6/30/11

The Approximability of Constraint Satisfaction Problems
Venkatesen Guruswami

$\boxed{\text{See Notes on Web}}$

Reduction from $(\delta, 1-\varepsilon_0)$-distinguishing Unique Games to $(\frac{1}{2}+\gamma, 1-3\varepsilon_0-\varepsilon)$-dist. 3 LIN

① Pick $r \in V$ at random, and $w_1, w_2, w_3 \in_R N(v)$ at random

② Pick $x, y \in_R \{0,1\}^R$ (u, a, v), $\mu \in_R \in \{0,1\}^R$ (ε-biased) $z = x \oplus y \oplus \mu$

③ (Pull back) $x' = x \circ \pi_{w_1 \to v}$ $y' = y \circ \pi_{w_2 \to v}$
$z' = z \circ \pi_{w_3 \to v}$

④ With prob. 1/2, check $f_{w_1}(x') \oplus f_{w_2}(y') \oplus f_{w_3}(z') = 0$

prob. 1/2 check $f_{w_1}(x') \oplus f_{w_2}(y') \oplus f_{w_3}(\overline{z'}) = 1$

Soundness Lemma: UG instance $\leq \delta$-satisfiable $\Rightarrow$ verifier accepts with prob. $\leq \frac{1}{2}+\gamma$

$$\gamma \leq O\left(\left(\frac{\delta}{\varepsilon}\right)^{1/5}\right).$$

Pf: Prob[verifier accepts] $= \rho = \frac{1}{2}\mathbb{E}\left[\frac{1+f_{w_1}(x')f_{w_2}(y')f_{w_3}(z')}{2}\right]$
$+\frac{1}{2}\mathbb{E}\left[\frac{1-f_{w_1}(x')f_{w_2}(y')f_{w_3}(\overline{z'})}{2}\right]$

$f_w : \{0,1\}^R \to \{1,-1\}$

$$g_v(x) = \mathbb{E}_{w \in N(v)}\left[f_w(x \circ \pi_{w \to v})\right]$$

$g_v : \{0,1\}^R \to [-1, 1]$

$$= \frac{1}{2} + \frac{1}{4}\mathbb{E}_{v,x,y,\mu}\left[g_v(x)g_v(y)g_v(z) - g_v(x)g_v(y)g_v(\overline{z})\right]$$

$z = x \oplus y \oplus \mu$ $\quad \mathbb{E}_{x,y}\left[g_v(x)g_v(y)g_v(x \oplus y)\right] = 1 \Rightarrow g_v$ is linear function.

Fact: The functions $\{\chi_\alpha\}_{\alpha \in \{0,1\}^R}$ $\chi_\alpha(x) = (-1)^{\alpha \cdot x}$ are all the linear functions

notes by Megan Heenehan

Fact: The $\{\chi_\alpha\}$ form an orthonormal basis of the vector space $\mathcal{F} = \{f : \{0,1\}^R \to \mathbb{R}\}$

$$\langle g, h\rangle = \frac{1}{2^R} \sum_{x \in \{0,1\}^R} g(x) h(x)$$

Fact: $\langle \chi_\alpha, \chi_\beta \rangle = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{otherwise} \end{cases}$

Every $g : \{0,1\}^R \to \mathbb{R}$ can be expressed uniquely as

$$g(x) = \sum_\alpha \underbrace{\hat{g}(\alpha)}_{\text{Fourier coefficient}} \chi_\alpha(x)$$

where $\hat{g}(\alpha) = \langle g, \chi_\alpha \rangle$

Fact: $\langle g, h \rangle = \sum_\alpha \hat{g}(\alpha) \hat{h}(\alpha)$ (Parseval's identity)

Cor $\sum_\alpha \hat{g}(\alpha)^2 = \frac{1}{2^R} \sum g(x)^2 \leq 1$

Lemma $\mathbb{E}_{x,y,\mu} [g(x) g(y) g(x \oplus y \oplus \mu)] = \sum_\alpha \hat{g}(\alpha)^3 (1 - 2\varepsilon)^{|\alpha|}$

$|\alpha| = |\mathrm{supp}(\alpha)| = |\{i \mid \alpha_i = 1\}|$.

