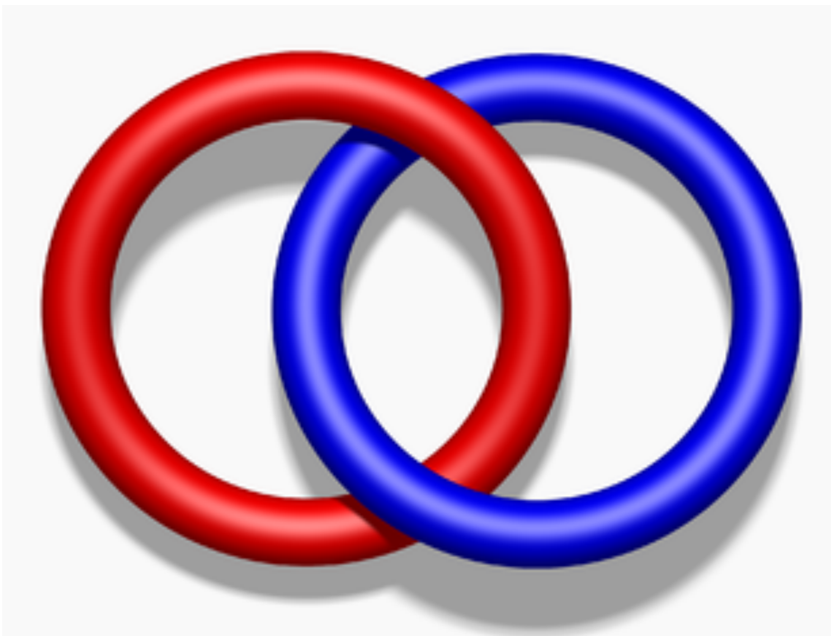
Pairs of Disjoint Cycles



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McGill

Based on joint work with
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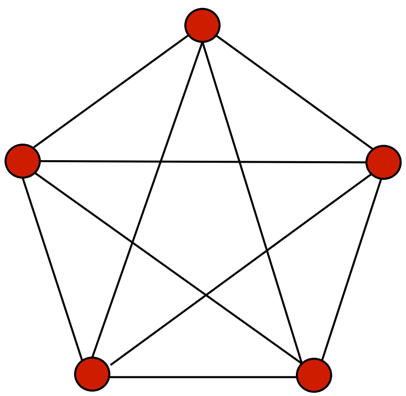
Robin Thomas (Georgia Tech)

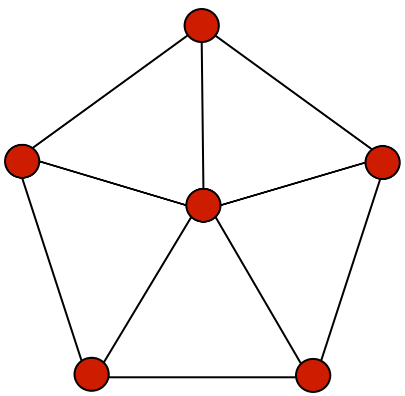
Conference in Honour of Bill Cunningham's 65th Birthday
June 13th, 2012

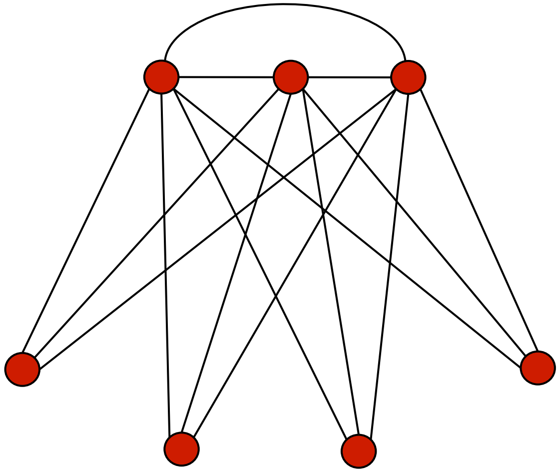
Two vertex-disjoint cycles

Theorem(Lovasz,1965): A graph G with minimum degree 3 contains no two vertex-disjoint cycles if and only if

- either G – v is a forest for some vertex v, or
- G is a wheel, or
- G is $K_{5,}$ or
- G – {v,u,w} is edgeless for some triple of vertices v,u,w.



