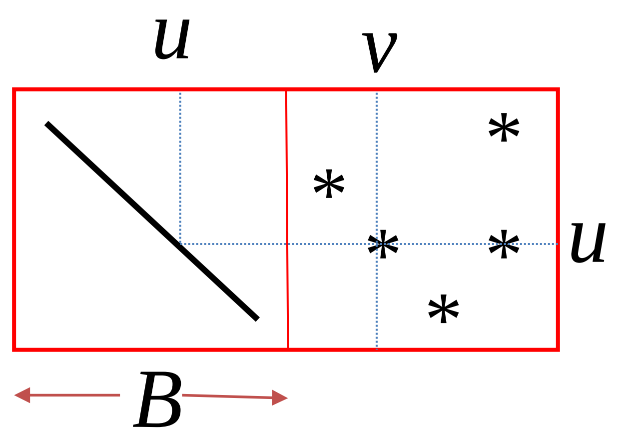
Gabow & Stallmann (1986)

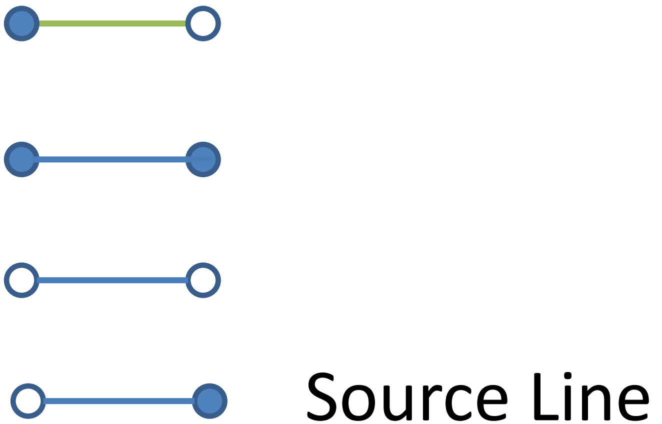
B: Base

$$G_B = (V, F \cup L)$$

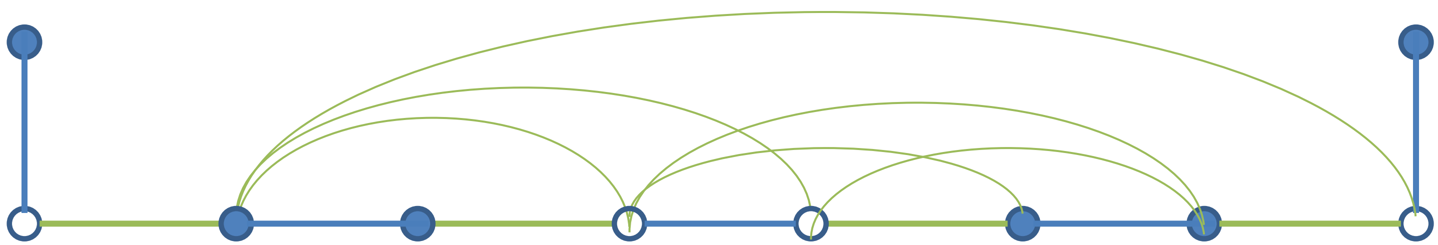
$$F = \{(u,v) \mid u \in B, v \in V \setminus B, B - u + v : \text{base}\}$$

