Background Lecture Notes
Fields Institute Summer Thematic Program on the
Mathematics of Constraint Satisfaction

Graph Theory and Combinatorics

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1 Introduction

First of all, a *graph* G is an ordered pair $G = (V, E)$, where V is a set (the vertex set) and $E \subseteq \binom{V}{2}$ (the set of edges). Here and throughout these notes, the symbol $\binom{V}{k}$ will mean the set of all k-element subsets of the set V; $\binom{V}{k} = \{W \subseteq V : |W| = k\}$.

All graphs considered in these notes are finite (i.e., their vertex set is finite), unless we explicitly allow infinite graphs.

The set of all positive integers less than or equal to n will be denoted by $[n] = \{1, 2, \ldots, n\}$.

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. A mapping $f : V \to V'$ is a *homomorphism* from G to H, if whenever $\{x, y\} \in E$, then $\{f(x), f(y)\} \in E'$. We can say that homomorphisms are mappings that preserve edges.

Notice that we have no other assumptions on homomorphisms than preserving edges: we do not require that the mapping is surjective nor that non-edges are mapped to non-edges.

Example 1.1. If G is a subgraph of G', then the identity mapping on $V(G)$ is a homomorphism from G to G'.

Example 1.2. A graph theorist's favorite graph is the Petersen graph P. Obviously $C_5 \to P$ but $K_3 \nrightarrow P$. It is not so obvious but true that $P \nrightarrow C_5$. A way to prove it is to check all 5^{10} possible mappings. A shorter proof consists in realizing that since a potential homomorphism cannot be injective and any two vertices of P lie in a copy of C_5, the homomorphic image would have to contain a triangle, but C_5 contains none.

Example 1.3. Homomorphisms are a generalization of graph coloring: a homomorphism of G to K_n is the same as a coloring of G with n colors.

Example 1.4. An important example: $C_{2k+1} \to C_{2l+1}$ if and only if $k \geq l$. Which part is easier to prove?

Example 1.5. A more involved example: The vertices of G are exams to be scheduled, and there is an edge between them if they share a student or a professor, so they cannot be scheduled on the same day. The vertices of H are slots for exams (described, for example, by the day, time and room where they take place), and there is an edge if the two slots are not on the same day. A homomorphism from G to H is understood as a feasible schedule of exams.

Example 1.6. The definition of a homomorphism can easily be extended to graphs with loops (note that allowing multiple edges does not bring any new concept!). The graph K_2^* has two vertices, 0 and 1, an edge connecting 0 and 1 and a loop at the vertex 0.

Any graph G is homomorphic to K_2^*: the constant mapping that maps all vertices of G to 0 is a homomorphism. However, the correspondence $f \mapsto f^{-1}(0)$ is a one-to-one correspondence of homomorphisms from G to K_2^* and independent sets of vertices in G. Therefore the number of homomorphisms from G to K_2^* is equal to the number of independent sets in G.

Homomorphisms can also be defined for other kinds of structures. Let $\Delta = (\delta_i; i \in I)$ be a finite sequence of positive integers. A *relational structure of type* Δ (or a Δ-structure) is a pair $A = (V, (E_i\colon i \in I))$, where V is a nonempty finite set and E_i are relations such that $E_i \subseteq V^{\delta_i}$ for all $i \in I$. In this way, directed graphs are relational structures for $\Delta = (2)$.

Let $A = (V, (E_i\colon i \in I))$ and $B = (W, (F_i\colon i \in I))$ be two relational structures of the same type Δ. A function $f : V \to W$ is a *homomorphism* from A to B, if for any $i \in I$ and any $(v_1, v_2, \ldots, v_{\delta_i}) \in E_i$ we have $(f(v_1), f(v_2), \ldots, f(v_{\delta_i})) \in F_i$. We write $f : A \to B$.

We say that A is *homomorphic* to B and write $A \to B$ if there exists a homomorphism $f : A \to B$. If $A \to B$ and $B \to A$, we say that A and B are *hom-equivalent* and write $A \sim B$. Notice that this is not the same as being isomorphic; e.g., all directed graphs containing a loop are pairwise hom-equivalent.

Example 1.7. An application in directed graphs: Let T_n be the transitive tournament on n vertices, i.e., $V(T_n) = \{1, 2, \ldots, n\}$ and $E(T_n) = \{(j,k)\colon j < k\}$. Then for a directed graph G with $|V(G)| = n$ we have $G \to T_n$ if and only if G contains no directed cycle. The homomorphism f is a linear extension or a topological sort of G.

It is an easy (but key) observation that the composition of two homomorphisms is a homomorphism as well.

2 Cores

A graph C is called a *core* if every homomorphism $f : C \to C$ is an automorphism. The following lemma states several easy properties of cores.

Lemma 2.1.

1. *If C and C' are two finite cores such that $C \sim C'$, then C and C' are isomorphic.*
2. *Every graph G has a unique (up to isomorphism) subgraph C such that $G \sim C$ and C is a core (the* core of G*).*
3. *The core C of a graph G is a retract of G, i.e., there exists a homomorphism $r : G \to C$ such that $r \restriction C$ is the identity mapping.*
4. *If two graphs G and H are hom-equivalent, then their cores are isomorphic.*

Proof. 1. Let $f : C \to C'$ and $g : C' \to C$. Then $g \circ f : C \to C$ is an automorphism and so g is a surjection. Similarly, as $f \circ g : C' \to C'$ is an automorphism, f is also a surjection. Therefore C and C' are isomorphic.

2. Let C be a graph with the least number of vertices in the set $\{f[G] : f : G \to G\}$. Uniqueness follows from 1 and the transitivity of $\sim$.

3. Let $f : G \to C$. There are distinct positive integers $m > n$ such that $f^m \restriction C = f^n \restriction C$. Let $r = f^{m-n}$.

4. A consequence of 1.

□

Remark. Deciding whether a graph is a core is a co-NP-complete problem. So is the problem of determining hom-equivalence of two graphs. Computing the core of a graph is hard.

Remark. Let $\mathscr{C}$ be the set of all non-isomorphic finite cores. Then the binary relation $\leq$, defined by $G \leq H$ if and only if $G \to H$, is a partial order on $\mathscr{C}$. We will mention several properties of this order later.

Example 2.1. Any vertex-critical k-chromatic graph G is a core, because if H is a subgraph of G with $|V(H)| < |V(G)|$, then $\chi(H) < k$ and therefore $G \nrightarrow H$.

The Petersen graph is a core.

3 Asymmetric and rigid graphs

A graph is called *asymmetric* if its only automorphism is the identity mapping. A graph is called *rigid* if it only has the identity endomorphism. Obviously a rigid graph is asymmetric.

3.1 Almost all graphs are asymmetric

Theorem 3.1. *Asymptotically almost all graphs are asymmetric, i.e., the probability that the random graph $G_{n,1/2}$ is asymmetric tends to* 1 *as n grows to infinity.*

Proof. Let $V = \{1, 2, \ldots, n\}$. We will count the number of graphs on V with a non-identity automorphism.

We can assume that all vertices have degree at least $\frac{n}{2}(1-\epsilon)$ and any two vertices have at most $\frac{n}{4}(1+\epsilon)$ common neighbors. Asymptotically almost all graphs satisfy this property, see below.

Let $\phi : V \to V$ be an automorphism such that $\phi(x) = y$ for some $x \neq y$. Let $M = \{v \in V : \phi(v) \neq v\}$ be the set of all vertices that are moved by ϕ, or the non-fixed points of ϕ. Let $V' = \binom{V}{2}$. We define the mapping $\phi' : V' \to V'$ by $\phi'(\{u, v\}) = \{\phi(u), \phi(v)\}$. The mapping ϕ' is a permutation on V'.

According to our assumption, there exist at least $\frac{n}{2}(1-\epsilon) - \frac{n}{4}(1+\epsilon) = \frac{n}{4}(1-3\epsilon)$ vertices that are connected by an edge to x but not to y. All these vertices are moved by the automorphism ϕ.

Hence $|M| \geq cn$ for a suitable constant c. Thus the number of pairs of vertices that are moved by this automorphism, or the number of non-fixed points of ϕ' is at least $\binom{cn}{2} - n \geq c'n^2$ for some constant c' and sufficiently large n. Therefore the number of cycles (or orbits) of ϕ' is at most $k = \binom{n}{2} - c'n^2/2$.

If ϕ is an automorphism of a graph G, then all pairs in one cycle of ϕ' are either edges or they are all non-edges of G. Therefore there are at most 2^k graphs such that ϕ is their automorphism.

There are $n!$ possible choices of ϕ, so we get that there are at most

$$n! \cdot 2^k = n! \cdot 2^{\binom{n}{2} - c'n^2/2}$$

graphs on V with a non-identity automorphism.

Thus the probability that a random graph on n vertices has a non-identity isomorphism is at most

$$\frac{n! \cdot 2^{\binom{n}{2} - c'n^2/2}}{2^{\binom{n}{2}}} \leq \frac{n^n}{2^{c'n^2/2}}$$

which tends to 0 as $n \to \infty$. □

An even stronger result can be proved.

Theorem 3.2 (Erdős, Rényi [2])**.** *For an arbitrary positive real number ϵ, asymptotically almost all random graphs $G_{n,1/2} = G = (V, E)$ have the following property: If $G' = (V', E')$ is a graph such that $V' = V$ and*

$$|E' \triangle E| \leq \frac{|V|}{2}(1-\epsilon),$$

then G' is asymmetric.

No explicit construction of such graphs (which are in majority) is known.