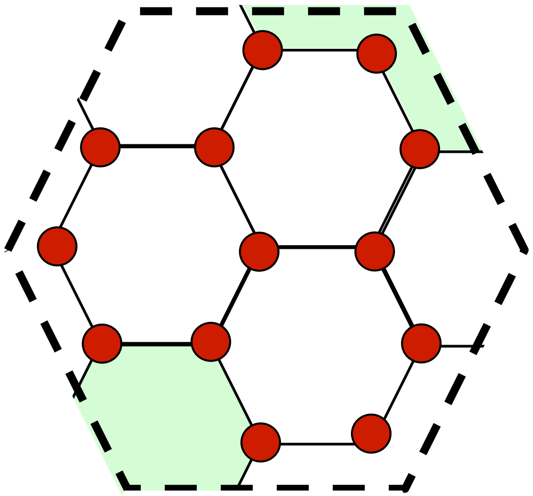




Two vertex-disjoint odd cycles

<u>Theorem(Slilaty, 2003)</u>: If a graph G contains no two vertex-disjoint odd cycles then

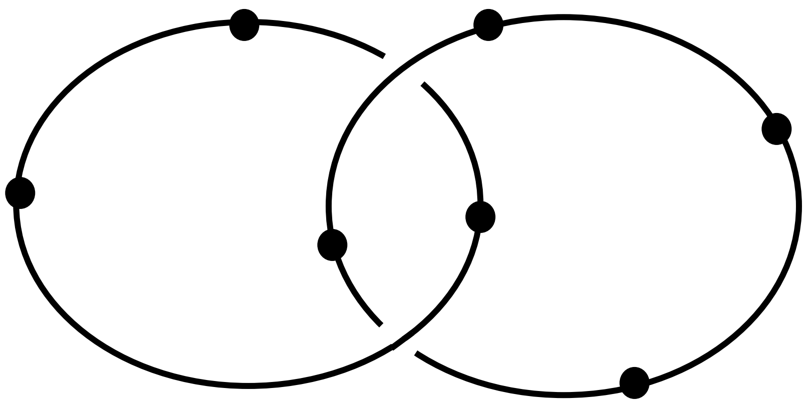
- either G – v is bipartite for some vertex v of G, or
- G is K_5, or
- G can be embedded in a projective plane with all cycles even, or
- G can be decomposed into graphs belonging to the above three classes.



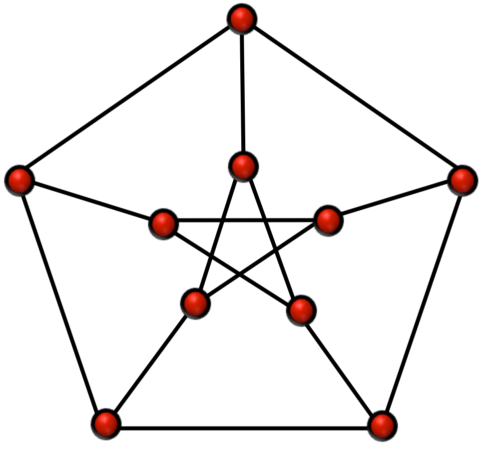
Two linked cycles

Theorem(Robertson, Seymour, Thomas,1995): A graph G does allow a *linkless embedding* in space if and only if G has no minor isomorphic a member of *the Petersen family.*

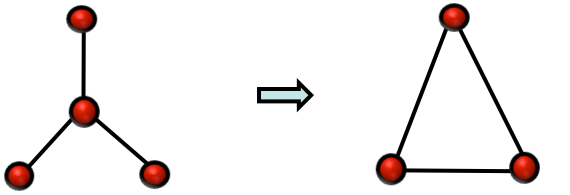
Theorem(van der Holst, 2003): There exists a polynomial time algorithm whether a given embedding has two linked cycles.



Linked cycles



Petersen family



Bilinear forms on the edge set

Let *G* be an undirected graph. Orient its edges arbitrarily. Given an oriented edge *e* we denote by *-e* an edge obtained from *e* by reversing the orientation. Let $\mathcal{E}(G)$ denote a set of linear combinations of oriented edges of *G* with integer coefficients. (It has the structure of a *Z*-module.)