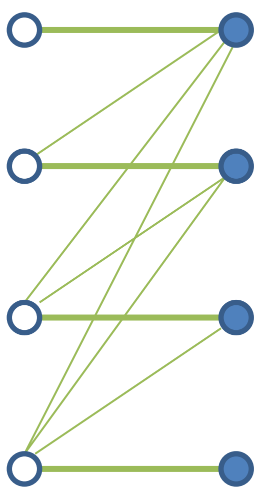
$A_B =$

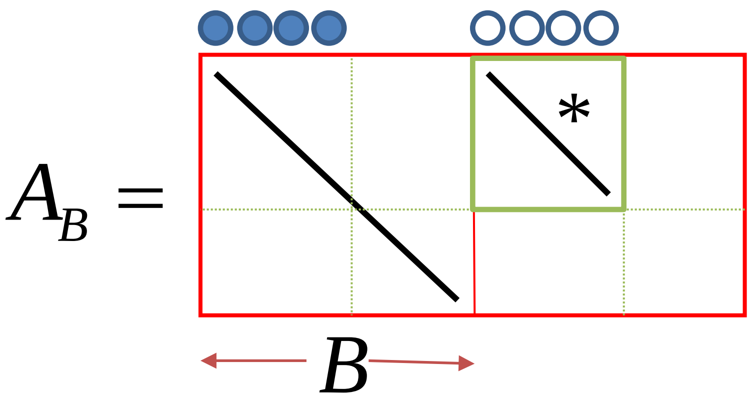




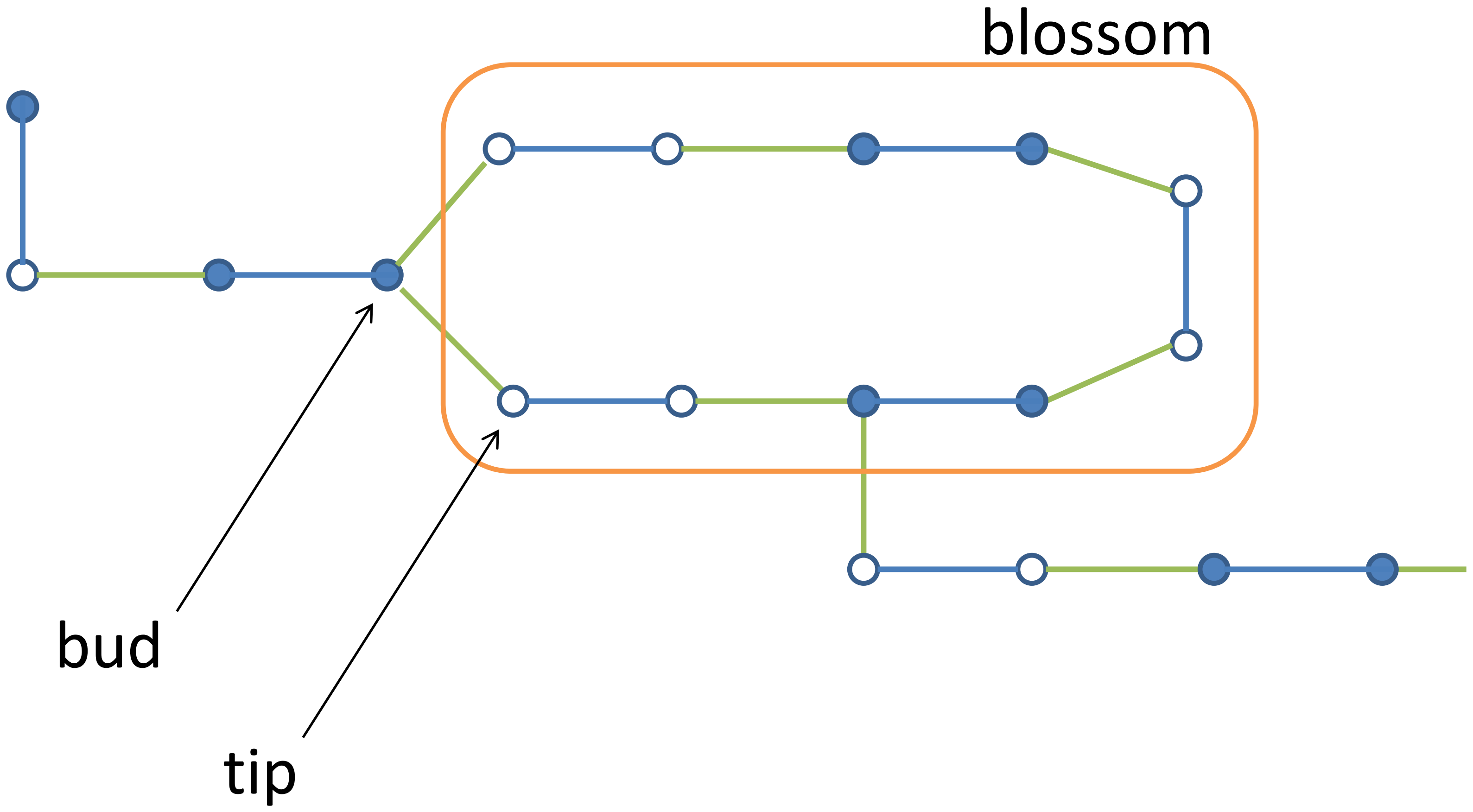
Augmenting Path Algorithm



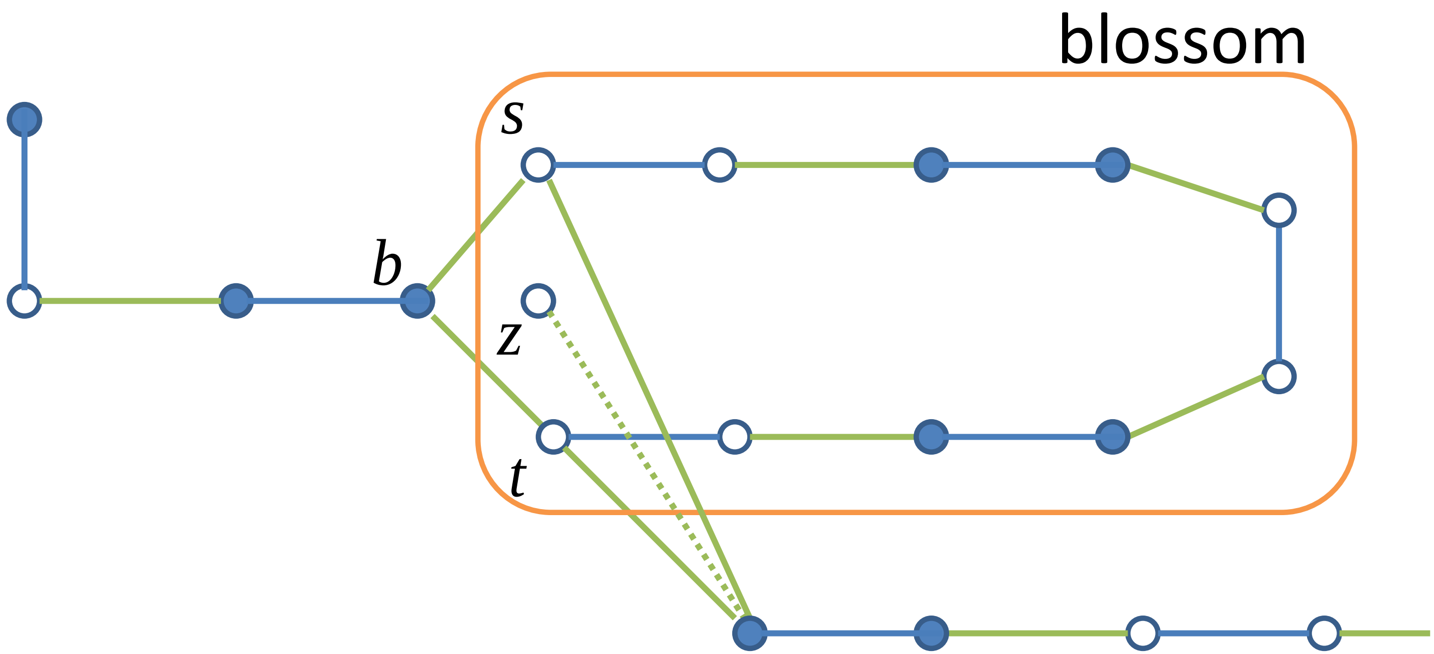


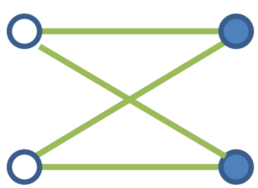


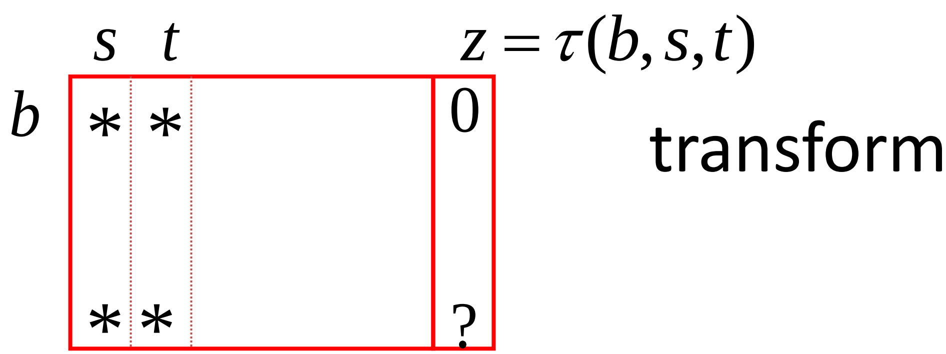
Augmenting Path Algorithm



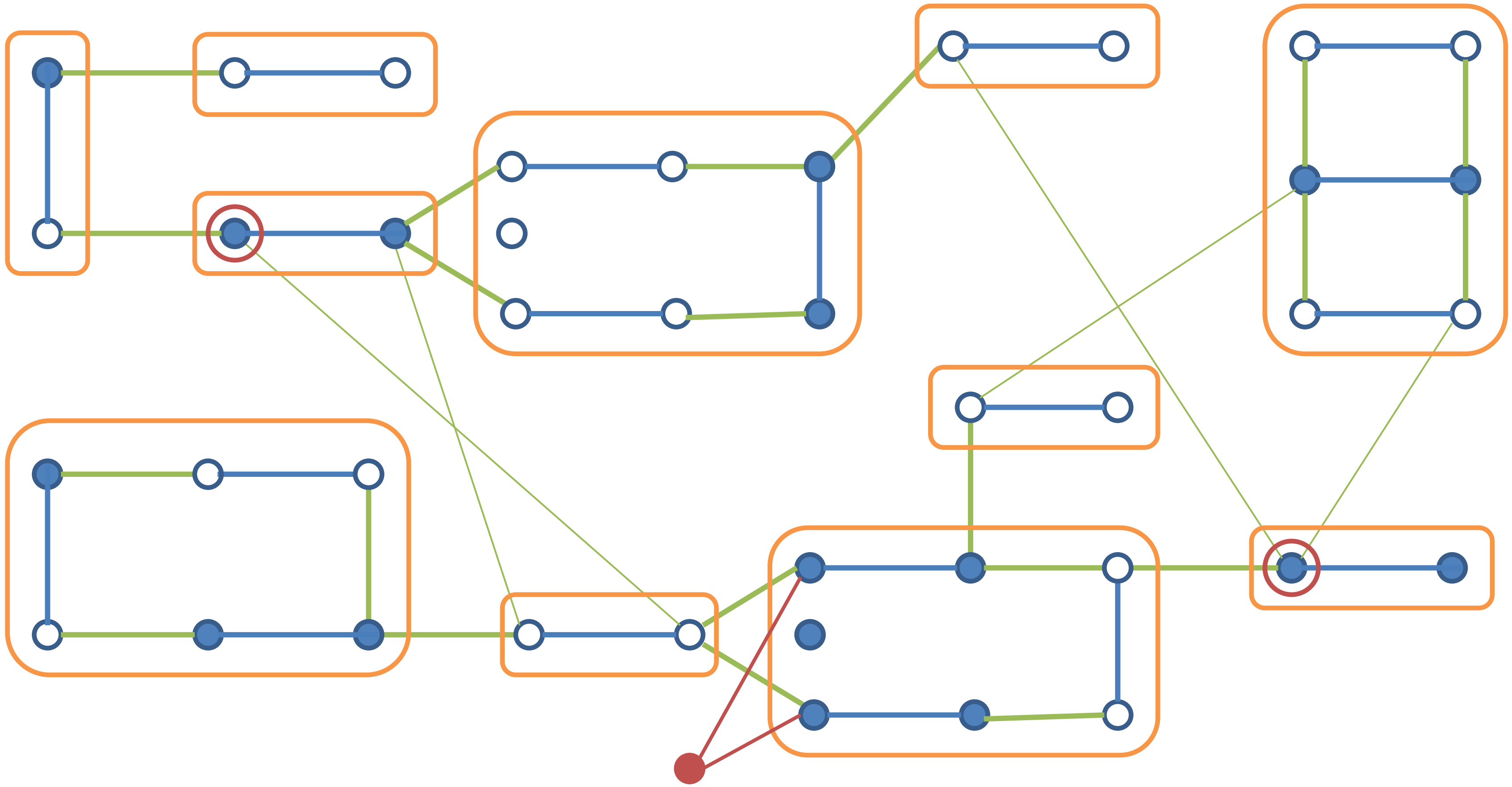
Augmenting Path Algorithm







Augmenting Path Algorithm



$$\nu(A) = \min\left\{\dim K + \sum_{i=1}^{k} \left\lfloor \frac{\dim L_i / K}{2} \right\rfloor\right\}$$

Primal-Dual Algorithm

B: Base $\quad G_B = (V, F \cup L)$