3.2 Almost all graphs are rigid

Theorem 3.3. *Asymptotically almost all graphs are rigid.*

Sketch of proof. It can be proved (e.g., using linearity of expectation and the Chernoff bound) that asymptotically almost all graphs G have the following properties for any positive ϵ:

1. for every vertex v, $\frac{n}{2}(1-\epsilon) \leq \deg_G v \leq \frac{n}{2}(1+\epsilon)$;

2. the number of common neighbors of any two vertices is at least $\frac{n}{4}(1-\epsilon)$ and at most $\frac{n}{4}(1+\epsilon)$;

3. both the largest clique and the largest independent set have fewer than $2\log_2(1+\epsilon n)$ vertices;

4. each set of $m > 30\log_2 n$ vertices induces a subgraph with at most $\frac{3}{4}\binom{m}{2}$ edges;

5. there is a set of $m > 60 \ln n$ disjoint pairs of vertices $a_i b_i$, $i = 1, 2, \dots, m$, such that for some $\frac{3}{4}\binom{m}{2}$ pairs at least one of $\{a_i, a_j\}$, $\{a_i, b_j\}$, $\{b_i, a_j\}$, $\{b_i, b_j\}$ is an edge.

If G and H are graphs on n vertices satisfying conditions 1–5, then every homomorphism $f : G \to H$ is injective. Therefore any graph satisfying conditions 1–5 is a core; so asymptotically almost all graphs are cores.

A graph that is both asymmetric and a core is rigid. □

3.3 Ulam's problem

How many mutually incomparable graphs can we find on a given set X? We are looking for a set $\mathcal{G}$ of graphs such that $G \nrightarrow G'$ for any two graphs $G, G' \in \mathcal{G}$.

For finite X, the probabilistic analysis sketched in the previous lecture together with Sperner's theorem gives the answer

$$\frac{\binom{\binom{n}{2}}{\frac{1}{2}\binom{2}{2}}(1-\epsilon)}{n!}$$

Now, however, we are interested in infinite X. (This is the only place where we consider infinite graphs but it is an interesting niche.)

The class of all cardinal numbers will be denoted by Card. Wherever we use κ, it will be a cardinal number. The letter ω is used to denote the cardinality of the set of all non-negative integers (or the set of all non-negative integers itself). The cofinality of a cardinal κ is the least cardinal α such that there exists a mapping $f : \alpha \to \kappa$ satisfying the condition that for every $\lambda \in \kappa$ there exists $\beta \in \alpha$ with $f(\beta) \geq \lambda$. We write $\mathrm{cf}(\kappa) = \alpha$.

Theorem 3.4 ([10])**.** *For any $\kappa \geq 1$ there exists a rigid digraph $G = (X, R)$ with $|X| = \kappa$.*

Proof. If κ is finite, let G be the oriented path of length $\kappa - 1$. If $\kappa = \omega$, let $R = \{(x, x+1) : x \in \omega\} \cup \{(0, 5)\}$. Then (ω, R) is a rigid graph.

Let $\kappa > \omega$. The vertex set X will be described gradually as we describe the whole graph. First, take $\kappa \times \{0, 1\} \cup \{a, b, c, a', b', c'\}$. When convenient, we will use α to denote $(\alpha, 0)$ and α' to denote $(\alpha, 1)$.

We start with edges $(0, a)$, (a, b), (b, c), $(c, 0)$, (a, c), $(0', a')$, (a', b'), (b', c'), $(c', 0')$, $(b', 0')$. Add edges (α, β) and (α', β') for all $\alpha < \beta < \kappa$ and (α', α) for all $\alpha < \kappa$.

For every $\beta < \kappa$ such that $\mathrm{cf}(\beta) = \omega$, fix some countable increasing sequence of cardinals $\{\beta_0, \beta_1, \dots\}$ so that $\lim \beta_n = \sup\{\beta_n : n < \omega\} = \beta$. Add an oriented path of length $n + 2$ joining β' and β_n for every $n < \omega$ and for every β with countable cofinality (all these paths are vertex-disjoint).

The graph has κ vertices because we start with $2\kappa + 6 = \kappa$ vertices and add $\kappa \times \omega = \kappa$ vertices.

Let $f : G \to G$ be a homomorphism. A complete graph on κ vertices must map to a complete graph again, so $\kappa \times \{0\}$ maps either to $\kappa \times \{0\}$ or to $\kappa \times \{1\}$. An oriented cycle must map to an oriented cycle, therefore 0 maps either to 0 or to $0'$. The edges $(0, b)$ and (a', c') enforce that $f(0) = 0$ and $f(0') = 0'$. As f is a homomorphism, we have $\alpha \leq \beta \Rightarrow f(\alpha) \leq f(\beta)$.

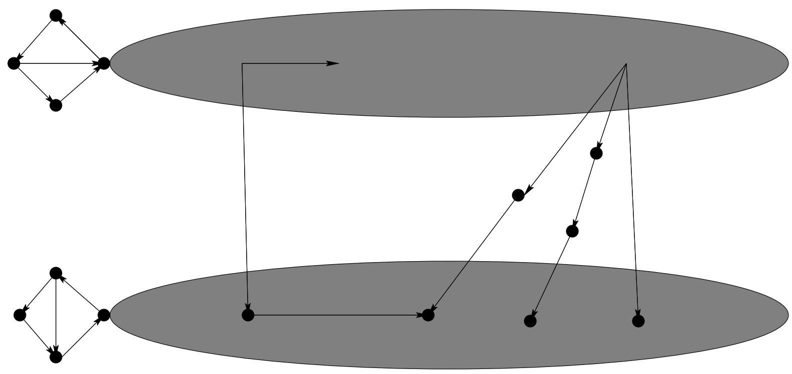

Figure 1: A rigid graph on an arbitrarily large set

If there was β such that $f(\beta) < \beta$, then, since f is a homomorphism, $\beta > f(\beta) > f(f(\beta)) > \dots$ would be an infinite decreasing sequence of cardinals. Such a sequence does not exist and therefore $f(\beta) \geq \beta$.

Suppose there is β such that $f(\beta) > \beta$. Let $\beta^0 = \beta$, $\beta^{n+1} = f(\beta^n)$. We have an increasing sequence $\beta^0 < \beta^1 < \beta^2 < \dots$. Let $\beta = \lim \beta^n = \sup\{\beta^n \colon n < \omega\}$. We have a fixed sequence $\{\beta_n\}$ with $\lim \beta_n = \beta$. The sequences $\{\beta^n\}$ and $\{\beta_n\}$ interlace (meaning that for any β_n there is $\beta^k > \beta_n$ and vice versa). The definition of β^n implies that the sequences $\{\beta^n\}$ and $\{f(\beta_n)\}$ interlace as well and therefore $\lim f(\beta_n) = \beta$.

The existence of oriented paths from β' to β_n implies that $f(\beta_n) = \beta_n$. Since the sequences $\{\beta^n\}$ and $\{\beta_n\}$ interlace, there have to be k and n such that $\beta^k \leq \beta_n < \beta^{k+1}$; that is a contradiction since $\beta^k \leq \beta_n$ but $f(\beta_n) = \beta_n < \beta^{k+1} = f(\beta^k)$.

We have shown that f is the identity mapping and therefore G is rigid. $\square$

Theorem 3.5. *For any cardinal κ, there exists a rigid (undirected) graph $G = (V, E)$ with $|V| = \kappa$.*

Proof. We use the *arrow construction.* Schematically, it is indicated in Fig. 2. Formally, if G is a directed graph and (I, a, b) is an undirected graph with two fixed (distinct) vertices, we define

$$V(G * (I, a, b)) = V(G) \cup ((V(I) \setminus \{a, b\}) \times E(G))$$

and

$$\begin{aligned} E(G * (I, a, b)) = \bigcup_{e \in E(G)} \Bigg(&\Big\{ \{(x, e), (y, e)\} \colon \{x, y\} \in E(I),\ \{x, y\} \cap \{a, b\} = \emptyset \Big\} \\ &\cup \Big\{ \{u, (x, e)\} \colon e = (u, v),\ \{a, x\} \in E(I) \Big\} \\ &\cup \Big\{ \{v, (x, e)\} \colon e = (u, v),\ \{b, x\} \in E(I) \Big\} \Bigg). \end{aligned}$$

Using a suitable rigid indicator (I, a, b) can force that homomorphisms $G \to H$ are in one-to-one correspondence with homomorphisms $G * (I, a, b) \to H * (I, a, b)$. Let G be the rigid digraph from Theorem 3.4. Then $G * (I, a, b)$ is rigid. $\square$

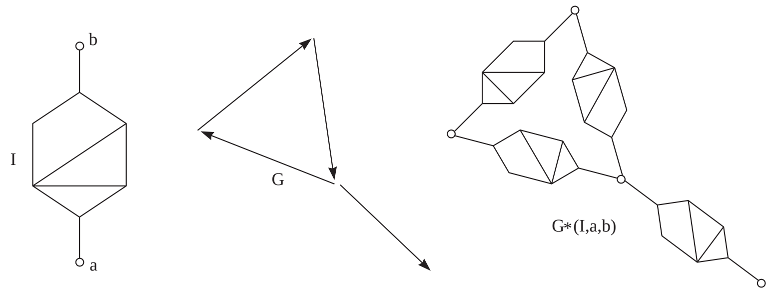

Figure 2: The arrow construction

Theorem 3.6. *For any cardinal κ, there exists a family of 2^κ mutually incomparable directed graphs.*

Proof. Take all orientations of a rigid undirected graph with κ edges, which exists by Theorem 3.5. ☐

Theorem 3.7. *For any cardinal κ, there exists a family of 2^κ mutually incomparable undirected graphs.*

Proof. Use the arrow construction once again on a family of 2^κ mutually incomparable directed graphs. ☐

Remark. It is undecidable in set theory, whether there exists a class of graphs $\mathscr{G} = \{G_\alpha : \alpha \in \text{Card}\}$ such that $|V(G_\alpha)| = \alpha$ and $G_\alpha \nrightarrow G_\beta$ for any $\alpha \neq \beta$.