Let Γ be a group and let $F: \mathcal{E}(G) \times \mathcal{E}(G) \rightarrow \Gamma$ be a bilinear map. We are interested in an algorithm for testing whether there exist two vertex disjoint cycles *C,D* in *G* such that

$$F(C,D) := \sum_{e\in C} \sum_{f\in D} F(e,f) \neq 0.$$

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Examples:

1) $\Gamma = \mathbf{Z}, F \equiv 1$. C and D are disjoint cycles.

2) $\Gamma = \mathbf{Z_2}, F \equiv 1$. C and D are disjoint odd cycles.

3) $\Gamma = \mathbf{Z}, F(s_1t_1, s_2t_2)=1, F \equiv 0$, otherwise. C and D correspond to a pair of paths one joining s_1 to t_1 and another s_2 to t_2.

One cycle case

Let $F : \mathcal{E}(G) \to \Gamma$ be a linear map.

Question: Does there exist a cycle C in G such that $F(C) := \sum_{e \in C} F(e) \neq 0$?

Let $\mathcal{C}(G)$ denote the cycle space : the span of the characteristic vectors of cycles.

Question': Is $F \equiv 0$ on $\mathcal{C}(G)$?

$\mathcal{C}(G)$ is well understood:

- It has dimension $|E(G)|-|V(G)|+1$ if G is connected.
- One can efficiently find a basis.
- It is determined by constraints:

  $\sum_{e \in \partial(v)} L(e) = 0$ for every $v \in V(G)$, where $\partial(v)$ denotes the set of edges going out of v.

2-circuits and 2-cycles

For cycles *C* and *D* we can define a map called 2-circuit

$$X_{C,D}: E(G) \text{ x } E(G) \to Z$$

by

$$X_{C,D}(e,f) = \begin{cases} 1, \; e \in C, f \in D, \text{ or -e} \in C, -f \in D \\ -1, \; \text{-e} \in C, f \in D, \text{ or e} \in C, -f \in D \\ 0, \; \text{otherwise.} \end{cases}$$

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For cycles *C* and *D* we can define a map called 2-circuit

$$X_{C,D} : E(G) \times E(G) \to Z$$

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Let $C_2(G)$ denote the span of 2-circuits. (It is a subspace of the space of $E(G) \times E(G)$ matrices.) Our bilinear form *F* can be considered as a linear map on this matrix space.

$$F(M) := \sum_{e,f \in E(G)} F(e,f)M(e,f). \text{ Then } F(X_{c,d}) = F(C,D).$$

It is enough to test if *F* is identically zero on $C_2(G)$.

2-circuits and 2-cycles

A bilinear map L: $E(G) \times E(G) \to \mathbb{Z}$ is a 2-cycle if

1) $L(e, f) = 0$ whenever e and f share a vertex;

2) $\sum_{e \in \partial(v)} L(e,f) = 0$ for every $v \in V(G), f \in E(G)$,

   where $\partial(v)$ denotes the set of edges going out of v.

3) $\sum_{e \in \partial(v)} L(f,e) = 0$ for every $v \in V(G), f \in E(G)$.

Let $\mathcal{L}_2(G)$ denote the span of 2-cycles.

Is it true that $\mathcal{L}_2(G) = C_2(G)$?

Kuratowski 2-cycles

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Let $G = K_5$. $V(G) = \{1,2,3,4,5\}$ and let $L(ij,kl)$=sgn(i,j,k,l,m). (E.g. L($12,35$)=sgn(12354)=-1)
L is the unique (up to rescaling) 2-cycle on K_5.

(sgn($12,34$) + sgn($12,35$) = 0)

There also exists an essentially unique 2-cycle on $K_{3,3}$.

Bilinear forms corresponding to these 2-cycles on subdivisions of K_5 and $K_{3,3}$ in general graphs are called Kuratowski 2-cycles.

Main theorem

Theorem(Van der Holst, N., Thomas): In a Kuratowski connected graph the space of 2-cycles $\mathcal{L}_2(G)$ has a basis consisting of 2-circuits and at most one Kuratowski 2-cycle. Moreover, one can find such a basis in polynomial time.