Pf $\mathbb{E}_{x,y,\mu}\left[\left(\sum_\alpha \hat{g}(\alpha)\chi_\alpha(x)\right)\left(\sum_\beta \hat{g}(\beta)\chi_\beta(y)\right)\left(\sum_\gamma \hat{g}(\gamma)\chi_\gamma(z)\right)\right]$

$$= \mathbb{E}_{x,y,\mu}\left[\sum_{\alpha,\beta,\gamma} \hat{g}(\alpha)\hat{g}(\beta)\hat{g}(\gamma)\chi_\alpha(x)\chi_\beta(y)\chi_\gamma(z)\right]$$

$$= \sum_{\alpha,\beta,\gamma} \hat{g}(\alpha)\hat{g}(\beta)\hat{g}(\gamma) \mathbb{E}_{x,y,\mu}\left[\chi_\alpha(x)\chi_\beta(y)\chi_\gamma(\mu)\right]$$

$$= \sum_{\alpha,\beta,\gamma} \hat{g}(\alpha)\hat{g}(\beta)\hat{g}(\gamma) \underbrace{\mathbb{E}_x[\chi_\alpha(x)\chi_\gamma(x)]}_{\langle \chi_\alpha, \chi_\gamma\rangle} \underbrace{\mathbb{E}_y[\chi_\beta(y)\chi_\gamma(y)]}_{\langle \chi_\beta, \chi_\gamma\rangle} \mathbb{E}_\mu[\chi_\gamma(\mu)]$$

$$= \sum_\alpha \hat{g}(\alpha)^3 \mathbb{E}_\mu\left[(-1)^{\sum \alpha_i \mu_i}\right]$$

$$= \sum_\alpha \hat{g}(\alpha)^3 \prod_i \mathbb{E}\left[(-1)^{\alpha_i \mu_i}\right]$$

$$= \sum_\alpha \hat{g}(\alpha)^3 (1 - 2\varepsilon)^{|\alpha|}$$

$\square$

$$g_v(x) = \frac{1}{2} + \frac{1}{4} \mathop{\mathbb{E}}_v \left[\sum_\alpha \hat{g}(\alpha)^3 (1-2\varepsilon)^{|\alpha|} - \sum_\alpha \hat{g}(\alpha)^3 (-1)^{|\alpha|} (1-2\varepsilon)^{|\alpha|} \right]$$

$$= \frac{1}{2} + \frac{1}{2} \mathop{\mathbb{E}}_v \left[\underbrace{\sum_{|\alpha| \text{ odd}} \hat{g}(\alpha)^3 (1-2\varepsilon)^{|\alpha|}}_{\Theta_v} \right]$$

Pf Soundness Continued

Prob[verifier accepts] $= \rho \geq \frac{1}{2} + \gamma$

$\mathop{\mathbb{E}}_v[\Theta_v] > 2\gamma$

$\Rightarrow$ at least γ frac. of $\underbrace{v\text{'s}}_{\text{such } v \text{ is a "good" } v}$ have $\Theta_v \geq \gamma$

Exercise: If v is good, $\exists \alpha$, $1 \leq |\alpha| \leq \underbrace{\frac{1}{\varepsilon} \log \frac{2}{\gamma}}_{\text{indep. of } R}$, s.t. $|\hat{g}_v(\alpha)| \geq \frac{\gamma}{2}$.

Labeling for UG

$v \in V$ if v is not good, $\ell(v) =$ arbitrary $= 1$; $\ell : V \cup W \to \{1, \dots, R\}$

if v is good let $\alpha \neq 0$ be s.t. $|\hat{g}_v(\alpha)| > \gamma/2$

$\ell(v) =$ any elt. in $\text{supp}(\alpha)$

$w \in W$

$$L_w = \bigcup_{\beta \in \{0,1\}^R} \text{supp}(\beta)$$

$|\hat{f}_w(\beta)| \geq \gamma/2$

$|\beta| \leq \frac{1}{\varepsilon} \lg \frac{2}{\gamma}$

$$\ell(w) = \begin{cases} 1 & \text{if } L_w = \emptyset \\ \text{random element from } L_w. \end{cases}$$

Fact: $\forall w$ ~~Exercise~~ $O\left(\frac{1}{\varepsilon \gamma^2} \log \frac{1}{\gamma}\right) \geq |L_w|$

"friendly to v"

Lemma: Suppose $v \in V$ is good. For at least $\frac{\gamma}{4}$ frac. of $w \in N(v)$. $\ell(v) \in \pi_{w \to v}(L_w) \triangleq \{\pi_{w \to v}(b) \mid b \in L_w\}$

$$\text{Prob}\left[\pi_{w\to v}(\ell(w)) = \ell(v)\right] = \frac{1}{|L_w|} \geq \Omega(\gamma^3 \varepsilon)$$

$$\geq \mathbb{E}\left[\text{frac of } \pi_{w\to v} \text{ satisfied}\right] \geq \gamma \cdot \frac{\gamma}{4} \cdot \Omega(\gamma^3\varepsilon) = \Omega(\gamma^5 \varepsilon).$$

$$\Rightarrow \gamma \leq O\left(\left(\frac{\delta}{\varepsilon}\right)^{1/5}\right).$$

<u>Pf of Lemma</u> Show $\ell(v) \in \pi_{v\to w}(L_B)$

$$\exists\, \alpha,\ |\alpha| \leq \lg \frac{2}{\gamma} \quad |\hat{g}_v(\alpha)| \geq \frac{\gamma}{2}$$

$$g_v(x) = \mathop{\mathbb{E}}_{w\in N(v)}\left[f_w(x \circ \pi_{w\to v})\right]$$