4 No-homomorphism lemma

For a graph G, let $\omega(G)$ denote its clique number (the size of its largest complete subgraph), let $\alpha(G)$ denote its independence number (the size of its largest independent set of vertices), and let $\chi(G)$ denote its chromatic number (see also next section).

If $G \to H$, then obviously $\chi(G) \leq \chi(H)$ and $\omega(G) \leq \omega(H)$. It is not true, however, that $G \to H$ implies that $\alpha(G) \leq \alpha(H)$ or that $\alpha(G) \geq \alpha(H)$. The following theorem shows an interesting relation between the sizes maximal independent sets and the sizes of the vertex sets of homomorphically comparable graphs.

We say that a graph H is *vertex-transitive*, if for any two of its vertices x and y there is an automorphism f of G such that $f(x) = y$.

Theorem 4.1 (no-homomorphism lemma)**.** *If G is an arbitrary graph and H is a vertex-transitive graph and if $G \to H$, then*

$$\frac{\alpha(G)}{|V(G)|} \geq \frac{\alpha(H)}{|V(H)|}.$$

Proof. Let
$$\mathscr{A} = \{A\colon\ A \subseteq V(H) \text{ independent}, |A| = \alpha(H)\},$$
let $m_x = |\{A \in \mathscr{A}\colon\ x \in A\}|$. Since H is vertex-transitive, for any two vertices x and y of H we have $m_x = m_y = m$.

Consider the number of pairs (v, A) such that v is a vertex of H and $v \in A \in \mathscr{A}$. Obviously,
$$|V(H)| \cdot m = |\{(v, A)\colon\ v \in A \in \mathscr{A}\}| = |\mathscr{A}| \cdot \alpha(H). \tag{$*$}$$

Let $f : G \to H$ be a homomorphism. Then we have
$$\begin{aligned} \left|f^{-1}[A]\right| &\leq \alpha(G), \\ \sum_{A\in\mathscr{A}} \left|f^{-1}[A]\right| &\leq \alpha(G) \cdot |\mathscr{A}|. \end{aligned}$$

On the other hand, we have
$$\sum_{A\in\mathscr{A}} \left|f^{-1}[A]\right| = \sum_{A\in\mathscr{A}} \sum_{v\in A} \left|f^{-1}(v)\right| = m \cdot |V(G)|.$$

Combining this and $(*)$ we get
$$\frac{\alpha(H)}{|V(H)|} = \frac{m}{|\mathscr{A}|} \leq \frac{\alpha(G)}{|V(G)|}.$$
□

Remark. The above proof implies that if $f : G \to H$, H is vertex-transitive and
$$\frac{\alpha(G)}{|V(G)|} = \frac{\alpha(H)}{|V(H)|},$$
then
$$\left|f^{-1}[A]\right| = \alpha(G)$$
for any $A \in \mathscr{A}$.

The no-homomorphism lemma is so-called because it is usually used to show that if H is vertex-transitive and the inequality of the independence ratios is not true then $G \nrightarrow H$.

Finally, we present two interesting applications of the no-homomorphism lemma.

Corollary 4.2. *The cycle C_{2k+1} is homomorphic to $C_{2k'+1}$ if and only if $k' \leq k$.*

Proof. The non-existence of a homomorphism is a consequence of the no-homomorphism lemma and the fact that, if $k' > k$,
$$\frac{\alpha(C_{2k+1})}{|V(C_{2k+1})|} = \frac{k}{2k+1} < \frac{k'}{2k'+1} = \frac{\alpha(C_{2k'+1})}{|V(C_{2k'+1})|}.$$
The existence is left as an exercise. □

For a positive integer k and $1 \le d \le k/2$, the *circulant clique* $K_{k/d}$ is be the graph whose vertex set is $\{0,1,\dots,k-1\}$ and $\{i,j\}$ is an edge if and only if $d \le |i-j| \le k-d$. It is easy to show that $\alpha(K_{k/d}) = d$.

The *Kneser graph* is the graph $K\binom{k}{d} = (V,E)$, where

$$V = \binom{\{0,1,\dots,k-1\}}{d}$$

and

$$E = \{\{A,B\}\colon\ A,B \in V,\ A \cap B = \emptyset\}.$$

Let $f : K_{k/d} \to K\binom{k}{d}$ with $f(i) = \{i, i+1, \dots, i+d-1\}$. It is easy to check that the Kneser graph is vertex-transitive and that f is a homomorphism. Using the no-homomorphism lemma, we get

$$\frac{\alpha\left(K\binom{k}{d}\right)}{\binom{k}{d}} = \frac{\alpha\left(K\binom{k}{d}\right)}{\left|V\left(K\binom{k}{d}\right)\right|} \le \frac{\alpha\left(K_{k/d}\right)}{|V\left(K_{k/d}\right)|} = \frac{d}{k}.$$

Realizing that an independent set in the Kneser graph is an intersecting family of d-element subsets of a k-element set, we obtain the famous Erdős-Ko-Rado theorem:

$$\alpha\left(K\binom{k}{d}\right) \le \binom{k}{d} \cdot \frac{d}{k} = \binom{k-1}{d-1}.$$

5 Graph coloring and chromatic numbers

5.1 The chromatic number

A *k-coloring* of a graph G is a partition of the vertex set into k disjoint sets satisfying the following property: whenever e is an edge of G, for every $1 \le i \le k$ it is true that $|e \cap V_i| \le 1$. Equivalently, a k-coloring may be regarded as a mapping $c : V \to \{1,2,\dots,k\}$ such that $c(x) \ne c(y)$ whenever $e = \{x,y\}$ is an edge of G; equivalently we can say that neighboring vertices are assigned different colors.

The *chromatic number* of a graph G is defined as

$$\chi(G) = \min\{k\colon\ G \text{ has a } k\text{-coloring}\}.$$

Example 5.1. The chromatic number of the complete graph $K_n = ([n], \binom{[n]}{2})$ is obviously $\chi(K_n) = n$ and the minimal coloring is the partition of the vertex set $[n]$ into n one-point sets.

Example 5.2. Another example, less trivial than the previous one, is C_5—the cycle of length 5. Its chromatic number is three. However, we have to prove two separate claims: firstly, that the chromatic number is at most three, i.e., that there is a 3-coloring of the graph C_5; and secondly, that the chromatic number is greater than two. In general, there is no known way to prove both these claims at the same time.

In the particular case of C_5 we are lucky. It can be proved that for a 2-colorable connected graph, there is always a unique coloring. So both claims are easy to see.

Having seen the two examples, one could easily become optimistic about the complexity of determining the chromatic number of graphs. That would not be wise. Notice that the task of determining whether for an input graph G, the chromatic number $\chi(G) \le 3$, is an NP-complete task. There are good bounds for certain classes of graphs (every planar graph can be colored by four colors). In many cases, however, it is not easy even to provide a good estimate for the chromatic number.

There is a simple algorithm for coloring a graph. First order the vertices of the graph. Then assign each vertex the least color that has not been used for any of its neighbors.

This algorithm implies that for any graph G, its chromatic number $\chi(G) \le \Delta(G) + 1$, where $\Delta(G) = \max\{\deg_G(x) \colon x \in V(G)\}$ is the maximum degree of the graph G. In fact, the famous Brooks' theorem states that $\chi(G) \le \Delta(G)$ unless the graph G (or any of its components) is a complete graph or an odd cycle.

Johansen and Kim proved using the probabilistic method that, if G does not contain triangles as subgraphs, $\chi(G) = O(d/\log d)$.

5.2 A scale of chromatic numbers

The notion of the circular chromatic number is very useful when solving various channel assignment problems. A (k,d)*-coloring* of a graph $G = (V,E)$ is any mapping $c \colon V \to \{0,1,\dots,k-1\}$ satisfying the condition that if $\{x,y\} \in E$ is an edge, then $d \le |c(x) - c(y)| \le k - d$. The *circular chromatic number* of a graph G is defined as

$$\chi_c(G) = \inf\left\{k/d \colon\ G \text{ has a } (k,d)\text{-coloring}\right\}.$$

We already know that a k-coloring of a graph G is the same as a homomorphism of G to K_k. It is easy to observe that a (k,d)-coloring of a graph G is the same as a homomorphism of G to $K_{k/d}$. Therefore

$$\chi_c(G) = \inf\{k/d \colon G \to G_{k/d}\}.$$

Notice that $\chi_c(K_{k/d}) = k/d$.

Theorem 5.1. *For positive integers $k \ge d$ and $k' \ge d'$, the graph $K_{k/d}$ is homomorphic to $K_{k'/d'}$ if and only if $k/d \le k'/d'$.*

Proof. If $K_{k/d}$ is homomorphic to $K_{k'/d'}$, then $k/d \le k'/d'$ by the no-homomorphism lemma.

Observe that $K_{k/d}$ is hom-equivalent to $K_{ck/cd}$. To see this, notice that $f : x \mapsto cx$ is a homomorphism $K_{k/d} \to K_{ck/cd}$ and $g : y \mapsto \lfloor y/c \rfloor$ is a homomorphism $K_{ck/cd} \to K_{k/d}$.