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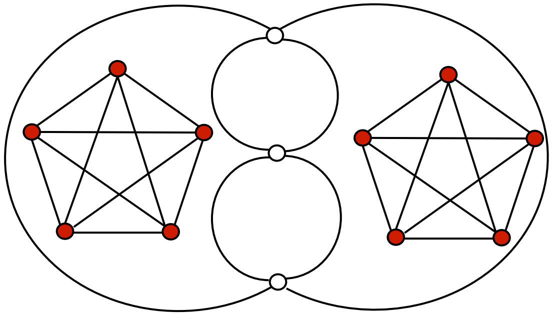
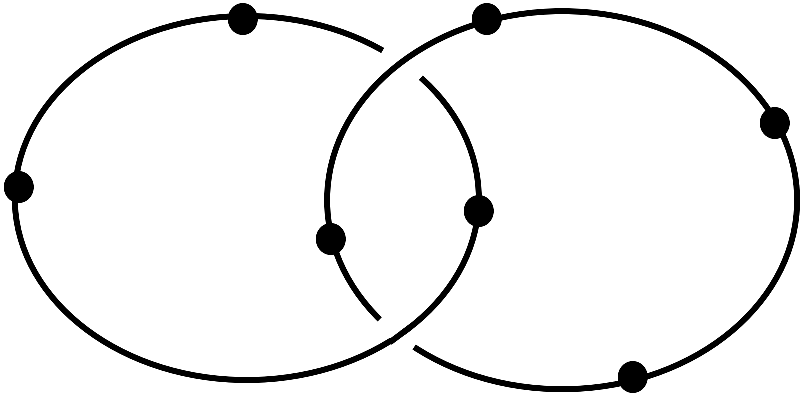
Kuratowski connected: No two subdivisions of K_5 and/or $K_{3,3}$ are separated by an ≤3-separation



Linking number as a bilinear form



Regular projection: displays an embedding in $\boldsymbol{R}^3$ as a drawing in the plane with crossings where for each crossing we record which edge is going “over” and which is going “under”.

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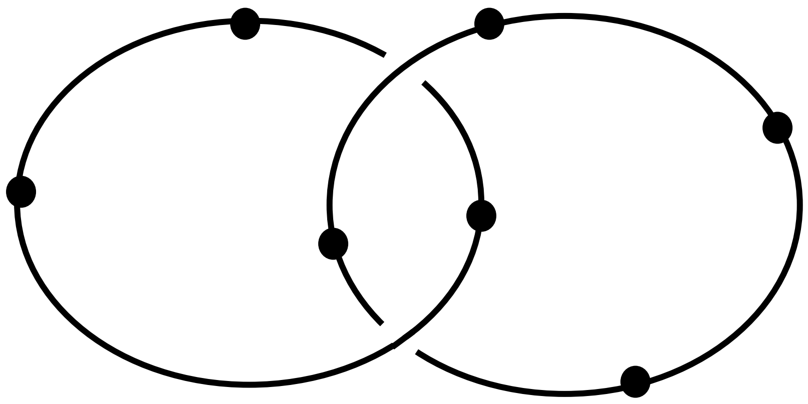

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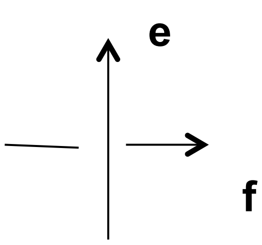
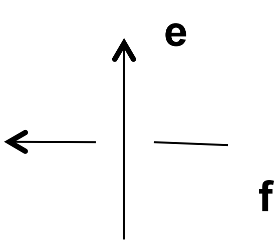


$F(e,f) = 1$



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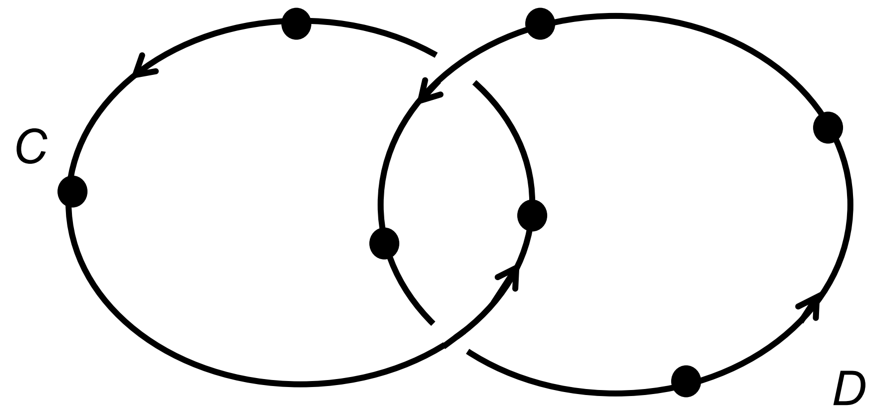