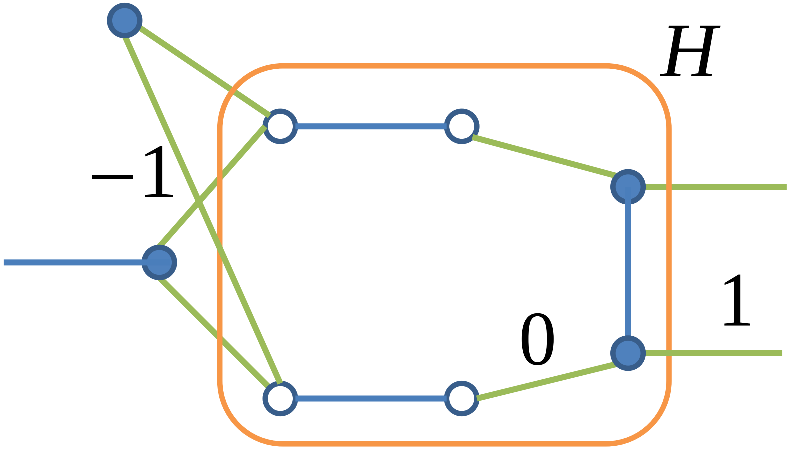
Laminar Family of Blossoms

$$\Lambda = \{H_1, H_2, \ldots, H_k\}$$

Dual Variables

$$p : V \to \mathbf{R}$$

$$q : \Lambda \to \mathbf{R}_+$$



$$Q_{uv} := \sum_{H \in \Lambda} \eta(H,u,v) q(H)$$

(DF1) $p(u) + p(\bar{u}) = w(\ell), \quad \forall \ell = \{u, \bar{u}\} \in L$

(DF2) $p(v) - p(u) \geq Q_{uv}, \quad \forall (u,v) \in F.$

(DF3) $p(z) = p(s), \quad \forall z = \tau(b,s,t).$

Initial Steps

$\Lambda := \phi$. Set p so that (DF1) is satisfied.

Find a base B that minimizes $p(B) = \sum_{u \in B} p(u)$.