If $kd' \le k'd$, then $K_{kk'/k'd}$ is a (spanning) subgraph of $K_{kk'/kd'}$, so we have

$$K_{k/d} \sim K_{kk'/k'd} \to K_{kk'/kd'} \sim K_{k'/d'}.$$

□

Theorem 5.1 and the fact that $K_{k/1} \cong K_k$ imply that the relation between the chromatic number of a graph and its circular chromatic number is very tight: $\chi(G) - 1 \leq \chi_c(G) \leq \chi(G)$. We shall prove that the circular chromatic number of a graph is always rational and that it is the minimum of the above set.

For $k \geq 2d$, the graph $K_{k/d}$ is a core if and only if k and d are relatively prime. Thus we get a natural isomorphism of the posets $(\{q \in \mathbb{Q}\colon\ q \geq 2\}, \leq)$ and $(\{K_{k/d}\colon\ k \geq 2d \wedge (k,d) = 1\}, \leq)$ where for any two graphs A and B we have defined $A \leq B$ if and only if $A \to B$.

We present another generalization (due to X. Zhu). Let C_r be the graph whose vertices are intervals of length at least $1/r$ on the unit-length circle. Two vertices are connected by an edge if and only if the intervals do not intersect. The following theorem can be proved.

Theorem 5.2. *For a graph G we have*

$$\chi_c(G) = \inf\{r\colon\ G \to C_r\}.$$

Another way to ascertain the circular chromatic number of a graph is the following. For a circuit C in the graph G, given an orientation of the circuit, let C^+ be the set of edges of C that are in positive direction (relative to the orientation of the circuit) and C^- the set of edges of C that are in negative direction.

Theorem 5.3. *For a graph G we have*

$$\chi_c(G) = \min_{\vec{G}} \max_{C} \left(\frac{|C^+|}{|C^-|} + 1 \right),$$

where the minimum is taken over all orientations $\vec{G}$ of G and the maximum is taken over all circuits C with a given orientation in $\vec{G}$.

There is an interesting connection to the (ordinary) chromatic number. Let I_r be the graph whose vertices are segments of the interval $[0,1]$ of length at least $1/r$. Two segments—vertices of the graph—are joined by an edge if and only if they are disjoint. Then we get

$$\chi(G) = \inf\{r\colon\ G \to I_r\}$$

since I_r is a perfect graph.

Let us now turn our attention to yet another chromatic number: the fractional chromatic number. It can be computed by means of linear programming and gives a lower bound on the (ordinary) chromatic number of a graph. Unfortunately, the linear program has exponentially many variables and does not provide an efficient algorithm for computing the fractional chromatic number. Indeed, the fractional chromatic number of a graph is NP-hard to compute.

We introduce the notion of multi-coloring—colors will now be sets of positive integers. A *k-tuple n-coloring* of a graph $G = (V,E)$ is a mapping $m : V \to \binom{[n]}{k}$ such that whenever v and v' are connected by an edge, $m(v) \cap m(v') = \emptyset$. The *fractional chromatic number* of a graph G is defined as

$$\chi_f(G) = \inf\left\{n/k\colon\ G \text{ has a } k\text{-tuple } n\text{-coloring}\right\}.$$

For any graph G, it is true that

$$\chi_{\mathrm{f}}(G) \le \chi_{\mathrm{c}}(G) \le \chi(G).$$

Not surprisingly, a d-tuple k-coloring of a graph G is just a homomorphism of G to the Kneser graph $K\binom{k}{d}$.

Theorem 5.4. *For a graph G we have*

$$\chi_{\mathrm{f}}(G) = \inf\left\{k/d : G \to K\tbinom{k}{d}\right\}.$$

The *measure graph* M_r is the graph whose vertices are subsets of the interval $[0,1]$ of Lebesgue measure at least $1/r$. Two vertices are connected if and only if they are disjoint sets.

Theorem 5.5. *If G is a graph, then*

$$\chi_{\mathrm{f}}(G) = \inf\{r : G \to M_r\}.$$

5.3 Chromatic number and girth

Is the chromatic number bounded for graphs without short cycles? The answer is no. It is a very famous result due to Erdős [1] who, using the probabilistic method, showed that for arbitrary positive integers k and g there exists a graph $G_{k,g}$ with girth at least g and chromatic number at least k.

Let us take a look at the so-called *shift graphs*. Given a positive integer n, the shift graph S_n is the graph with the vertex set $V(S_n) = \binom{[n]}{2}$ (the vertices of the shift graph can be regarded as the edges of the complete graph on n vertices). Two vertices $\{i,j\}$ and $\{k,l\}$ of the shift graph are connected by an edge if $i < j = k < l$.

It is easy to see that shift graphs contain no triangles and that they contain 4-cycles as their subgraphs (if $n \ge 4$). We will prove in the next lecture that $\chi(S_n) = \lceil \log n \rceil$, so the chromatic number of the shift graphs is unbounded.

We can generalize the shift graphs in this way: $S_{n,l}$ has $\binom{[n]}{l}$ as its vertex set. Two vertices v and v' are joined by an edge if $v = \{i, j_1, j_2, \dots, j_{l-1}\}$, $v' = \{j_1, j_2, \dots, j_{l-1}, k\}$ and $i < j_1 < j_2 < \dots < j_{l-1} < k$. It can be shown that

$$\chi(S_{n,l}) \sim \underbrace{\log\log\dots\log}_{l \text{ times}} n,$$

so the chromatic number is again unbounded and $S_{n,l}$ does not contain odd cycles of length less than $2n$.

5.4 Shift graphs

We shall define the shift graphs in a more general way now. Let $G = (V,E)$ be a simple directed graph, i.e., $E \subseteq V^2$ and G has no loops or multiple edges. The *shift graph* of the graph G will be the (directed) graph $\partial G = (E, E(\partial G))$, where $E(\partial G) = \{((x,y),(y,z)) : (x,y),(y,z) \in E\}$.

Note that the shift graph of the transitive tournament T_n on n vertices is an orientation of the shift graph S_n introduced in the first lecture.

The chromatic number of a directed graph is defined in the same way as for undirected graphs. A k-coloring of a graph G has the property that two distinct vertices x and y of the graph G get different colors whenever (x, y) is an edge or (y, x) is an edge.

Theorem 5.6. *For an arbitrary simple directed graph G, it is true that*

$$\log \chi(G) \leq \chi(\partial G) \leq \min\left\{k\colon \chi(G) \leq \tbinom{k}{\lfloor k/2 \rfloor}\right\}.$$

Both the lower and the upper bound are the best possible.

Proof. First let us prove the lower bound. It suffices to show that if ∂G is t-colorable, then G is 2^t-colorable.

Let $V(\partial G) = E(G) = E_1 \cup E_2 \cup \cdots \cup E_t$ be a t-coloring of the graph ∂G. In the graph (V, E_i), every vertex is either a source or a sink (otherwise there would be a monochromatic edge in the graph ∂G). So there is a homomorphism f_i of the graph (V, E_i) to P_1, the directed path of length one. The mapping $f : V(G) \to \{0, 1\}^t$ such that $f(v) = (f_1(v), f_2(v), \ldots, f_t(v))$ is a 2^t-coloring of the graph G.

The lower bound is the best possible: suppose that $n = 2^t$ and consider $G = T_n$ (G is the transitive tournament with the vertex set $V(G) = \{0, 1, \ldots, n-1\}$ whose edges are pairs (k, l) such that $k < l$). Define a t-coloring of ∂G so that $V(\partial G) = E(G) = E_1 \cup \cdots \cup E_t$ with

$$E_i = \{(k, l)\colon k < l \text{ and } i \text{ is the first binary digit where } k \text{ and } l \text{ differ}\}.$$

Thus $\chi(\partial G) \leq t = \log n = \log \chi(G)$.

Remark. As the shift graph $S_{n,3}$ is an orientation of $\partial\partial T_n$, $S_{n,4}$ is an orientation of $\partial\partial\partial T_n$, etc. we obtain the claim from the first lecture that

$$\chi(S_{n,l}) \sim \underbrace{\log\log\ldots\log}_{l \text{ times}} n.$$

The proof of the upper bound is more difficult and at the same time more interesting. We need to introduce a number of new ideas related to partial orders.

For a partially ordered set $P = (X, \leq)$, the symbol $\mathscr{A}(P)$ will denote the set of all antichains (independent sets) in P.

Remark. Determining the number of antichains in a partially ordered set is an unsolved problem known as Dedekind's problem.

We define a partial order $\preceq$ on the set $\mathscr{A}(P)$ so that

$$A \preceq B \text{ if and only if } (\forall a \in A)(\exists b \in B)\ a \leq b.$$

Let $G = (V, E)$ be a digraph and $P = (X, \leq)$ a partially ordered set. A mapping $f : V \to X$ is an *A-map* if $f(x) \not\leq f(y)$ whenever (x, y) is an edge of G. The fact that there exists an A-map of G to P will be denoted by $G \xrightarrow{A} P$.

Let D_k denote the poset $D_k = (\{1,2,\dots,k\},\{(x,x)\colon 1 \le x \le k\})$, the k-element antichain. Notice that there exists an A-map of G to D_k if and only if G is k-colorable.

To finish the proof of the upper bound, we will make use of the following theorem.