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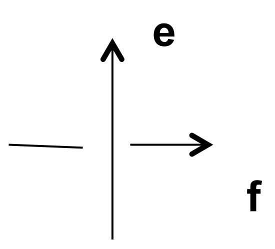

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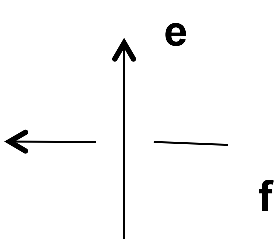

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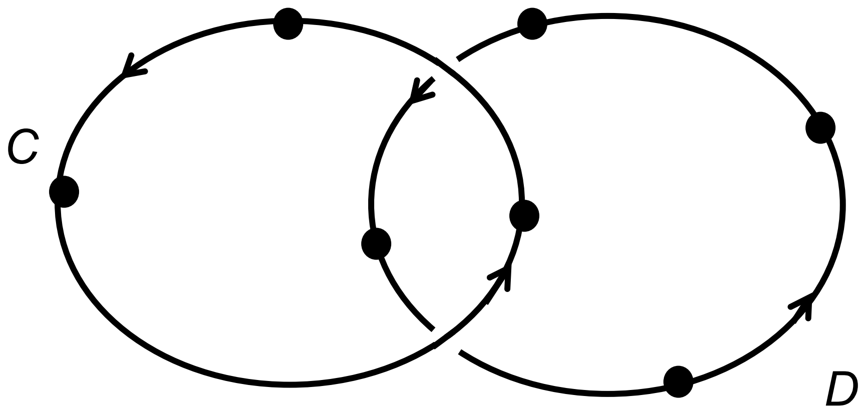

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Linkless embeddings and Kuratowski 2-cycles

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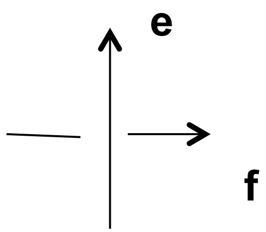

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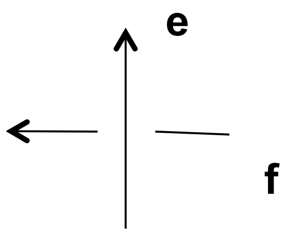

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Suppose G admits a linkless embedding. Let F be the linking bilinear form in a regular projection of this embedding. Then F is identically zero on $C_2(G)$ ($F(C,D) = 0$ for any two cycles.) But any drawing of K_5 or $K_{3,3}$ contains an odd number of crossings between independent edges, so F is non-zero on any Kuratowski 2-cycle. Thus $L_2(G) \neq C_2(G)$.

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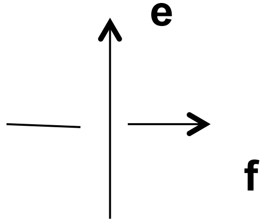

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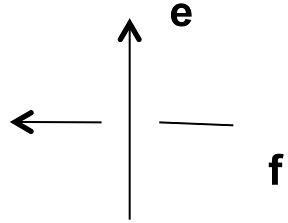

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Theorem: Let G be Kuratowski-connected. Then $\mathcal{L}_2(G) = \mathcal{C}_2(G)$ if and only if G is planar or G does not admit a linkless embedding in $\boldsymbol{R}^3$.

Open problem

Pfaffian orientations: It is #P-hard to compute the number of perfect matchings in a graph. But if a graph admits a Pfaffian orientation then one can compute it efficiently.

Theorem(N.): A graph has a Pfaffian orientation if and only if it admits a drawing in the plane with crossings in which every perfect matching crosses itself even number of times.

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Can we test if a drawing has this property?

If $F : E(G) \times E(G) \rightarrow \mathbf{Z_2}$ is the crossing bilinear form ($F(e, f)=1$ if and only if e and f cross) then we want to determine whether $F(M,M)=0$ for every perfect matching M.

It is enough to find a basis of the span of $E(G) \times E(G)$-matrices X_M, where

$$X_M(e, f)=1 \text{ if and only if } e,f \in M.$$

Can this basis be found in polynomial time?

Thank you!