Extract the set F^* of tight edges in F.

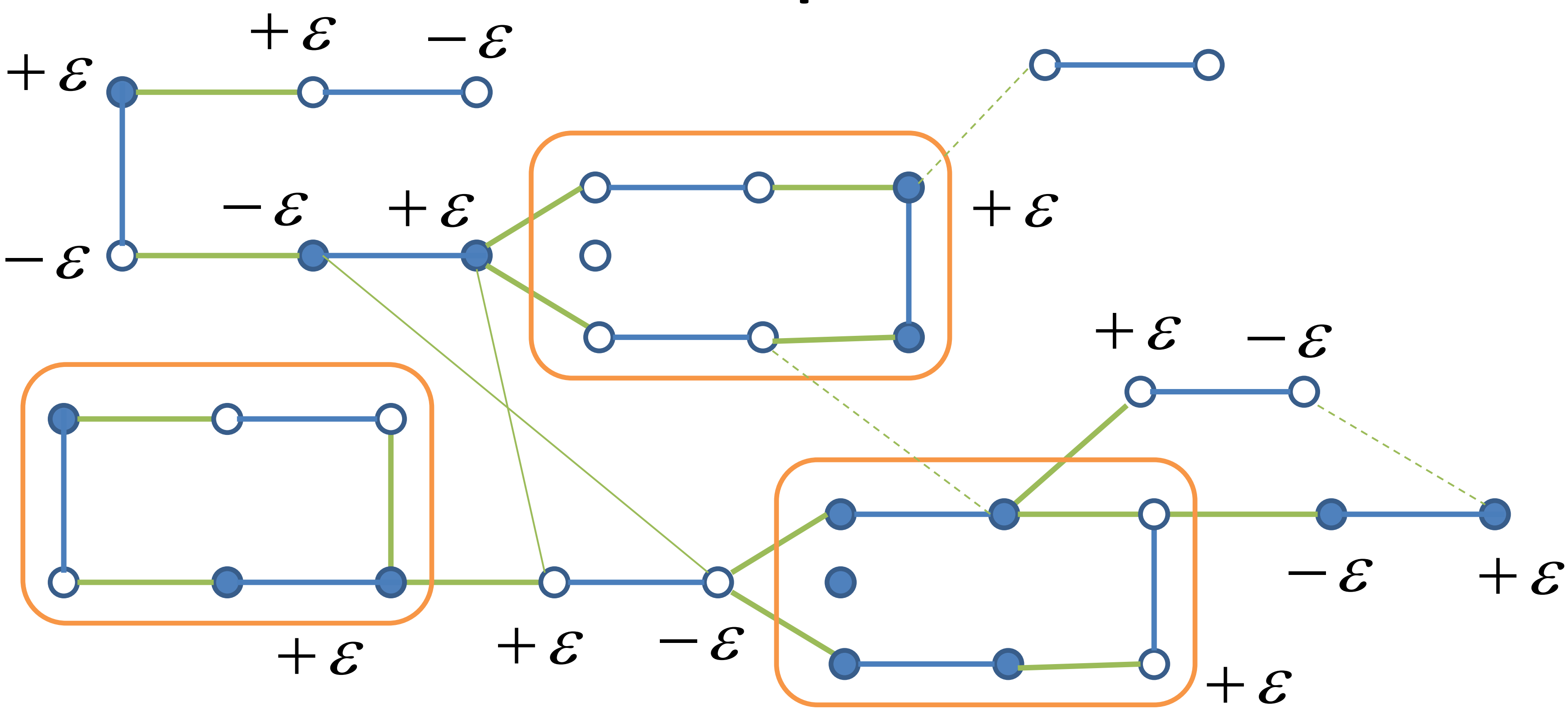
Search for an augmenting path in $G_B^* = (V, F^* \cup L)$.

(DF1) $p(u) + p(\overline{u}) = w(\ell), \quad \forall \ell = \{u, \overline{u}\} \in L$