Theorem 5.7. *Let G be a simple digraph and P a partially ordered set. Then $G \xrightarrow{A} \mathscr{A}(P)$ if and only if $\partial G \xrightarrow{A} P$.*

Now we finish the proof of the upper bound. We know that $\chi(\partial G) \le k$ if and only if $\partial G \xrightarrow{A} D_k$ if and only if $G \xrightarrow{A} \mathscr{A}(D_k)$. However, $\mathscr{A}(D_k)$ is isomorphic to the poset $(2^{[k]},\subseteq)$. According to Sperner's theorem, there is an independent system of $t = \binom{k}{\lfloor k/2\rfloor}$ subsets of $2^{[k]}$. So if we can find a t-coloring of the graph G, we can also find an A-map $f : V(G) \to 2^{[k]}$ to satisfy the condition that $f(x) \not\subseteq f(y)$ for every edge (x,y) of the graph G. Therefore if G is t-colorable, then ∂G is k-colorable.

To show that the upper bound is tight, consider a graph $G = (V,E)$ such that $(x,y) \in E$ if and only if $(y,x) \in E$. Then ∂G is k-colorable if and only if there is a mapping $f : V(G) \to 2^{[k]}$ such that $f(x) \parallel f(y)$ whenever (x,y) is an edge of G. Sperner's theorem implies that if ∂G is k-colorable, then G is t-colorable for $t = \binom{k}{\lfloor k/2\rfloor}$. $\square$

Proof (of Theorem 5.7). First, let $f : G \xrightarrow{A} \mathscr{A}(P)$. If $e = (x,y) \in E(G) = V(\partial G)$, then $f(x) \not\le f(y)$, so there exists an element $u \in f(x)$ such that $u \le v$ for no $v \in f(y)$. Set $F(e) = u$ (if there are more such elements, pick any of them). The mapping $F : V(\partial G) \to X$ is an A-map, because if $((x,y),(y,z))$ is an edge of ∂G, then $F(y,z) \in f(y)$ and therefore $F(x,y) \not\le F(y,z)$.

Conversely, let $F : \partial G \xrightarrow{A} P$. For $x \in V(G)$ define $S_x = \{(x,y)\colon (x,y) \in E(G)\}$. Define the mapping $f : V(G) \to \mathscr{A}(P)$ so that $f(x)$ is the set of maximal elements of $F[S_x]$.

Let (x,y) be an edge of the graph G. There exists an element u of $f(x)$ such that $F(x,y) \le u$. If there was an element $v = F(y,z) \in f(y)$ such that $u \le v$, then $F(x,y) \le u \le v = F(y,z)$, but since $((x,y),(y,z))$ is an edge of ∂G, that would contradict the fact that F is an A-map. Therefore $f(x) \not\le f(y)$ and f is an A-map. $\square$

5.5 Coloring order density

Let $\mathscr{C}$ be the set of all non-isomorphic finite cores. Then the binary relation $\le$, defined by $G \le H$ if and only if $G \to H$, is a partial order on $\mathscr{C}$. We write $G < H$ if $G \to H$ and $H \nrightarrow G$.

Theorem 5.8 (Welzl [16])**.** *If G_1 and G_2 are two core graphs such that $G_1 < G_2$, then there exists a graph G such that $G_1 < G < G_2$ unless $(G_1,G_2) \in \{(K_0,K_1),(K_1,K_2)\}$.*

Proof. If $(G_1,G_2) = (K_0,K_2)$, let $G = K_1$. Otherwise G_2 is not bipartite and so it contains an odd cycle. Let H be a graph with odd girth greater than the odd girth of G_2 with

$$\chi(H) > |V(G_1)|^{|V(G_2)|}.$$

We know such a graph exists—we may take a suitable shift graph $S_{n,k}$ described earlier.

Let $G = G_1 \cup (G_2 \times H)$. Obviously $G_1 \to G$ and $G \to G_2$. We have to show that $G_2 \nrightarrow G$ and $G \nrightarrow G_1$.

We can assume that G_2 is connected. We know that $G_2 \nrightarrow G_1$. The product $G_2 \times H$ is not bipartite because otherwise G_2 or H would have to be bipartite. Let $p : G_2 \times H \to H$ be the projection. If C is an odd cycle in $G_2 \times H$, then its image $p[C]$ in H contains an odd cycle that is at most as long as C. Therefore the odd girth of $G_2 \times H$ is greater or equal to the odd girth of H and that is greater than the odd girth of G_2. This implies that $G_2 \nrightarrow G_2 \times H$ and so $G_2 \nrightarrow G$.

Suppose that there is a homomorphism $f : G_2 \times H \to G_1$. To each vertex x of the graph H, assign the mapping $f_x : V(G_2) \to V(G_1)$ such that $f_x(u) = f(u, x)$. Consider the mappings f_x as colors assigned to the vertices of the graph H. There are $k = |V(G_1)|^{|V(G_2)|}$ possible mappings of $V(G_2)$ to $V(G_1)$, but $\chi(H) > k$, so there must be an edge $\{x, y\}$ of H with $f_x = f_y$. If $\{u, v\} \in E(G_2)$, then $\{(u, x), (v, y)\} \in E(G_2 \times H)$ and so $\{f(u, x), f(v, y)\} \in E(G_1)$ because f is a homomorphism. However, $f(u, x) = f_x(u)$ and $f(v, y) = f_y(v) = f_x(v)$, so $\{f_x(u), f_x(v)\} \in E(G_1)$ and f_x is a homomorphism of G_2 to G_1, contradicting the assumption that $G_2 \nrightarrow G_1$. Therefore $G_2 \times H \nrightarrow G_1$ and $G \nrightarrow G_1$ either. □

5.6 The classes $\mathscr{C}_k$

We shall now show how each k-colorable graph is an induced subgraph of a power of a single graph.

We define the *join* $G + G'$ of two graphs $G = (V, E)$ and $G' = (V', E')$ (supposing that $V \cap V' = \emptyset$) in the following way:

$$\begin{aligned} V(G+G') &= V \cup V' \\ E(G+G') &= E \cup E' \cup \{\{v, v'\} : v \in V \wedge v' \in V'\} \end{aligned}$$

Let $\mathscr{C}$ be a set of graphs. The symbol $SP(\mathscr{C})$ will denote the class of all induced subgraphs of (finite) products of some elements of $\mathscr{C}$. If the set $\mathscr{C}$ consists of a single graph, we will use $SP(G)$ instead of $SP(\{G\})$. The symbol P_3 denotes the path of length three.

Theorem 5.9. *Let $k \geq 2$, let $\mathscr{C}_k$ be the class of all k-colorable graphs. Let $A_k = P_3 + K_{k-2}$. Then $\mathscr{C}_k = SP(A_k)$.*

Proof. First we prove that if a graph G is k-colorable, it is an element of $SP(A_k)$. Let $F = \{f : f : G \to A_k\}$ be the set of all homomorphisms of G to A_k. The set F is nonempty, as G is k-colorable and therefore admits a homomorphism to K_k, which is a subgraph of A_k.

Let $H = (A_k)^{|F|}$. We will use the elements of F as indices for the vertices of H; each vertex x of H will thus be an $|F|$-tuple $(x_{f_1}, x_{f_2}, \ldots, x_{f_{|F|}})$, where $f_1, f_2, \ldots, f_{|F|}$ are all the elements of F. Two vertices $x, y \in V(H)$ are connected with an edge if and only if for each $f \in F$ the vertices x_f and y_f form an edge in the graph A_k.

Now we define a mapping $\phi : G \to H$ and show that it is an embedding of G to H. For $v \in V(G)$ and $f \in F$ let $\phi(v)_f = f(v)$.

Let $x, y \in V(G)$, $x \neq y$. If $\{x, y\} \in E(G)$, then for any $f \in F$ it is true that $\{f(x), f(y)\} \in E(A_k)$ because f is a homomorphism. Therefore $\{\phi(x), \phi(y)\} \in E(H)$.

If, on the other hand, $\{x,y\} \notin E(G)$, find a k-coloring c of G. The mapping c is a homomorphism of G to K_k, which is a subgraph of A_k. Consider a homomorphism $f: G \to A_k$ such that $f(z) = c(z)$ for $z \notin \{x,y\}$, and $f(x)$ and $f(y)$ are two distinct nonadjacent vertices on the path P_3 in the graph A_k. Then we have $\phi(x)_f \neq \phi(y)_f$ and $\phi(x)_f \not\sim \phi(y)_f$

We have shown that ϕ is an embedding of G to H, so $\mathscr{C}_k \subseteq SP(A_k)$. It is an easy exercise to show that $SP(A_k) \subseteq \mathscr{C}_k$—just notice that 1. A_k is k-colorable, 2. the product of k-colorable graphs is k-colorable, and 3. any subgraph of a k-colorable graph is k-colorable. □

Remark. We can generalize the notion of k-coloring to H-coloring, where H is a graph. An H-coloring of a graph G is simply a homomorphism of G to H and we ask the question whether $G \to H$. Let $\mathscr{C}_H$ be the class of all graphs that admit a homomorphism to H. For every graph H there exists a finite set of graphs $\mathscr{A}$ such that $\mathscr{C}_H = SP(\mathscr{A})$. The proof is left for the reader as homework.

It is rather surprising that the complementary class $\bar{\mathscr{C}}_k$ of all graphs that are not k-colorable can be generated easily as well.