<u>Exercise</u>: $\hat{g}_v(\alpha) = \mathop{\mathbb{E}}_{w}\left[\hat{f}_w(\alpha \circ \pi_{w\to v})\right]$

$$\Rightarrow \mathop{\mathbb{E}}_{w\in N(v)} \left|\hat{f}_w(\alpha \circ \pi_{w\to v})\right| \geq \gamma/2$$

By averaging $\geq \frac{\gamma}{4}$ frac. of $w \in N(v)$ have

$$\left|\hat{f}_w(\alpha \circ \pi_{w\to w})\right| \geq \frac{\gamma}{4}.$$

$\text{supp}(\alpha \circ \pi_{w\to v}) \subseteq L_w$

$\ell(v) \in \text{supp}(\alpha)$

$\pi^{-1}_{w\to v}(\ell(v)) \in L_w$

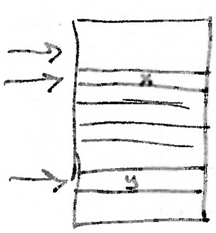
$\Leftrightarrow \ell(v) \in \pi_{w\to v}(L_w)$

$\square$

<u>Dictatorship Test</u> or <u>Projection Test</u>

$g: \{0,1\}^R \to \{0,1\}$.

<u>Max cut</u>



Pick $(x, y) \in D \subseteq \{0,1\}^R \times \{0,1\}^R$

Check $g(x) \neq g(y)$

① <u>Completeness</u>: If $g(x) = x_i$, test accepts with prob. β.

② <u>Soundness</u>: If g is far from a dictator prob. test accepts $\leq \alpha + \gamma$.

Defn| $\mathrm{Inf}_i(g) = \Pr_{x \in_R \{0,1\}^R} [g(x) \neq g(x \oplus e_i)]$

Exercise| $\mathrm{Inf}_i(g) = \sum_{\substack{\alpha: \\ \mathrm{supp}\,\alpha \ni i}} \hat{g}(\alpha)^2$

"far from a dictator" means $\forall\, i \in \{1, \dots, R\}$
$\mathrm{Inf}_i(g) \leq \tau \quad (\tau = \tau(\gamma))$

Max Cut
$(x,y) \in D \qquad f(x) \neq f(y)$

$f(x) = x_i$ would like $x_i \neq y_i$ with prob. β.

Pick x at random
$y_i = \begin{cases} x_i & \text{with prob. } 1-\beta \\ \bar{x}_i & \text{with prob } \beta \end{cases}$

noise sensitivity

Soundness $\Pr_{\substack{x \in \{0,1\}^R \\ y \in N_\beta(x)}} [f(x) \neq f(y)] \triangleq NS_\beta(f)$

Exercise| $NS_\beta(f) = \dfrac{1 - \sum_\alpha \hat{f}(\alpha)^2 (1-2\beta)^{|\alpha|}}{2}$

$NS_\beta(\mathrm{Proj}_i) = \beta$.

Majority $\Pr_{(x,y)} [\mathrm{Maj}(x) \neq \mathrm{Maj}(y)]$

$f: \{1,-1\}^R \to \{1,-1\}$

$\mathrm{Maj}(x) = \mathrm{sgn}(\sum x_i)$

$$\Pr_{\substack{x \in \{-1,1\}^R \\ y \in N_\beta(x)}} \Big[\underbrace{\mathrm{sgn}\Big(\frac{1}{\sqrt{R}} \sum x_i\Big)}_{\downarrow N(0,1)} \neq \underbrace{\mathrm{sign}\Big(\frac{1}{\sqrt{R}} \sum y_i\Big)}_{\mathrm{sgn}\left(\frac{1}{\sqrt{R}} \sum \mu_i g_i\right)} \Big]$$

$\mathrm{sgn}\Big(\frac{1}{\sqrt{R}} \sum g_i\Big)$

g_i Gaussians

$$a = \left(\frac{1}{\sqrt{R}}, \frac{1}{\sqrt{R}}, \dots, \frac{1}{\sqrt{R}}\right) \qquad b = \left(\frac{\mu_1}{\sqrt{R}}, \dots, \frac{\mu_R}{\sqrt{R}}\right)$$

$$\Pr\left[\operatorname{sgn}\left(\sum x_i\right) \neq \operatorname{sgn}\left(\sum y_i\right)\right] \approx \Pr\left[\operatorname{sgn}(a \cdot g) \neq \operatorname{sgn}(b \cdot g)\right]$$

$$= \frac{\angle a\, b}{\pi} = \frac{\arccos(a\, b)}{\pi} \approx \frac{\cos^{-1}(1-2\beta)}{\pi}$$

Dictators pass test with prob. β

Far from dictators pass test with prob $\alpha = \frac{\cos^{-1}(1-2\beta)}{2\pi} + \gamma$