(DF2) $p(v) - p(u) \geq Q_{uv}, \quad \forall (u, v) \in F.$

(DF3) $p(z) = p(s), \quad \forall z = \tau(b, s, t).$

Dual Update

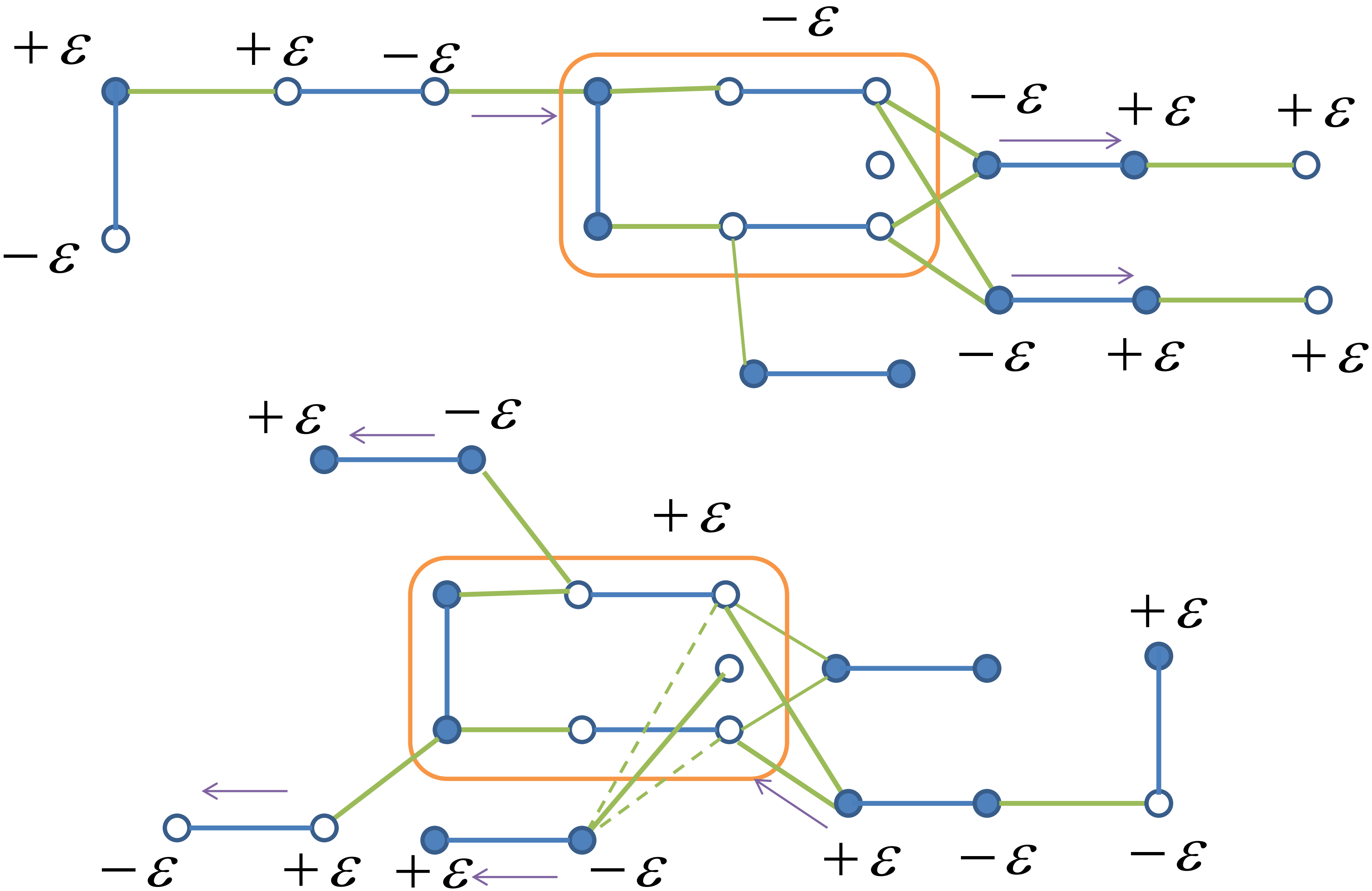


(DF1) $p(u)+p(\bar{u})=w(\ell), \quad \forall \ell=\{u,\bar{u}\}\in L$

(DF2) $p(v)-p(u)\geq Q_{uv}, \quad \forall (u,v)\in F.$

(DF3) $p(z)=p(s), \quad \forall z=\tau(b,s,t).$

Blossoms



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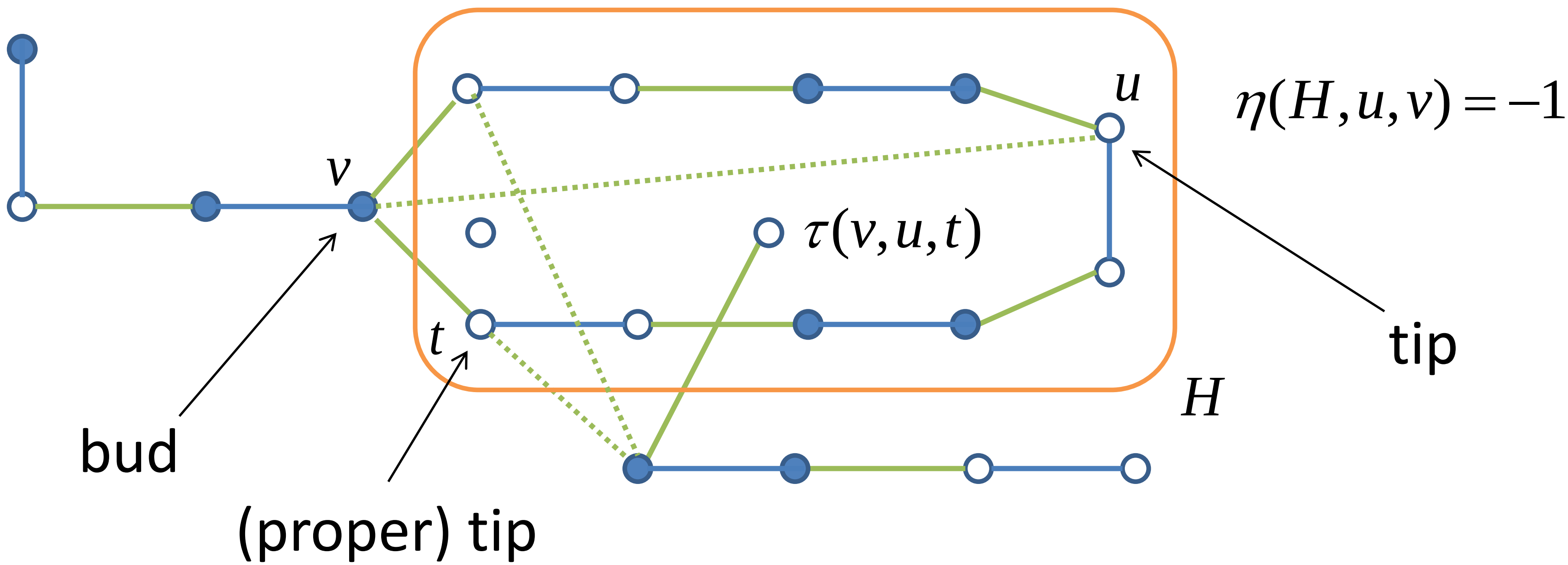
Blossoms