We need to define a graph operation to be used in the following theorem. Given two graphs G and G' with disjoint vertex sets, and given two vertices x and y of the graph G and two vertices x' and y' of the graph G' such that $\{x,y\} \in E(G)$ and $\{x',y'\} \in E(G')$, we define the *H-join* (Hajós join) of the two graphs to be the graph created from the union of G and G' by identifying x with x', deleting the edges $\{x,y\}$ and $\{x',y'\}$ and adding the edge $\{y,y'\}$; so it is the graph (V,E), where

$$\begin{aligned}
V &= (V(G) \cup V(G')) \setminus \{x'\}, \\
E &= (E(G) \cup E(G') \cup \{\{x,z'\}: \{x',z'\} \in E(G')\} \cup \{\{y,y'\}\}) \setminus \\
&\quad (\{\{x',z'\}: \{x',z'\} \in E(G')\} \cup \{\{x,y\},\{x',y'\}\}).
\end{aligned}$$

Theorem 5.10 (Hajós [4])**.** *Every graph $G \in \bar{\mathscr{C}}_k$ can be generated from the complete graph K_{k+1} by a finite (possibly empty) sequence of the following four operations:*

1. *adding a new vertex,*
2. *adding a new edge connecting two existing vertices,*
3. *contracting a non-edge, i.e., identifying two nonadjacent vertices (and replacing any multiple edges that may appear by a single edge),*
4. *H-joins.*

Conversely, no graph generated from K_{k+1} by a finite sequence of the above four operations is k-colorable.

Proof. We prove the first part of the theorem by induction on $|V|$ and on $\binom{n}{2} - |E|$. Suppose that there exists a graph $G = (V,E)$ such that G is not k-colorable but it cannot be generated by the four operations (we will say that it is not constructible); moreover suppose that all graphs that are not k-colorable and have less than $|V|$ vertices or have $|V|$ vertices and more than $|E|$ edges are constructible.

Obviously, $|E| < \binom{n}{2}$, because complete graphs are constructible (just using operations 1. and 2.). We may assume that G contains $K_1 \cup K_2$ as an induced subgraph—otherwise it would be a complete multipartite graph, contain K_{k+1} and therefore be constructible. So there are vertices x, y and y' such that $\{y, y'\}$ is an edge of the graph G and $\{x, y\}$ and $\{x, y'\}$ are not. Let G' be a copy of G with the edge $\{x, y\}$ added, let G'' be a copy of G with the edge $\{x, y'\}$ added. Both G' and G'' are constructible, because they are not k-colorable and have more edges than G.

Let H be the H-join of the graphs G' and G'', identifying x in G' with x in G'', deleting $\{x, y\}$ in G' and $\{x, y'\}$ in G'' and adding the edge $\{y_{G'}, y'_{G''}\}$. Then H is constructible and identifying all pairs of corresponding vertices in G' and G'' (operation 3.) yields the graph G. This implies that G is constructible—a contradiction. □

Remark (open problem). As a consequence of Hajós' theorem, we get that the shift graphs are constructible. However, the sequence of operations leading to shift graphs has not been described yet.

6 Graph product

6.1 The definition

Many types of products have been described by mathematicians so far. The product we will be talking about is called the *direct product* or the *category product* or simply just the *product* of graphs. The vertex set of the product $G \times H$ is the Cartesian product of the vertex sets of the factors. Two vertices $x = (g, h)$ and $y = (g', h')$ are joined by an edge in the product if and only if $\{g, g'\} \in E(G)$ and $\{h, h'\} \in (H)$. Formally,

$$
\begin{aligned}
V(G \times H) &= V(G) \times V(H), \\
E(G \times H) &= \{\{(g, h), (g', h')\} \colon \{g, g'\} \in E(G) \wedge \{h, h'\} \in E(H)\}.
\end{aligned}
$$

Example 6.1. The product $K_2 \times K_3$ is isomorphic to the cycle C_6 and it is shown in Figure 3.

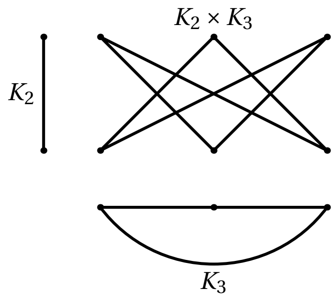

Figure 3: Illustrating the direct product of graphs

6.2 Independence problem

Given a graph G, does there exist a graph H such that G is *incomparable* with H, i.e., $G \nrightarrow H$ and $H \nrightarrow G$? Given a finite set of graphs $\mathcal{G}$, does there exist a graph H that is incomparable with any graph in $\mathcal{G}$? The answer to both questions is positive.

Theorem 6.1. *Let $\mathcal{G}$ be a finite set of graphs such that $\chi(G) \geq 3$ for each $G \in \mathcal{G}$. Then there exists a graph H incomparable with every $G \in \mathcal{G}$.*

Proof. Let $n = \max\{|V(G)|: G \in \mathcal{G}\}$, $g = \max\{\text{girth}(G);\ G \in \mathcal{G}\}$ and $k = \max\{\chi(G):\ G \in \mathcal{G}\}$. Let H' be a graph with odd girth greater than g and chromatic number greater than n^{k+1}. Finally, let $H = K_{k+1} \times H'$.

Now the argument is similar to that we used in the proof of the density theorem. For any $G \in \mathcal{G}$, we have $G \nrightarrow H$, because H has no short odd cycles.

Suppose that there is a graph $G \in \mathcal{G}$ such that $f: H \to G$. For $x \in V(H')$, let $f_x : V(K_{k+1}) \to V(G)$ be the mapping defined by $f_x(u) = f(u,x)$. As there are $|V(G)|^{k+1}$ mutually different mappings of $V(K_{k+1})$ to $V(G)$ and $\chi(H') > n^{k+1} \geq |V(G)|^{k+1}$, there must be an edge $\{x, y\}$ of H' such that $f_x = f_y$. The mapping f_x is a homomorphism of K_{k+1} to G because for $\{u, v\} \in E(K_{k+1})$, the set $\{(u,x),(v,y)\}$ is an edge of $K_{k+1} \times H'$ and so $\{f_x(u), f_x(v)\} = \{f(u,x), f(v,y)\}$ is an edge of G. Since G is k-colorable, we have $K_{k+1} \to G \to K_k$ and that is a contradiction. □

Remark. The previous theorem is not true for directed graphs. There is an infinite number of of finite maximal antichains, classified by Foniok, Nešetřil and Tardif [3].

Remark. If G is a countable graph, then there exists a countable graph H incomparable with G (proved by Nešetřil and Shelah [12]). The generalization for a finite set of countable graphs is not true, however. There is no countable graph incomparable both with K_3 and a universal K_3-free graph. The latter can be constructed by induction: $V_1 = \{0\}$, $E_1 = \emptyset$. If we have V_k and E_k, let

$$\begin{aligned} V' &= \left\{u:\ u: V_k \to \{0,1\} \wedge \tbinom{u^{-1}(1)}{2} \cap E_k = \emptyset\right\} \\ E' &= \{\{u,x\}:\ u \in V' \wedge x \in V_k \wedge u(x) = 1\} \\ V_{k+1} &= V_k \cup V' \\ E_{k+1} &= E_k \cup E' \end{aligned}$$

Then

$$G = \Big(\bigcup_{k=1}^{\infty} V_k, \bigcup_{k=1}^{\infty} E_k\Big)$$

is a universal K_3-free graph (every countable K_3-free graph is embeddable in G).

6.3 Homomorphism cancellation property

Let F and G be graphs. We say that F is *pointed* for G if for any two homomorphisms f and f' of G to F and each vertex x of G it holds that $f(y) = f'(y)$ for all $y \neq x$ implies that $f(x) = f'(x)$ as well.

Theorem 6.2. *Let F, G and H be graphs with $\chi(H) > |V(F)|^{|V(G)|}$. Then $G \times H \to F$ if and only if $G \to F$.*

Moreover, if G is connected and F is pointed for G, then homomorphisms $G \to F$ and homomorphisms $G \times H \to F$ are in a one-to-one correspondence.

Proof. Proof of the existence is the same as in the previous theorem.

Let $A = \{g\colon g : G \to F\}$ and $B = \{f\colon f : G \times H \to F\}$. We are looking for a bijection of A to B.

Let $f : G \times H \to F$, $f_x(u) = f(u, x)$ for $x \in V(H)$. We prove an auxiliary claim that $f_x = f_y$ whenever $\{x, y\} \in E(H)$ and f_x is a homomorphism. Let there be an edge $\{x, y\}$ of H such that $f_x \neq f_y$. Then there is a vertex $u \in V(G)$ that $f_x(u) \neq f_y(u)$. We define a new homomorphism $f' : G \to H$ by $f'(u) = f_y(u)$ and $f'(v) = f_x(v)$ for $v \neq u$ (it is easy to check that f' is a homomorphism). This is a contradiction because f' and f_x only differ at u but F is pointed for G.

Let $p : G \times H \to G$ be the projection. We define $\phi : A \to B$ by $\phi : g \mapsto g \circ p$. Obviously ϕ is an injection. To see that ϕ is a surjection, let $f \in B$. Take an arbitrary $x \in V(H)$. Then we have

$$(\phi(f_x))(u, y) = (f_x \circ p)(u, y) = f_x(u) = f_y(u) = f(u, y),$$

where the third equality is a consequence of the auxiliary claim and the fact that G is connected. Therefore we get $\phi(f_x) = f$. $\square$

Remark. Setting $F = G$ we get that there are uniquely F-colorable graphs with arbitrarily large odd girth, because F is pointed for F whenever F is a core.

Setting $F = G = K_k$ we get that there are uniquely k-colorable graphs with arbitrarily large odd girth.

6.4 Hedetniemi's conjecture

The homomorphisms $\pi_1 : G_1 \times G_2 \to G_1$ and $\pi_2 : G_1 \times G_2 \to G_2$ defined by $\pi_1(v_1, v_2) = v_1$ and $\pi_2(v_1, v_2) = v_2$ are called *projections*.

Proposition 6.3. *Whenever G is a graph such that $g_1 : G \to G_1$ and $g_2 : G \to G_2$ are homomorphisms, then there exists a unique homomorphism $f : G \to G_1 \times G_2$ such that $\pi_j \circ f = g_j$ for $j = 1, 2$.*

The proof is left as an exercise.

Corollary 6.4. *A graph G is homomorphic to the product $G_1 \times G_2$ if and only if $G \to G_1$ and $G \to G_2$.*

Corollary 6.5. $\chi(G_1 \times G_2) \le \min\{\chi(G_1), \chi(G_2)\}$.

Hedetniemi [5] conjectured that the inequality above actually holds with equality: $\chi(G_1 \times G_2) = \min\{\chi(G_1), \chi(G_2)\}$. In spite of substantial effort of many mathematicians the problem remains wide open to this day. The conjecture is trivially true when the minimum is at most 3 and nontrivially true when the minimum is 4.

Let $f(k) = \min\{\chi(G_1 \times G_2)\colon \chi(G_1), \chi(G_2) \ge k\}$; this function is called the *Poljak-Rödl function.* Hedetniemi's conjecture states that $f(k) = k$ for all k. At present, however, f is not even known to be unbounded. It is known that either $f(k)$ is unbounded or $f(k) \le 9$ for all k [15, 17].

7 Isomorphisms of graphs

Let us now concentrate on the issue of deciding, given two finite graphs G and H, whether G is isomorphic to H ($G \cong H$).

7.1 Lovász' theorem

The symbol $\langle F, G\rangle$ will denote the number of all homomorphisms of F to G. Let $F_1, F_2, F_3, \ldots$ be all non-isomorphic finite graphs. The *Lovász vector* of a graph G is $\langle G\rangle = (n_1, n_2, n_3, \ldots)$, where $n_k = \langle F_k, G\rangle$.

Theorem 7.1 (Lovász [7])**.** *Two finite graphs G and H are isomorphic if and only if $\langle G\rangle = \langle H\rangle$.*

Proof. It is evident that if $G \cong H$, then $\langle G\rangle = \langle H\rangle$.

Let $\langle\langle F, G\rangle$ denote the number of all monomorphisms (injective homomorphisms) of F to G.

Suppose that $\langle G\rangle = \langle H\rangle$. Then for an arbitrary graph F, it is true that $\langle\langle F, G\rangle = \langle\langle F, H\rangle$. Let us prove this claim by induction on the number of vertices of the graph F. First, if $|V(F)| = 1$, then $\langle\langle F, G\rangle = \langle F, G\rangle = \langle F, H\rangle = \langle\langle F, H\rangle$. If $|V(F)| > 1$, then

$$\begin{aligned}\langle F, G\rangle &= \sum_{\Theta \in Eq(V(F))} \langle\langle F/\Theta, G\rangle \\ &= \langle\langle F, G\rangle + \sum_{\substack{\Theta \in Eq(V(F)) \\ \Theta \neq \mathrm{id}}} \langle\langle F/\Theta, G\rangle,\end{aligned}$$

where $Eq(V(F))$ is the set of all equivalence relations on $V(F)$ and F/Θ is the graph whose vertex set is the set of all equivalence classes of Θ and an edge connects two classes c and c' if there are vertices $u \in c$ and $u' \in c'$ so that $\{u, u'\}$ is an edge of F. (Note that loops may occur in F/Θ.) This is because every homomorphism $f : F \to G$ corresponds to a monomorphism of F/Θ to G for $\Theta = \{(u, u') : f(u) = f(u')\}$.

Similarly, we get

$$\langle F, H\rangle = \langle\langle F, H\rangle + \sum_{\substack{\Theta \in Eq(V(F)) \\ \Theta \neq \mathrm{id}}} \langle\langle F/\Theta, H\rangle.$$

By induction, we know that for any $\Theta \in Eq(V(F))$, $\Theta \neq \mathrm{id}$,

$$\langle\langle F/\Theta, G\rangle = \langle\langle F/\Theta, H\rangle,$$

since $|V(F/\Theta)| < |V(F)|$. Therefore we have $\langle\langle F, G\rangle = \langle\langle F, H\rangle$.

Applying the equality for $F = G$ and $F = H$ we get $\langle\langle G, H\rangle = \langle\langle G, G\rangle \geq 1$ and $\langle\langle H, G\rangle = \langle\langle H, H\rangle \geq 1$. If there is a monomorphism of G to H and a monomorphism of H to G, then G and H are isomorphic. $\square$

Lovász' theorem has a number of interesting consequences.

Corollary 7.2. *Let G and H be graphs. If $G^2 \cong H^2$, then $G \cong H$.*

Proof. Let F be a graph. Every homomorphism $f : F \to G^2$ corresponds to a pair of homomorphisms (f_1, f_2) of F to G; if $f(u) = (x_1, x_2)$, then $f_i(u) = x_i$. Moreover, the correspondence is one-to-one (due to the categorical properties of the product). Therefore

$$\langle F, G\rangle^2 = \langle F, G^2\rangle = \langle F, H^2\rangle = \langle F, H\rangle^2$$

and so $\langle F, G\rangle = \langle F, H\rangle$. □

Corollary 7.3. *Let A, B and C be graphs, let C have a loop. If $A \times C \cong B \times C$, then $A \cong B$.*

Proof. For a graph F, we have $\langle F, A\times C\rangle = \langle F, A\rangle \cdot \langle F, C\rangle$ and $\langle F, B\times C\rangle = \langle F, B\rangle \cdot \langle F, C\rangle$. Since C has a loop, $\langle F, C\rangle \neq 0$. Therefore $\langle F, A\times C\rangle = \langle F, B\times C\rangle$ implies $\langle F, A\rangle = \langle F, B\rangle$. □

Corollary 7.4. *Let A, B, C and D be graphs. If $D \to C$ and $A \times C \cong B \times C$, then $A \times D \cong B \times D$.*

Proof. Let F be a graph. If $\langle F, D\rangle = 0$, then

$$\langle F, A\times D\rangle = \langle F, A\rangle \cdot \langle F, D\rangle = 0 = \langle F, B\rangle \cdot \langle F, D\rangle = \langle F, B\times D\rangle.$$

If $\langle F, D\rangle \neq 0$, then $\langle F, C\rangle \neq 0$ and $\langle F, A\rangle = \langle F, B\rangle$ as before. This yields

$$\langle F, A\times D\rangle = \langle F, A\rangle \cdot \langle F, D\rangle = \langle F, B\rangle \cdot \langle F, D\rangle = \langle F, B\times D\rangle.$$

□

Remark. It is not generally true that $A\times C \cong B\times C$ implies $A \cong B$. A counterexample: A consists of two isolated loops, $B = C = K_2$. Another counterexample: $A = K_3$, $B = C_6$ (the cycle of length 6), $C = K_2$.

Remark. If A, B and C are not bipartite, then they have the cancellation property: $A \times C \cong B \times C \implies A \cong B$.

Remark. If we set $\langle G\rangle' = (\langle G, F_i\rangle;\ i = 1, 2, 3, \ldots)$, then it can be proved that $\langle G\rangle' = \langle H\rangle'$ if and only if $G \cong H$. The proof uses the inclusion-exclusion principle.

7.2 Ulam's conjecture

For a graph G, let $S(G)$ be the set of all proper induced subgraphs of G. Ulam's conjecture states that G is isomorphic to H if and only if there is a bijection $\iota : S(G) \to S(H)$ such that $\iota(G') \cong G'$ (the existence of such a bijection will be denoted by $S(G) \leftrightarrow S(H)$). This conjecture is called *vertex reconstruction.*

Notice that $S(G)$ contains *all* proper induced subgraphs of G, although some of them may be isomorphic.

Edge reconstruction is a similar conjecture, taking all (not just induced) subgraphs. So $S'(G)$ is the set of all proper subgraphs of G and the conjecture is that $G \cong H$ if and only if $S'(G) \leftrightarrow S'(H)$.

A different formulation of the conjecture is the following: assign each graph G a set $s(G)$ of its subgraphs, $s(G) = \{G - v\colon\ v \in V(G)\}$. The edge version is $s'(G) = \{G - e\colon\ e \in E(G)\}$.

Knowing $S(G)$ is equivalent to knowing $s(G)$. If we know $S(G)$, then $s(G)$ is formed by the elements of $S(G)$ with the greatest number of vertices. Conversely, if we know $s(G)$, take the multiset $M = s(G) \cup \bigcup_{H \in s(G)} S(H)$. Every element H of $S(G)$ is contained $|V(G)| - |V(H)|$ times in M.

Similarly, knowing $S'(G)$ is equivalent to knowing $s'(G)$.

The only counterexample known so far for vertex reconstruction is that K_2 and the graph consisting of two isolated vertices have the same proper subgraphs. However, vertex reconstruction does not hold for directed graphs.

It can be shown that vertex reconstruction implies edge reconstruction.

Vertex reconstruction is known to be true for trees, disconnected graphs (with some exceptions) and planar graphs. Edge reconstruction is true for regular graphs as well.

We will prove the following theorem.

Theorem 7.5 (Müller)**.** *Let* $G = (V,E)$ *and* $G' = (V',E')$*. Let* $|V| = n$ *and* $|E| \geq 1 + n(\log n - 1)$*. Then* $G \cong H$ *if and only if* $s'(G) \leftrightarrow s'(H)$*.*

Proof. Suppose that $s'(G) \leftrightarrow s'(H)$. Then, of course, G and H have the same number of vertices and the same number of edges $|E| = |E'| = m$. For a bijection $f : V \to V'$, we define the *defect* of f to be the set $D(f) = \{e \in E \colon f[e] \notin E'\}$. For $D \subseteq E$, let $\langle\langle G,H\rangle_D$ be the number of all bijections of V to V' whose defect is D, and for a non-negative integer d let $\langle\langle G,H\rangle_d$ be the number of all bijections $f : V \to V'$ such that $|D(f)| = d$. Note that $D(f) = \emptyset$ if and only if f is a monomorphism.