(DF1) $p(u)+p(\bar{u})=w(\ell), \quad \forall \ell=\{u,\bar{u}\}\in L$

(DF2) $p(v)-p(u)\geq Q_{uv}, \quad \forall (u,v)\in F.$

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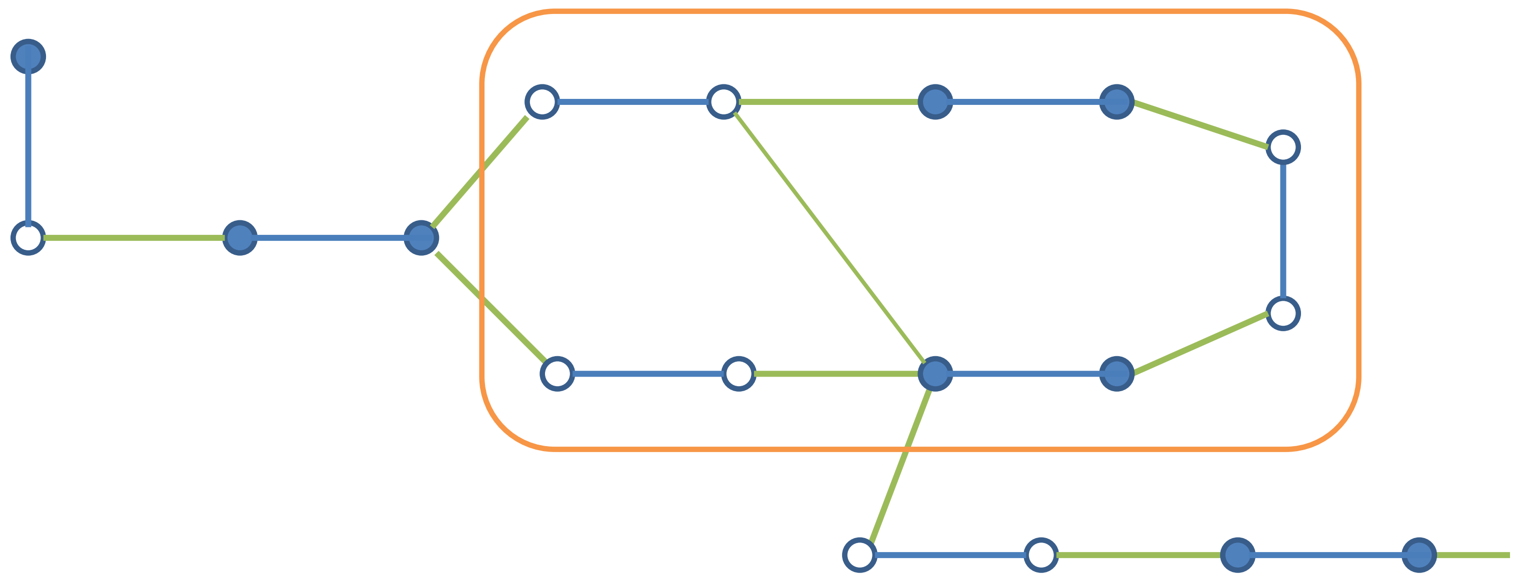
Augmentation



(DF1) $p(u) + p(\bar{u}) = w(\ell), \quad \forall \ell = \{u, \bar{u}\} \in L$

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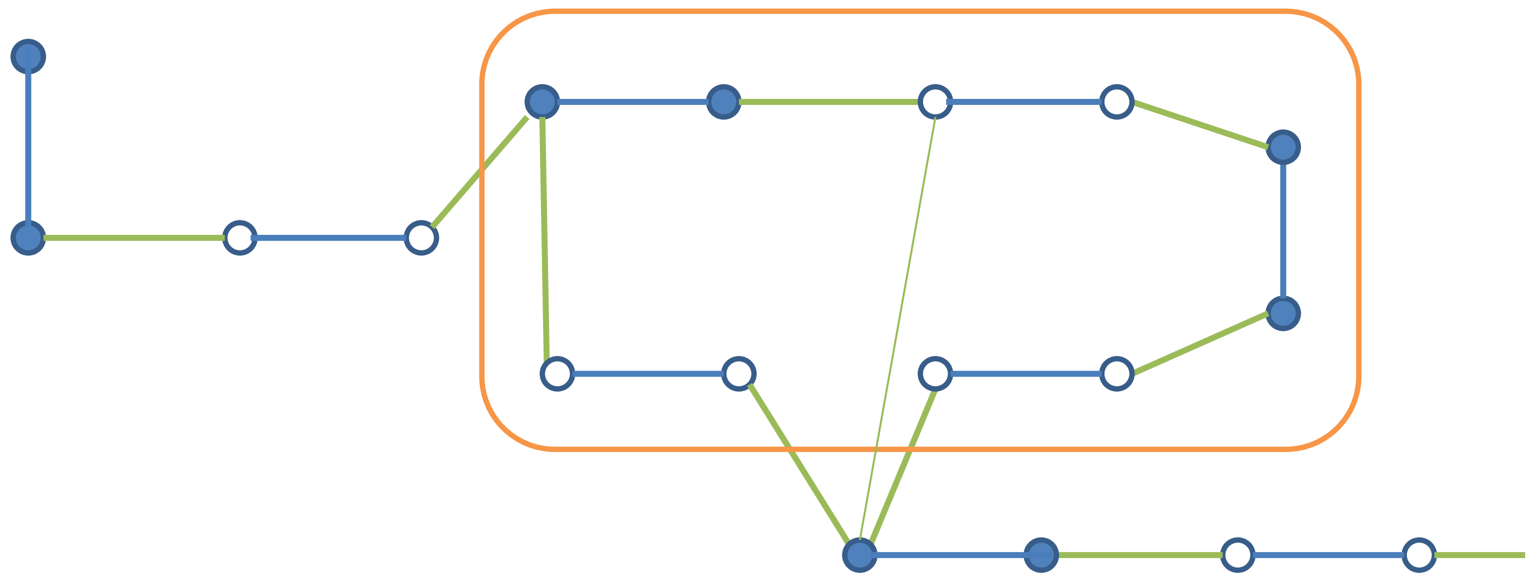
Augmentation



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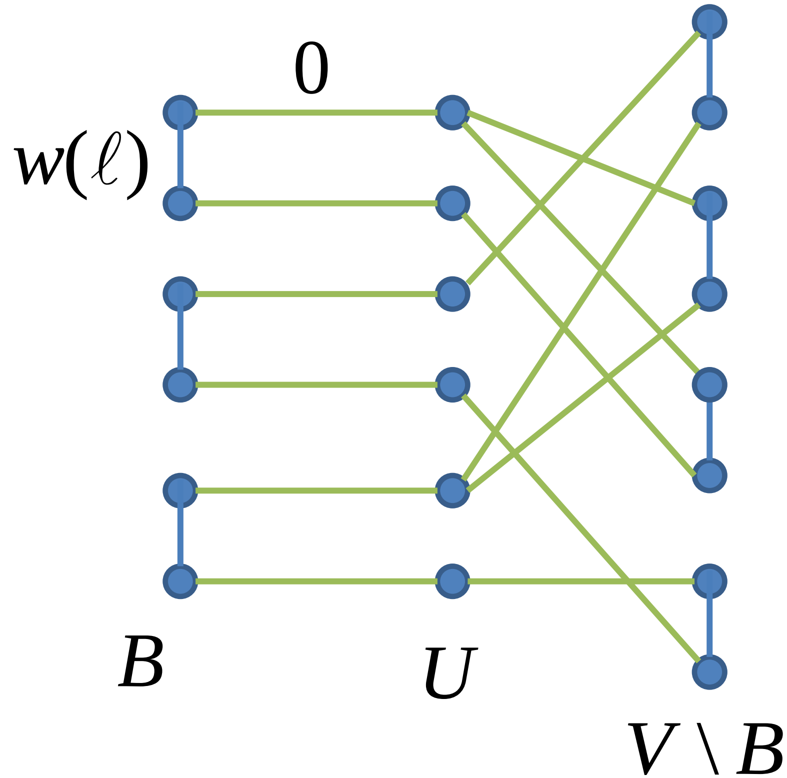
Optimality

$$\hat{A}(\sigma) = \begin{pmatrix} O & A \\ -A^{\mathrm{T}} & \begin{matrix} D_1 & & & O \\ & D_2 & & \\ & & \ddots & \\ O & & & D_{n/2} \end{matrix} \end{pmatrix}$$