Further, notice that

$$\sum_{D \subseteq E(G)} \langle\langle G,H\rangle_D = \sum_{d=0}^{m} \langle G,H\rangle_d = n!$$

Assume, by way of contradiction, that $G \ncong H$ and, without loss of generality, that $\langle\langle G,H\rangle = 0$. Let $\bar{H} = (V', \binom{V'}{2} \setminus E')$ be the complement of H. Using the inclusion-exclusion principle we have

$$\begin{aligned} \langle\langle G,\bar{H}\rangle &= \langle\langle (V,\emptyset),H\rangle - \sum_{e \in E} \langle\langle (V,\{e\}),H\rangle + \sum_{D \in \binom{E}{2}} \langle\langle (V,D),H\rangle \\ &\quad - \sum_{D \in \binom{E}{3}} \langle\langle (V,D),H\rangle + \cdots + (-1)^m \langle\langle G,H\rangle \end{aligned}$$

and

$$\begin{aligned} \langle\langle H,\bar{H}\rangle &= \langle\langle (V',\emptyset),H\rangle - \sum_{e \in E'} \langle\langle (V',\{e\}),H\rangle + \sum_{D \in \binom{E'}{2}} \langle\langle (V',D),H\rangle \\ &\quad - \sum_{D \in \binom{E'}{3}} \langle\langle (V',D),H\rangle + \cdots + (-1)^m \langle\langle H,H\rangle. \end{aligned}$$

Since G and H have the same proper subgraphs,

$$|\langle\langle G,\bar{H}\rangle - \langle\langle H,\bar{H}\rangle| = |\langle\langle G,H\rangle - \langle\langle H,H\rangle| = \langle\langle H,H\rangle \neq 0$$

and therefore $m \le \binom{n}{2}/2$, because otherwise $\langle\langle G, \bar{H}\rangle = \langle\langle H, \bar{H}\rangle = 0$ as the existence of a monomorphism of a graph A to a graph B implies that $|E(A)| \le |E(B)|$.

However, we want to show that m is even smaller, and that is what we need the defects for.

Let $D = \{e_1, e_2, \dots, e_d\}$. Then

$$\begin{aligned}
\langle\langle G, \bar{H}\rangle_D &= \langle\langle (V, D), H\rangle - \sum_{e \in E\setminus D} \langle\langle (V, D \cup \{e\}), H\rangle + \sum_{D' \in \binom{E\setminus D}{2}} \langle\langle (V, D \cup D'), H\rangle \\
&\quad - \sum_{D' \in \binom{E\setminus D}{3}} \langle\langle (V, D \cup D'), H\rangle + \cdots + (-1)^{m-d} \langle\langle G, H\rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle\langle G, \bar{H}\rangle_d &= \sum_{D \in \binom{E}{d}} \langle\langle G, \bar{H}\rangle_D \\
&= \sum_{D \in \binom{E}{d}} \langle\langle (V, D), H\rangle - \binom{d+1}{d} \sum_{D' \in \binom{E}{d+1}} \langle\langle (V, D'), H\rangle \\
&\quad + \binom{d+2}{d} \sum_{D' \in \binom{E}{d+2}} \langle\langle (V, D'), H\rangle \\
&\quad - \binom{d+3}{d} \sum_{D' \in \binom{E}{d+3}} \langle\langle (V, D'), H\rangle \\
&\quad + \cdots + (-1)^{m-d} \binom{m}{d} \langle\langle G, H\rangle.
\end{aligned}$$

The same is true for $\langle\langle H, \bar{H}\rangle_d$:

$$\begin{aligned}
\langle\langle H, \bar{H}\rangle_d &= \sum_{D \in \binom{E'}{d}} \langle\langle (V', D), H\rangle - \binom{d+1}{d} \sum_{D' \in \binom{E'}{d+1}} \langle\langle (V', D'), H\rangle \\
&\quad + \binom{d+2}{d} \sum_{D' \in \binom{E'}{d+2}} \langle\langle (V', D'), H\rangle \\
&\quad - \binom{d+3}{d} \sum_{D' \in \binom{E'}{d+3}} \langle\langle (V', D'), H\rangle \\
&\quad + \cdots + (-1)^{m-d} \binom{m}{d} \langle\langle H, H\rangle.
\end{aligned}$$

By subtracting the two formulas, we get

$$|\langle\langle G, \bar{H}\rangle_d - \langle\langle H, \bar{H}\rangle_d| = \binom{m}{d} |\langle\langle G, H\rangle - \langle\langle H, H\rangle| = \binom{m}{d} \langle\langle H, H\rangle.$$

Therefore

$$\sum_{d=0}^{m} |\langle\langle G, \bar{H}\rangle_d - \langle\langle H, \bar{H}\rangle_d| = 2^m \cdot \langle\langle H, H\rangle \geq 2^m.$$

On the other hand,

$$\begin{aligned}\sum_{d=0}^{m} |\langle\langle G, \bar{H}\rangle_d - \langle\langle H, \bar{H}\rangle_d| &\leq \sum_{d=0}^{m} \langle\langle G, \bar{H}\rangle_d + \sum_{d=0}^{m} \langle\langle H, \bar{H}\rangle_d \\ &= 2n! < 2 \cdot \left(\frac{n}{2}\right)^n,\end{aligned}$$

if $n > 2$.

We have proved that if $s'(G) \leftrightarrow s'(H)$ and $G \not\cong H$, then $2^m < 2 \cdot \left(\frac{n}{2}\right)^n$ and so $m < 1 + n(\log_2 n - 1)$. ◻

8 Projective graphs

A graph G is *projective*, if every homomorphism $f : G \times G \times \cdots \times G \to G$ satisfying the property that $f(x, x, \ldots, x) = x$ is a projection.

Remark. Every projective graph is connected.

Theorem 8.1 (Larose, Tardif)**.** *A graph G is projective if and only if every homomorphism $f : G \times G \to G$ satisfying the property that $f(x, x) = x$ is a projection.*

Remark. A rigid graph is projective if and only if every homomorphism $f : G \times G \to G$ is a projection.

A graph G has the *k-extension property*, if for any $A, B \subseteq V(G)$ such that $A \cap B = \emptyset$ and $|A| + |B| \leq k$, there exists a vertex x of G ($x \notin A \cup B$) connected by an edge to all vertices in A and not connected to any vertex in B.

A graph G is *fair*, if for any four distinct vertices u, v, w and x of G there exists a vertex $z \notin \{u, v, w, x\}$ connected to x, u and v and not connected to x.

Theorem 8.2 (Erdős)**.** *Asymptotically almost all graphs have the k-extension property.*

Corollary 8.3. *Asymptotically almost all graphs are fair.*

Remark. Cherlin's problem: It is not known whether there exist arbitrarily large triangle-free graphs with the k-extension property for triangle-free graphs (i.e., for any $A, B \subseteq V(G)$ such that A is an independent set in G, $A \cap B = \emptyset$ and $|A| + |B| \leq k$, there exists a vertex x of G, $x \notin A \cup B$, connected by an edge to all vertices in A and not connected to any vertex in B).

Lemma 8.4. *If G is fair and $f : G \times G \to G$ satisfies $f(x, x) = x$, then $f(v, w) \in \{v, w\}$.*

Proof. Suppose $f(v, w) = u \notin \{v, w\}$. As G is fair, there exists a vertex t connected to v and w and not connected to u. Then $(t, t) \sim (v, w)$ in $G \times G$ and $t = f(t, t) \sim f(u, v) = u$ is a contradiction. ◻

Lemma 8.5. *If G is fair, $f : G \times G \to G$ satisfies $f(x,x) = x$ and v, w, r and s are distinct vertices of G such that $f(v,w) = v$, then $f(r,s) = r$.*

Proof. For contradiction, suppose that $f(r,s) = s$. Since G is fair, there exists a vertex s' of G connected to r, v and w and not connected to s and a vertex v' connected to r, s and w and not connected to v.

Then both $\{(s',v'),(v,w)\}$ and $\{(s',v'),(r,s)\}$ are edges of $G \times G$ and therefore $f(s',v')$ is connected to both $f(v,w) = v$ and $f(r,s) = s$. That is a contradiction since we know that $f(s',v') \in \{s',v'\}$. □

Theorem 8.6 (Łuczak, Nešetřil [8])**.** *Asymptotically almost all graphs are projective.*

Proof. Asymptotically almost all graphs are fair (Corollary 8.3). All fair graphs are projective (Lemma 8.4, Lemma 8.5). □

Remark. Recall that if a graph F is pointed for a connected graph G, then for a graph H with sufficiently large chromatic number, homomorphisms of G to F are in a one-to-one correspondence with homomorphisms of $G \times H$ to F.

Applying this to $F = K_k$ and $G = K_k^t$ yields that there are graphs with arbitrarily large odd girth which have exactly t non-equivalent k-colorings.

More generally, we can get the following: if H is a projective graph, A is a set, ρ_1, ρ_2,..., ρ_t are partitions of A into $|V(H)|$ parts, then there exists a graph G (with large odd girth) such that $A \subseteq V(G)$ and any homomorphism $f : G \to H$ coincides on A with one of the partitions.

Remark. Nešetřil and Zhu [14] proved that for $k \geq 2d$ the graph $K_{k/d}$ is projective and therefore the complete graph K_k is projective.

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