$\deg_{\sigma} \mathrm{Pf}\, \hat{A}(\sigma) \leq$ Maximum Weight of a Perfect Matching

Optimality

$$y(u) := p(u) \quad (u \in V)$$
$$y(u') := -p(u) \quad (u' \in U)$$

Dual Linear Program

$$W = U \cup V$$

$$\text{Minimize} \quad \sum_{v \in W} y(v) - \sum_{S \in \Omega} z(S)$$

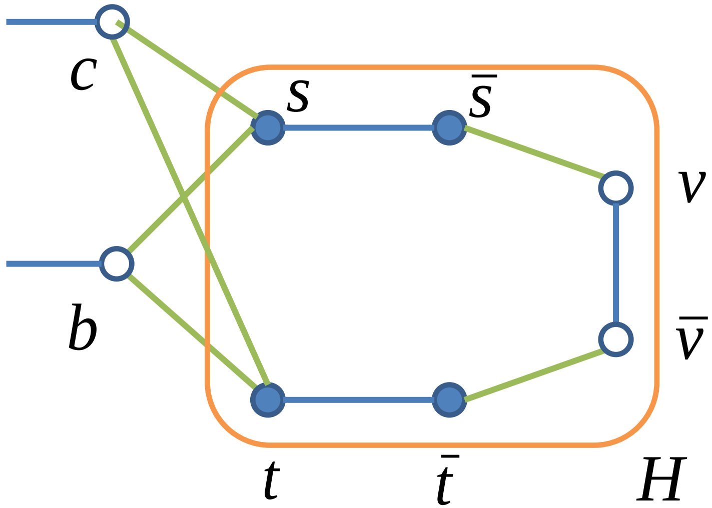
$$\text{subject to} \quad \sum_{u \in \partial e} y(u) - \sum_{S \in \Omega, e \in \Delta S} z(S) \geq w(e) \quad (\forall e \in E)$$

$$z(S) \geq 0 \quad (\forall S \in \Omega)$$

$\deg_\sigma \mathrm{Pf}\, \hat{A}(\sigma) \leq$ Maximum Weight of a Perfect Matching

$\leq$ Dual Objective Value

Optimality



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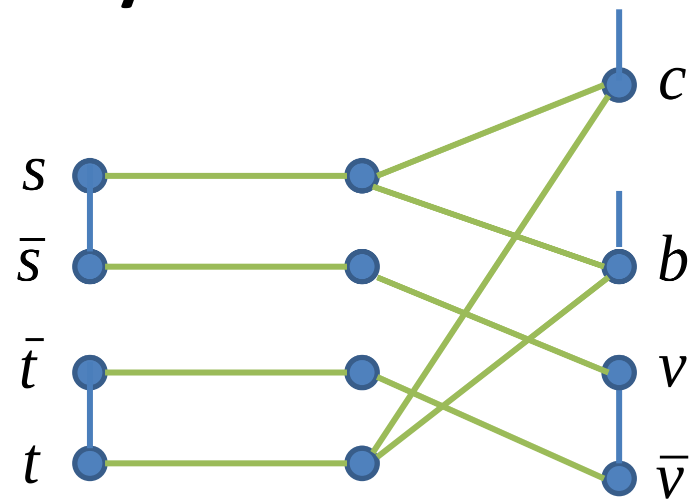

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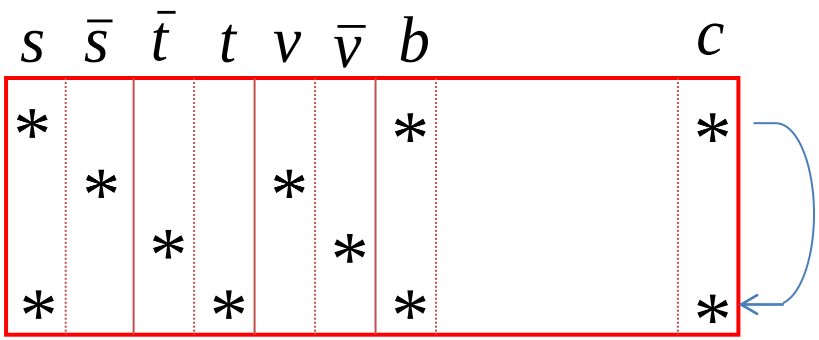

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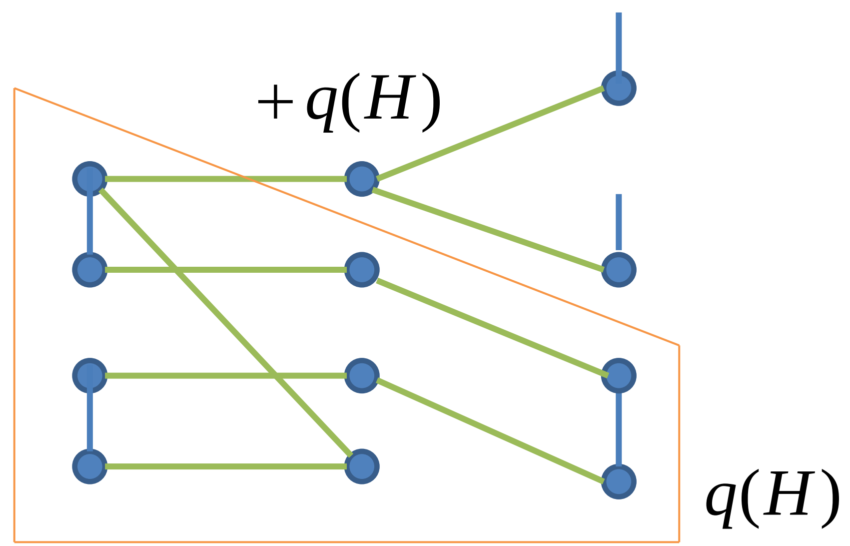
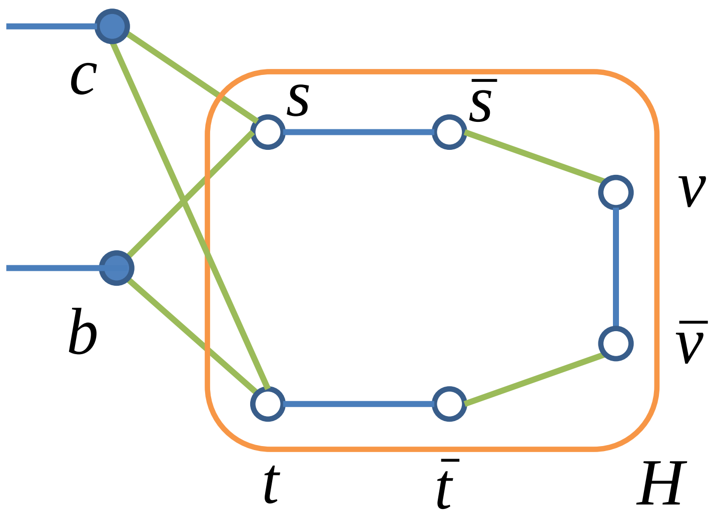


$+q(H)$
$q(H)$

Optimality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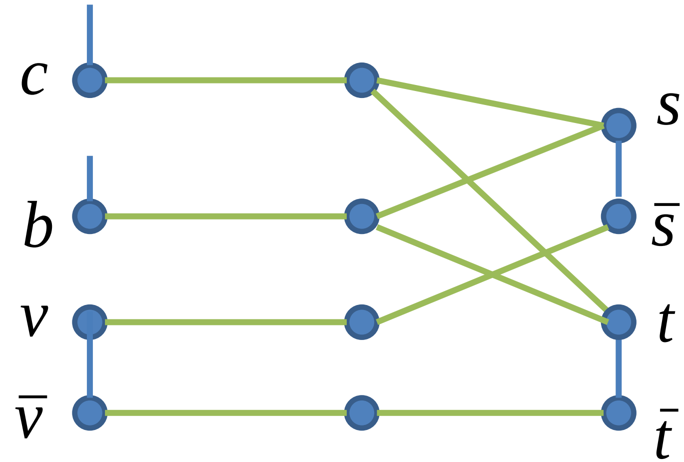
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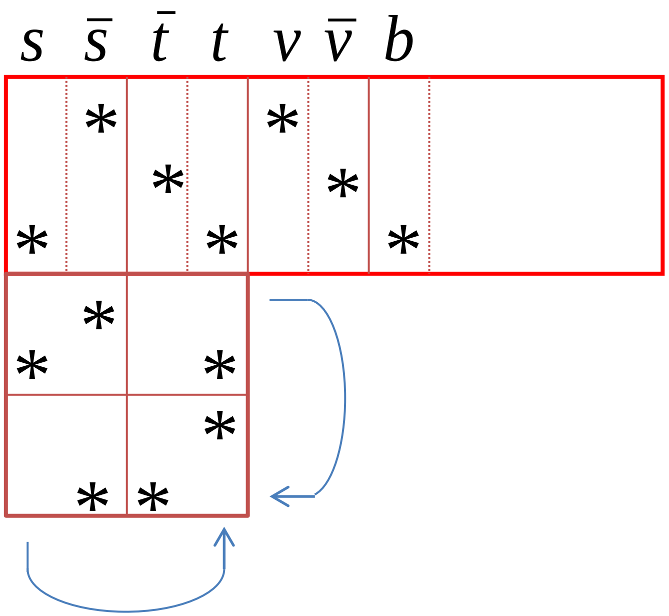
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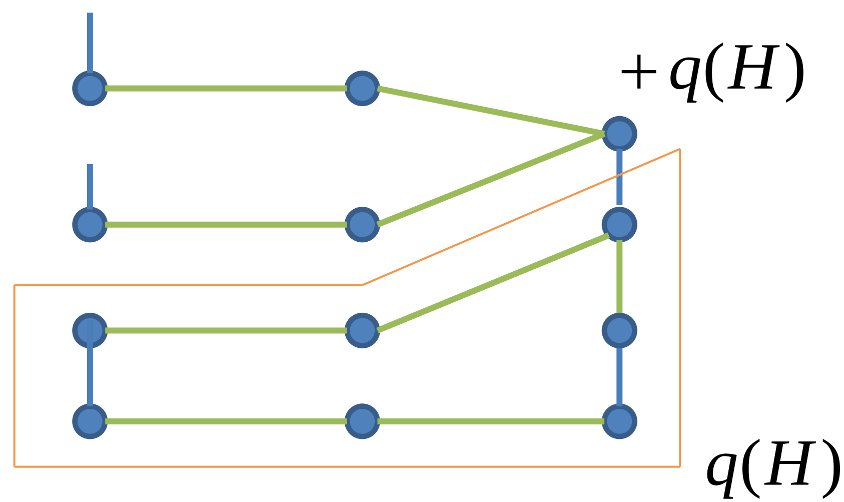
$+q(H)$
$q(H)$

Optimality

$\deg_\sigma \operatorname{Pf} \hat{A}(\sigma) \leq$ Maximum Weight of a Perfect Matching

$$\leq \sum_{v \in W} y(v) - \sum_{S \in \Omega} z(S)$$

$$= \sum_{v \in V \setminus B} p(v) = \sum_{\ell \subseteq V \setminus B} w(\ell)$$

$$\zeta(A) = \sum_{\ell \in L} w(\ell) - \deg_\sigma \operatorname{Pf} \hat{A}(\sigma)$$

$$\therefore \zeta(A) \geq \sum_{\ell \subseteq B} w(\ell)$$

Good Characterization

There exist nonsingular matrices Y and Z such that

$$\hat{A}'(\sigma) := \begin{bmatrix} Y^{\mathrm{T}} & O \\ O & Z^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} O & A \\ -A^{\mathrm{T}} & D \end{bmatrix} \begin{bmatrix} Y & O \\ O & Z \end{bmatrix}$$ satisfies

$\deg_\sigma \operatorname{Pf} \hat{A}'(\sigma) =$ Maximum Weight of a Perfect Matching in $G(\hat{A}'(\sigma))$.

$$\zeta(A) = \sum_{\ell \in L} w(\ell) - \deg_\sigma \operatorname{Pf} \hat{A}(\sigma)$$

Combinatorial Relaxation Method by Murota (1990)

Questions

- Polyhedral Description ?

  Gyula Pap's talk

- Applications to Approximate Algorithms

  5/3-Approx. Algorithm for Steiner Tree

  Prömel & Steger (1998)

  How about Metric TSP ?